Facultad de Ciencias Exactas, Ingeniería y Agrimensura

Escuela de Posgrado y Ed. Continua

TATG - 2016

Clases de grafos

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1. Preliminares

Definición de clases o familias de grafos:

1. de acuerdo a su *dibujo*. Ejemplos: caminos, ciclos, árboles, orugas (Fig. 1), paw, claw ($K_{1,3}$), complete suns (Fig. 3), windmills (Fig. 4), etc.

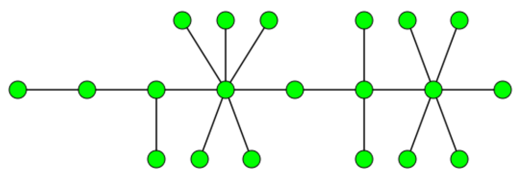


Figura 1: Oruga

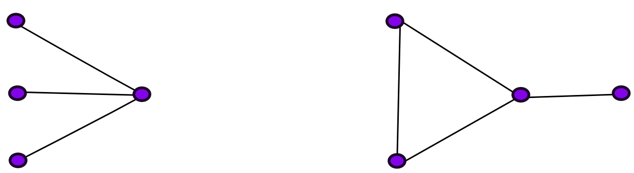


Figura 2: Claw, Paw

2. por su utilidad en la resolución de un problema determinado. Ejemplos: grafos perfectos, grafos planares.

3. por su construcción. Ejemplos: grafos de intersección, grafos serie-paralelo, grafos distancia hereditaria.

4. por *subestructuras* prohibidas. Pueden ser *subgrafos* prohibidos (grafos perfectos) o *menores* prohibidos (planares). Si F es una familia de grafos fija, finita o no, se llaman F-free o F-minor-free, respectivamente.

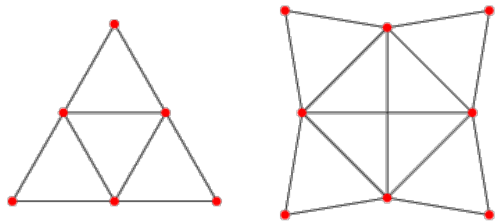


Figura 3: Complete sun

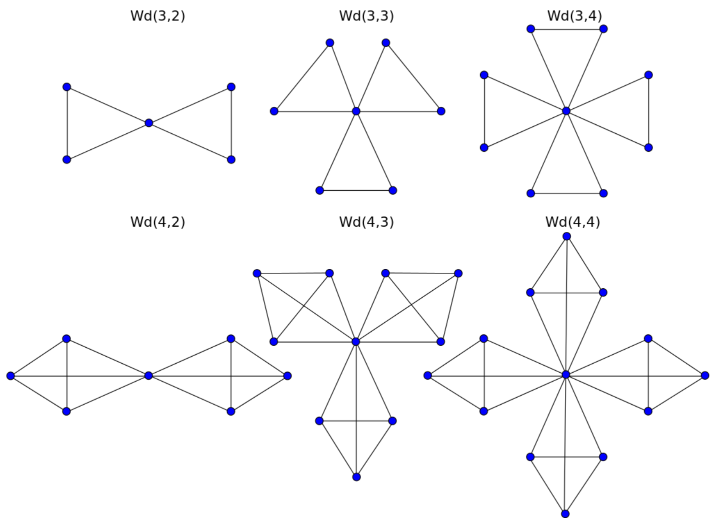


Figura 4: Windmill

Notación:

1. S_n estable con n vértices, K_n clique (o completo) con n vértices
2. $K_{n,m}$ bipartito completo con n y m vértices
3. P_n camino con n vértices, C_n ciclo sin cuerdas con n vértices. Si $n \geq 5$ se llama agujero (hole)
4. F_n fan con $n+1$ vértices (P_n y vértice universal)
5. W_n wheel con $n+1$ vértices (C_n y vértice universal)

G grafo, $U \subseteq V(G)$: G_U o $G[U]$ subgrafo de G inducido por U. Una familia de grafos $\mathcal{F}$ se dice *hereditaria* si para todo grafo $G \in F$ y todo $U \subseteq V(G)$, el grafo $G[U] \in \mathcal{F}$.

