Non Conservative Ultimate Bound Estimation in LTI Perturbed Systems

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Abstract

A closed formula for ultimate bound estimation in LTI systems with non–vanishing perturbations is presented. The formula, based on geometrical properties of the system, provides an alternative to Lyapunov's second method based study. Although it is not proven that the bound obtained is always less conservative than the Lyapunov's bound, some examples are introduced where the estimations are significantly improved. An extension of the idea for nonlinear systems is also sketched.

Key words: perturbation analysis, lyapunov methods, ultimate boundedness.

1 Introduction

Several problems of dynamical system analysis include the effects of perturbations. These perturbations can result from modeling errors, aging or uncertanties and disturbances which exist in any realistic problem [4].

In some cases, the perturbations do not disappear when the state approximates the origin. In presence of these *nonvanishing perturbations* asymptotic stability is no longer possible. However, under certain conditions, *ultimately boundedness* of the trajectories can be ensured.

Nonvanishing perturbations can represent, for instance, effects of quantization [6], unknown disturbance signals [7], unmodeled dynamics [8], limitations in networked control systems [11,2,1,3] and errors in numerical methods [5]. In all those problems, it is always important estimating the ultimate bound as a measure of the undesirable effect of the perturbation.

This estimation is usually obtained analyzing a Lyapunov function of the non perturbed system. However, the results can be very conservative due to the loss of the system and perturbation structure.

A different approach is used in [5] to study the effects of state perturbations introduced by quantization–based numerical integration methods in LTI systems. There, the Lyapunov approach is replaced by an analysis of the diagonalized system.

This work generalizes the mentioned study for general LTI systems with bounded input and state perturbations. After deriving the closed formulae from the classic Lyapunov analysis (Sec.3), the new methodology is introduced (Sec.4) and the basic ideas to extend it to nonlinear system are presented. Finally, some examples are discussed comparing both approaches.

2 Preliminaries

2.1 Problem statement

Consider a LTI perturbed system

$$\dot{x}(t) = A(x(t) + \Delta x(t)) + B\Delta u(t) \qquad (1)$$

where the perturbation components satisfy

$$|\Delta x_i(t)| \leq \Delta x_{max_i} \;\; ; \; |\Delta u_j(t)| \leq \Delta u_{max_j} \qquad (2)$$

with $1 \leq i \leq n$, $1 \leq j \leq m$, and where Δx_{max_i} and Δu_{max_j} are non-negative constants.

The goal is to obtain closed expressions for the ultimate bound of (1) subject to (2)

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2.2 Notation

The symbol $|\cdot|$ will indicate the componentwise module of a matrix or vector. If T is a matrix with components $T_{1,1}, \dots, T_{n,m}$, then $|T|$ will be a new matrix of the same size than T with components $|T_{1,1}|, \dots, |T_{n,m}|$.

For vectors of the same dimension, the inequality $x \le y$ implies that $x_i \le y_i$ for every component of x and y.

With this notation, Ineqs.(2) become

$$|\Delta x(t)| \le \Delta x_{max}; \quad |\Delta u(t)| \le \Delta u_{max} \tag{3}$$

3 Lyapunov-based Analysis

Let $U(x) = x^T P x$ where $P = P^T > 0$ satisfies $A^T P + PA = -Q$ with $Q = Q^T > 0$. Then

$$\dot{U}(x) = -x^T Q x + 2x^T PA\Delta x + 2x^T PB\Delta u \tag{4}$$

The last two terms can be bounded by

$$2x^T PA\Delta x \le 2\|x\| \cdot \|PA\| \cdot \|\Delta x_{max}\| \tag{5}$$
$$2x^T PB\Delta u \le 2\|x\| \cdot \|PB\| \cdot \|\Delta u_{max}\| \tag{6}$$

3.1 Analysis in Norm 2

If $\|x\|_2 \ge \rho$ where

$$\rho \triangleq \frac{2}{\lambda_{min}(Q)}(\|PA\|_2\|\Delta x_{max}\|_2 + \|PB\|_2\|\Delta u_{max}\|_2)$$

it results from (5) and (6) that

$$x^T Q x \ge \|x\|_2^2 \lambda_{min}(Q) \ge 2x^T PA\Delta x + 2x^T PB\Delta u$$

and from (4) we have that $\dot{U}(x) < 0$. Let

$$c \triangleq \max_{\|x\|_2=\rho} U(x) = \rho^2 \lambda_{max}(P) \tag{7}$$

Then, it can be ensured that the trajectories will finish inside the level surface given by $U(x) = c = \rho^2\lambda_{max}(P)$ and the ultimate bound will be μ_2 so that

$$\min_{\|x\|_2=\mu_2} U(x) = \mu_2^2 \lambda_{min}(P) = c$$

Using (7) and the definition of ρ in the last equation, we have

$$\mu_2 = \sqrt{\frac{\lambda_{max}(P)}{\lambda_{min}(P)}} \cdot \frac{2}{\lambda_{min}(Q)} \cdot (\|PA\|_2 \cdot \|\Delta x_{max}\|_2 + \|PB\|_2 \cdot \|\Delta u_{max}\|_2) \tag{8}$$

