ON COMPLETENESS OF INTEGRAL MANIFOLDS OF NULLITY DISTRIBUTIONS

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ABSTRACT. We give a conceptual proof of the fact that if M is a complete submanifold of a space form, then the maximal integral manifolds of the nullity distribution of its second fundamental form through points of minimal index of nullity are complete.

1. INTRODUCTION

Let M be a submanifold of a space form and let \mathcal{N} be the nullity distribution of its second fundamental form. The index of nullity of M at p is the dimension of \mathcal{N}_p . It is well known, from Codazzi equation, that \mathcal{N} is an autoparallel distribution restricted to the open and dense subset of M where the index of nullity is locally constant.

If M is complete and one restricts to the open subset U of points of M where the index of nullity is minimal, then the integral manifolds of \mathcal{N} through points of U are also complete, from a result of Ferus [Fer71]. We will give a conceptual proof of this result, as a corollary of a general theorem, whose proof involves very simple geometric ideas.

2. Main results

Lemma. Let M be a Riemannian manifold and let $f : M \to N$ be a differentiable function of constant rank such that f(M) is an embedded submanifold of N (this can always be assumed locally). Assume that the distribution ker(df) is autoparallel and let Σ be an integral manifold of ker(df). Let $\gamma(t)$ be a geodesic in Σ , $v \in T_{f(\gamma(0))}f(M)$ and let J(t) be the horizontal lift of v along γ , i.e., $J(t) \in \text{ker}(df_{\gamma(t)})^{\perp}$ and df J(t) = v. Then J(t)is a Jacobi vector field along γ .

Proof. Let c(s) be a (short) curve in f(M) such that c'(0) = v. The horizontal lift of c(s) through points of Σ gives rise to a perpendicular variation $\tilde{c}_q(s)$ by totally geodesics submanifolds, which must be by isometries. Therefore $\tilde{c}_{\gamma(t)}(s)$ is a variation by geodesics whose associated variation field is J(t).

Theorem. Let M be a complete Riemannian manifold, $f : M \to N$ be a differentiable function and let U be the open subset of M where the rank of f is maximal. Assume that $\ker(df)_{|U}$ is autoparallel. Then its integral manifolds are complete.

Proof. Let Σ be a totally geodesic integral manifold of ker(df) through a point $p \in U$ and let $\gamma : [0, b) \to \Sigma$ be a maximal geodesic in Σ .

Observe that f has maximal rank in a neighborhood of each point of Σ . From the local form of maps of constant rank it is not difficult to see that given $t_1, t_2 \in [0, b)$ there are open neighborhoods V_1 and V_2 of $\gamma(t_1), \gamma(t_2)$ such that $f(V_1)$ and $f(V_2)$ are embedded submanifolds of N and $f(V_1) \cap f(V_2)$ contains an open neighborhood of $f(\gamma(t_1)) = f(\gamma(t_2))$ in both $f(V_1)$ and $f(V_2)$. In particular, $T_{f(\gamma(t_1))}f(V_1) = T_{f(\gamma(t_2))}f(V_2) =: \mathbb{V}$.

Let $v \in \mathbb{V}$ and apply the previous lemma to define a Jacobi field J along γ that projects down to v. Since M is complete $\gamma(b)$ and J(b) are well defined and J(b), by the continuity of df, also projects down to v. Then $df_{\gamma(b)}(T_{\gamma(b)}M)$ contains \mathbb{V} . So, rank $(df_{\gamma(b)}) = \operatorname{rank}(df_{\gamma(0)})$ and therefore $\gamma(b) \in \Sigma$.

If M^n is a submanifold of the Euclidean space \mathbb{R}^{n+k} , the Gauss map of M is the map $G: M \to G_k(\mathbb{R}^{n+k})$ definded by $p \mapsto \nu_p M$, where $\nu_p M$ denotes the normal space of M at p. If M^n is a submanifold of the sphere $\mathbb{S}^{n+k} \subset \mathbb{R}^{n+k+1}$, then the Gauss map of M is defined to be the map $G: M \to G_k(\mathbb{R}^{n+k+1})$ that sends each point to its normal space in the sphere, regarded as a subspace of \mathbb{R}^{n+k+1} (see the remark bellow). A similar construction can be made for a submanifold M^n of the hyperbolic space H^{n+k} , regarded as a submanifold of the Lorentz space $\mathbb{R}^{n+k,1}$.

It is well known that in the three cases, the nullity distribution of M coincides with the kernel of its Gauss map. Therefore we get,

Corollary. Let M be a complete submanifold of a space form. Then any maximal integral manifold of the nullity distribution of M through a point of minimal index of nullity is complete.

Remark. Let M be a submanifold of a space form and let ν_1 be a parallel sub-bundle of the normal bundle. One can regard to the nullity of the second fundamental form projected to this sub-bundle. This is equivalent to regard the common kernell of all shape operators of vectors in this sub-bundle. This generalized nullity space coincides with the kernell of the generalized Gauss map $p \mapsto \nu_1(p)$. So, as in the corollary, if M is complete one has completeness of the integral manifolds where the kernel has minimal dimension.

References

[Fer71] D. Ferus. On the completeness of nullity foliations. Michigan Math. J., 18:61–64, 1971.

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