Ejemplo: planares. Antiejemplo: ciclos.

2. Algunas familias particulares

2.1. Grafos de interseccion

Given a family of sets S_i, $i = 1, 2, \ldots, n$ its *intersection graph* is the graph G such that $V(G) = \{S_i, i = 1, 2, \ldots, n\}$ and $E(G) = \{S_i, S_j : S_i \cap S_j \neq \emptyset\}$.

In particular

- An *interval* graph is defined as the intersection graph of intervals on the real line, or of connected subgraphs of a path graph.
- A *circular arc* graph is defined as the intersection graph of arcs on a circle.
- A *unit disk* graph is defined as the intersection graph of unit disks in the plane.
- *Knesser* graphs: $K_{n,k}$, $k \leq \frac{n}{2}$

Let G be a graph. Its *line graph* $L(G)$ is defined as the intersection graph of the edges of G, where we represent each edge as the set of its two endpoints. Ant its *clique graph* $C(G)$ is the intersection graph of its maximal cliques.

2.2. Distancia hereditarios

A graph G is *distance-hereditary* if it is connected and the distance function in every connected induced subgraph of G is the same as in G itself.

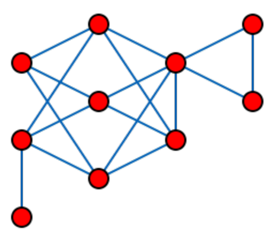


Figura 5: Grafo distancia hereditario

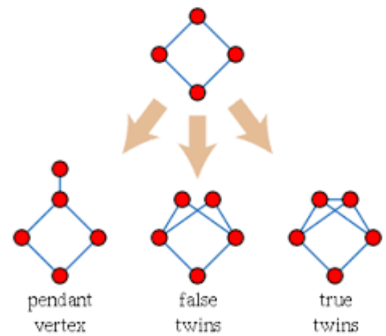
Teorema 1. *The following definitions are equivalent:*

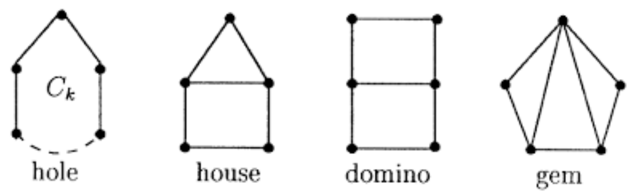
1. *G is distance-hereditary*
2. *G can be construct from a single vertex by the following operations:*

   Adding a pendant vertex: A new vertex that is adjacent to precisely one existing vertex of the graph.

   Creating false twins: For an existing vertex x add a new vertex y with neighbours $N(x) = \{y \in V(G) : xy \in E(G)\}$.

   Creating true twins: For an existing vertex x add a new vertex y with neighbours $N[x] = N(x) \cup \{x\}$.





3. *G is HHDG-free (hole, house, domino, gem)*

Demostración. Ver [2] □

2.3. Grafos cordales

A graph is *chordal* if every cycle of length at least 4 has a chord (C_4-free).

Teorema 2. *The following definitions are equivalent:*

1. *G is chordal*

2. *G is the intersection graph of subtrees of a tree T. In particular T can be chosen such that each vertex of T corresponds to a maximal clique of G and the subtrees T_v consist of precisely those maximal cliques in G that contain v. T is then called the clique tree of G.*

3. *Every minimal cutset in every induced subgraph of G is a clique.*

Demostración. Ver [4] y [5]. □

A vertex v of G is called simplicial in G if $N(v)$ is a clique in G.

The ordering $v_1, \ldots, v_n$ of the vertices of G is a perfect elimination order of G if for all i, v_i is simplicial in $G[v_1, \ldots, v_i]$.

Teorema 3. *G is chordal iff G has a perfect elimination order.*

Demostración. Ver en D.J. Rose, R.E. Tarjan, G.S. Lueker Algorithmic aspects of vertex elimination on graph SIAM J. Computing 5 1976 266–283 □

If G is a graph with (induced) subgraphs G_1, G_2 and S such that $G = G_1 \cup G_2$ and $S = G_1 \cap G_2$ we say that G arises from G_1 and G_2 by *pasting* these graphs along S.