3.2 Analysis in Norm ∞

Using the Constrained Quadratic Lemma [10], the minimum of the quadratic function when the component $x_i = r$ is $r^2/e_i Q^{-1} e_i^T = r^2/(Q^{-1})_{i,i}$. Thus, it results that

$$x^T Q x \ge \frac{\|x\|_\infty^2}{\max_i (Q^{-1})_{i,i}} \tag{9}$$

Then, following a similar idea to the analysis in norm 2, the bound obtained is

$$\mu_\infty = 2b_q\sqrt{b_p\|P\|_\infty} \cdot (\|PA\|_\infty \|\Delta x_{max}\|_\infty + \|PB\|_\infty \|\Delta u_{max}\|_\infty) \tag{10}$$

with $b_q \triangleq \max_{1\le i\le n}(Q^{-1})_{i,i}$; $b_p \triangleq \max_{1\le i\le n}(P^{-1})_{i,i}$

4 Non Conservative Ultimate Bound

4.1 Trajectories Starting from the Origin

Lemma 1 *Consider the first order equation with complex coefficient*

$$\dot{x} = a(x + \Delta x) + B\Delta u \tag{11}$$

where a, x, $\Delta x \in \mathbb{C}$, $\Delta u \in \mathbb{C}^k$ and $B \in \mathbb{C}^{1\times k}$. Assume also that $\mathbb{R}e(a) < 0$, $|\Delta x| \le \Delta x_{max}$ and $|\Delta u| \le \Delta u_{max}$.

Let $x(t)$ be a solution of (11) from the initial condition $x(t_0) = 0$. Then, for all $t \ge t_0$ it results that

$$|x(t)| \le \left|\frac{a}{\mathbb{R}e(a)}\right| \Delta x_{max} + \left|\frac{B}{\mathbb{R}e(a)}\right| \Delta u_{max}$$

PROOF. Let $x \triangleq \rho \cdot e^{j\theta}$ with $\rho, \theta \in \mathbb{R}$. Replacing with this definition and operating, equation (11) becomes

$$\dot{\rho} + j\rho \cdot \dot{\theta} = a(\rho + \Delta x \cdot e^{-j\theta}) + B\Delta u \cdot e^{-j\theta}$$

Taking the real part of the equation above it results that

$$\begin{aligned}\dot{\rho} &= \mathbb{R}\mathrm{e}(a)\rho + \mathbb{R}\mathrm{e}(a\Delta x \cdot e^{-j\theta}) + \mathbb{R}\mathrm{e}(B\Delta u \cdot e^{-j\theta}) \\ &\le \mathbb{R}\mathrm{e}(a)\rho + |a|\Delta x_{max} + |B|\Delta u_{max}\end{aligned}$$

Thus, when

$$\rho = |x(t)| = \frac{|a|\Delta x_{max} + |B|\Delta u_{max}}{|\mathbb{R}\mathrm{e}(a)|}$$

it results that $\dot{\rho} \le 0$ and $|x(t)|$ cannot become greater than the given bound.

Applying Lemma 1 to each component of a decoupled system, the following corollary is obtained

Corollary 2 *Consider system* (1) *where* x, $\Delta x \in \mathbb{C}^n$, $A \in \mathbb{C}^{n\times n}$, $\Delta u \in \mathbb{C}^k$ *and* $B \in \mathbb{C}^{n\times k}$. *Assume also that* A *is a diagonal matrix with* $\mathbb{R}e(A_{i,i}) < 0$ *and consider the inequalities* (3). *Let* $x(t)$ *be a solution of* (1) *from* $x(t_0) = 0$. *Then, for all* $t \geq t_0$ *we have*

$$|x(t)| \leq |\mathbb{R}e(A)^{-1}A|\Delta x_{max} + |\mathbb{R}e(A)^{-1}B|\Delta u_{max}$$

With this corollary, the following theorem can be derived

Theorem 3 *Consider system* (1) *where* x, $\Delta x \in \mathbb{R}^n$, $A \in \mathbb{R}^{n\times n}$, $\Delta u \in \mathbb{R}^k$ *and* $B \in \mathbb{R}^{n\times k}$. *Assume that* A *is a diagonalizable Hurwitz matrix and suposse that the perturbation terms satisfy* (3). *Let* $x(t)$ *be a solution of the system starting from* $x(t_0) = 0$. *Then, for all* $t \geq t_0$

$$|x(t)| \leq \epsilon_{lin} \triangleq |V| \cdot (|\mathbb{R}e(\Lambda)^{-1}\Lambda||V^{-1}|\Delta x_{max} + |\mathbb{R}e(\Lambda)^{-1}V^{-1}B|\Delta u_{max}) \quad (12)$$

where Λ *is a diagonal eigenvalues matrix of* A *and* V *is an associated matrix of eigenvectors, i.e.,* $V^{-1}AV = \Lambda$

PROOF. Let $x = Vz$. Replacing in (1) we obtain

$$\dot{z} = \Lambda(z + V^{-1}\Delta x) + V^{-1}B\Delta u \quad (13)$$

where

$$|V^{-1}\Delta x| \leq |V^{-1}|\Delta x_{max}; \quad |V^{-1}B\Delta u| \leq |V^{-1}B|\Delta u_{max}$$

Since Λ is diagonal with $\mathbb{R}\mathrm{e}(\Lambda_{i,i}) < 0$ and considering the last inequalities, system (13) satisfies the hypothesis of Corollary 2. Then, for all $t \geq t_0$ we can ensure that

$$|z(t)| \leq |\mathrm{Re}(\Lambda)^{-1}\Lambda||V^{-1}|\Delta x_{max} + |\mathrm{Re}(\Lambda)^{-1}V^{-1}B|\Delta u_{max} \quad (14)$$

and finally, we have

$$|x(t)| = |Vz(t)| \leq |V||z(t)|$$

and replacing $|z(t)|$ with (14) we retrieve (12).

4.2 Ultimate Bound of a Perturbed LTI System

Theorem 4 *System* (1)*, under the hypothesis of Theorem 3 is globally ultimately bounded with ultimate bound* $\mu = \|\epsilon_{lin}\|$*, with* ϵ_{lin} *defined in* (12). *Moreover, it exists a finite time* $t_1 = t_1(c, x_0)$ *so that for each positive constant* c *the solutions satisfy*

$$|x(t)| \leq (1 + c)\epsilon_{lin} \quad (15)$$

for all $t \geq t_1$ *and for an arbitrary initial condition* x_0.

PROOF. Let $x(t)$ be a solution of (1) starting from an arbitrary initial condition $x(0) = x_0$, and let $\tilde{x}(t)$ be the solution also starting from x_0 of the nominal system

$$\dot{\tilde{x}} = A\tilde{x} \quad (16)$$

Let $e(t) \triangleq x(t) - \tilde{x}(t)$. Then, it results that $e(0) = 0$ and $e(t)$ satisfies equation (1), which implies that it satisfies the hypothesis of Theorem 3. Then,

$$|e(t)| \leq \epsilon_{lin} \quad (17)$$

Since A is Hurwitz, the nominal system (16) is exponentially stable. Then, for each positive constant c a finite time t_1 exists so that for all $t > t_1$ we have

$$|\tilde{x}(t)| \leq c \cdot \epsilon_{lin} \quad (18)$$

Using the fact that $x(t) = e(t) + \tilde{x}(t) \Rightarrow |x(t)| \leq |e(t)| + |\tilde{x}(t)|$ and replacing with (17) and (18) we arrive to (15), which completes the proof.

4.3 Application to Nonlinear Systems

Let us consider now the nonlinear version of (1)

$$\dot{x}(t) = f(x(t) + \Delta x(t), \Delta u(t)) \quad (19)$$

with f being a continuously differentiable function. Eq.(19) can be rewritten as

$$\dot{x}(t) = A \cdot (x(t) + \Delta x(t)) + B \cdot \Delta u(t) + g(x(t) + \Delta x(t), \Delta u(t)) \quad (20)$$

where

$$A = \left.\frac{\partial f(x, u)}{\partial x}\right|_{(0,0)} \qquad B = \left.\frac{\partial f(x, u)}{\partial u}\right|_{(0,0)} \quad (21)$$

Assuming that (19) is exponentially stable, it results that A is Hurwitz. Moreover, if Δx_{max} and Δu_{max} are small enough we can ensure that the solutions $x(t)$ (starting inside the region of attraction) are ultimately bounded. Thus, a vector of positive values x_{max} exists so that if $t > T$ then $|x(t)| < x_{max}$.