Teorema 4. *A graph is chordal iff it can be constructed recursively by pasting along complete subgraphs, starting from complete graphs.*

Demostración. Ver [4] $\square$

2.4. Cografos

G is a *cograph* (short for complement-reducible graph) if can be constructed from isolated vertices by disjoint union and join operations.

Teorema 5. *The following statements are equivalent:*

1. *G is a cograph.*
2. *Every nontrivial induced subgraph of G has at least one pair of twins (T o F).*
3. *In every induced subgraph H of G, the intersection of any maximal clique and any maximal independent set contains precisely one vertex.*
4. *G is P_4-free.*
5. *The complement of every nontrivial connected induced subgraph of G is disconnected.*
6. *Every connected induced subgraph of G has diameter at most 2.*

Demostración. Las pruebas de estas equivalencias están en distintas publicaciones.

- $1 \Rightarrow 2 \Rightarrow 3 \Rightarrow 4$ y $5 \Rightarrow 1$ [3]
- $4 \Leftrightarrow 5$ en [7]
- $4 \Leftrightarrow 6$ en D.P. Sumner, Dacey graphs, J. Australian Math, 1974.

$\square$

2.4.1. Graphs with *few* P_4's

Una generalización

Let G be a graph and A a P_4 in G. A *partner* of A is a vertex $v \in V(G) - A$ such that $A \cup \{v\}$ induces at least 2 P_4's in G.

- P_4-reducible: no vertex belongs to more than one P_4.
- P_4-sparse: no induced P_4 has a partner.
- P_4-tidy: every induced P_4 has at most one partner.

- *partner limited* (PL): any induced P_4 has at most two partners.
- P_4-lite: every set of at most six vertices induces at most two P_4 or a spider.
- P_4-laden: every set of at most six vertices induces at most two P_4 or a split.

Otra generalización

G is a (q,t)-*graph* if no set of at most q vertices induces more than t distinct P_4's (Babel and Olariu).

In particular, $(q, q-4)$ graphs:

- $q = 4$: Cograph.
- $q = 5$: P_4-sparse graphs.

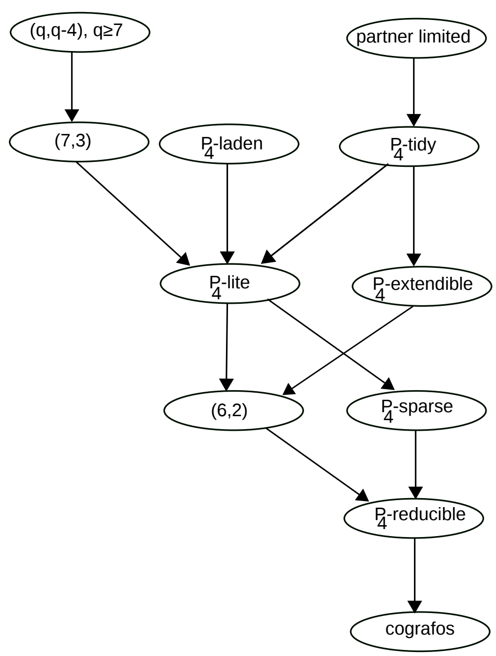


Figura 6: Grafos con pocos P_4

Estas familias de grafos se relacionan como se muestra en la Figura 6.

2.5. (multi) Grafos series-parallel

A graph is an sp-graph, if it may be turned into K_2 by a sequence of the following operations:

1. Replacement of a pair of parallel edges with a single edge that connects their common endpoints.
2. Replacement of a pair of edges incident to a vertex of degree 2 other than s or t with a single edge.

The S operation entails deleting a vertex of degree 2 and adding a new edge between its neighbours. The P operation entails deleting an edge parallel to another edge.

Teorema 6. *Let G be a multigraph. The following definitions are equivalent:*

1. *G is sp.*

2. *its 2-connected components can be generated from a loop with S and P operations.*

3. *G is K_4-minor-free.*

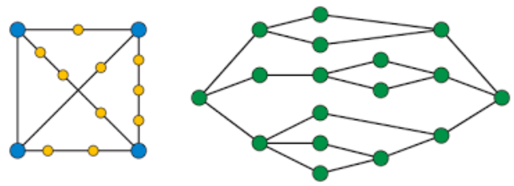


Figura 7: Grafo no sp y sp

3. Operaciones de grafos

Hasta ahora vimos

- pasting (o k-suma)
- Unión disjunta y join
- Agregado de: vértices pendientes, true twins, false twins, aristas paralelas
- Subdivisión de aristas

3.1. Reemplazo de un vértice de G por un grafo H

For disjoint graphs G and H and $v \in V(G)$, $G[H/v]$ denotes the graph obtained by the *substitution* in G of v by H, i.e.

$V(G[H/v]) = (V(G) \cup V(H)) - \{v\}$ and

$E(G[H/v]) = E(H) \cup \{e : e \in E(G)$ and e is not incident with $v\} \cup \{uw : u \in V(H),\ w \in V(G)$ and w is adjacent to v in $G\}$.

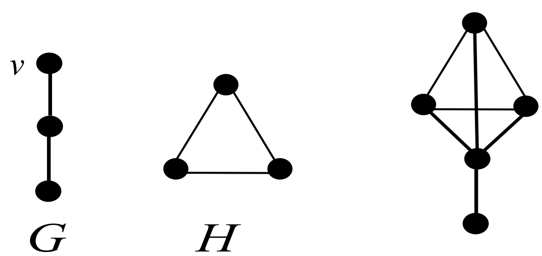


Figura 8: Substitution of a vertex

En particular:

1. Agregado de true twin de un vértice v = reemplazo de v por K_2
2. Agregado de false twin de un vértice v = reemplazo de v por S_2

3.2. Productos

The *product* $G * H$ of the graphs G and H is a graph such that the vertex set of $G * H$ is the cartesian product $V(G) \times V(H)$.

There exist several products according to the definition of $E(G * H)$:

- **Lexicographic product** $G \times H$ (u, v) and (x, y) are adjacent in $G \times H$ if and only if either $ux \in E(G)$ or
  $u = x$ and $vy \in E(H)$.

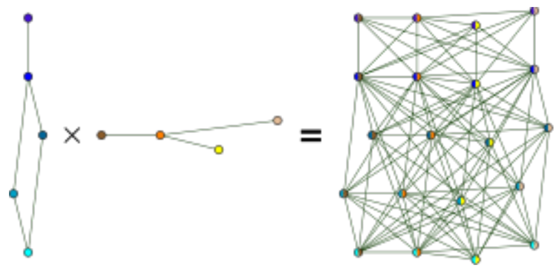


Figura 9: Lexicographic product

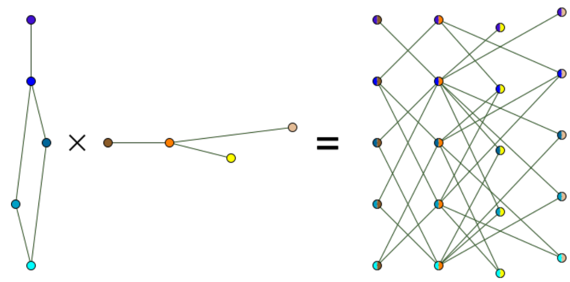


Figura 10: Categorical product

- **Categorical product** $G \bullet H$ (u, v) and (x, y) are adjacent in $G \bullet H$ if and only if $ux \in E(G)$ and $vy \in E(H)$.
- **Cartesian product** $G \Box H$: (u, v) and (x, y) are adjacent in $G \Box H$ if and only if
  $u = x$ and $vy \in E(H)$, or
  $v = y$ and $ux \in E(G)$.

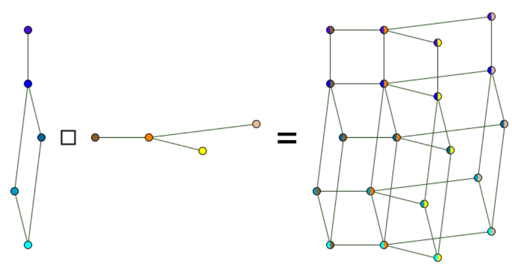


Figura 11: Cartesian product

- **Strong product** $G \boxtimes H$ (u, v) and (x, y) are adjacent in $G \boxtimes H$ if and only if
  $u = x$ and $vy \in E(H)$, or
  $v = y$ and $ux \in E(G)$, or
  $vy \in E(H)$ and $ux \in E(G)$.
  Observación: $E(G \boxtimes H) = E(G \bullet H) \cup E(G \Box H)$

4. Ejercicios

1. Probar que si un grafo es de línea, no contiene los subgrafos de la Figura 12 como subgrafos inducidos.

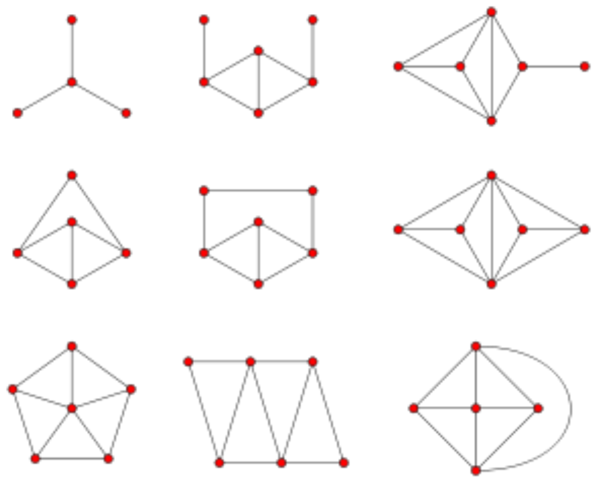


Figura 12: Grafos ejercicio 1.

2. Let G be an interval graph. Prove that (con la notación de [1])

   a) G is $(C_{n+4}, T_2, X_{31}, XF_{2n+1}, XF_{3n})$-free.

   b) G is chordal.

3. A block graph is a graph in which every biconnected component (block) is a clique, por ejemplo, windmill y el de Figura 13.

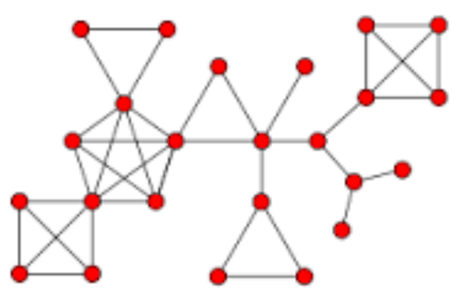


Figura 13: Grafo block

   a) they are the diamond-free chordal graphs

   b) Block graphs are chordal and distance-hereditary.

4. Probar que todo cografo es distancia hereditario.

5. Let G be a connected graph. Prove that G is a tree iff G is K_3-minor-free.

6. A graph is a *split graph* if it can be partitioned in an independent set and a clique.

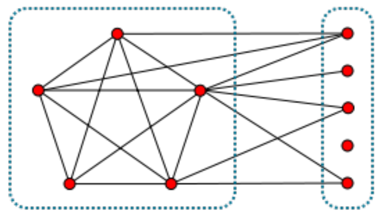


Figura 14: Grafo split

Ejemplos particulares:

- arañas sin cabeza: G such that $V(G)$ can be partitioned into $S = \{s_1, \ldots, s_r\}$, $C = \{c_1, \ldots, c_r\}$, $r \geq 2$,
  - si s_i is adjacent to $c_j \iff i = j$, araña flaca.
  - si s_i is adjacent to $c_j \iff i \neq j$, araña gorda.
- complete suns de Figura 3.

a) Probar que si G es split entonces es G is $(2K_2, C_4, C_5)$-free.

b) Split graphs are chordal graphs the complements of which are also chordal

7. Probar (1) $\Rightarrow$ (2) y (1) $\Rightarrow$ (3) en Teorema 6.

8. Las contenciones de la Figura 6 son estrictas.

Referencias

[1] Graph Classes http://www.graphclasses.org/index.html

[2] H. J. Bandelt, H. M. Mulder, *Distance-hereditary graphs*, J. Comb. Theory (B) 41 1986 182–208.

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