Boundedness of g ensures (for fixed Δx_{max} and Δu_{max}) the existence of a function g_{max} so that $|g(x(t) + \Delta x(t), \Delta u(t))| < g_{max}(x_{max})$, $\forall t > T$

Then, function g in (20) acts as a bounded perturbation. Defining ϵ_{lin} as in (12) we can repeat the procedure of Theorems 3 and 4 starting from $t = T$ concluding that

$$|x(t)| < \epsilon_{lin} + |V| \cdot |\mathrm{Re}(\Lambda)^{-1}V^{-1}| \cdot g_{max}(x_{max}) \quad (22)$$

$\forall t > T + t_1$. When the right hand side of (22) is a contraction map, its fixed point

$$x_{max} = \epsilon_{lin} + |V| \cdot |\mathbb{R}\text{e}(\Lambda)^{-1} V^{-1}| \cdot g_{max}(x_{max}) \quad (23)$$

gives an upper bound of x_{max}.

This new bound is now imlicitely defined, but it could be easily calculated by fixed point iterations starting from $x_{max} = \epsilon_{lin}$.

5 Numerical Examples

Consider the perturbed system

$$\dot{x}(t) = A[x(t)+\Delta x(t)] = \begin{bmatrix} 0 & 100 \\ -100 & -10001 \end{bmatrix} [x(t)+\Delta x(t)]$$

where the perturbation components are bounded by $|\Delta x_1(t)| \leq 0.01$ and $|\Delta x_2(t)| \leq 0.0001$. This example, taken from [5], represents the dynamics of the error in the simulation of a stiff system with the QSS method.

Lyapunov formulas (8) and (10) are minimized taking [1]

$$Q_2 = \begin{bmatrix} 1 & -1.4631 \\ -1.4631 & 197.568 \end{bmatrix} \quad Q_\infty = \begin{bmatrix} 1 & -1.448 \\ -1.448 & 196.124 \end{bmatrix}$$

obtaining bounds $\mu_2 = 20.1861$ and $\mu_\infty = 14.45$.

For the same example, our approach –Theorem 4– gives $\tilde{\mu}_2 = 0.0100085$ (bound in norm 2) and $\tilde{\mu}_\infty = 0.01$. The estimation is about 2000 times less conservative in norm 2 and 1400 times in norm ∞. The calculations were made using the eigenvectors and eigenvalues matrices obtained with function 'eig' of Matlab.

Moreover, Theorem 4 concludes after certain time t_1 we will have $|x_1(t)| < (1+c)0.01$ and $|x_2(t)| < (1+c)0.0003$ for any given positive constant c. Thus, the estimation for the second component is even much better.

Lyapunov's poor performance is due, in part, to the fact that the eigenvalues of P are very different and then the level surfaces of $U(x)$ are very flat ellipses. Hence, the radius ρ of the ball where the Lyapunov function derivative is negative differs significantly from the maximum norm in that ellipse (μ_2). Also, the problem structure is lost and the way in which the perturbation terms act is not exploited.

[1] The minimum were obtained with Nelder–Mead simplex method (function 'fminsearch' of Matlab).

Consider now the system resulting of adding a nonlinear term to the previous system

$$\dot{x}(t) = f(x+\Delta x) = A[x(t)+\Delta x(t)] + \begin{bmatrix} 0 \\ k(x_2 + \Delta x_2)^3 \end{bmatrix}$$

According to (12), for the same perturbation, we have $\epsilon_{lin} = [0.01, 0.0003]^T$. The nonlinear term is bounded by

$$g_{max}(x_{max}) = [0 \;\; k(x_{max_2} + 0.0001)^3]^T$$

Then, taking $k = 5 \times 10^9$, Eq.(23) has a fixed point $x_{max} = [0.01437 \; 0.0003437]^T$. Then, when $x(0)$ is in the region of attraction, a constant T exists so that $|x_1(t)| < 0.01437$ and $|x_2(t)| < 0.003437$, $\forall t > T$.

As before, any Lyapunov analysis using quadratic functions gives more conservative results.

6 Conclusions

A new approach for the estimation of the ultimate bound in perturbed LTI systems, which can be extended to nonlinear cases, was introduced. In the cases analyzed, the estimation significantly improved Lyapunov based results, even when the choice of Q was optimized. Another important benefit is the possibility of obtaining separated bounds for each state variable.

There are many works in the recent literature that introduce novel design techniques attempting to guarantee ultimate bounds in hybrid and sampled data systems (see for instance [2,9,3,1]). In all these examples, the estimation of the ultimate bound is derived from a Lyapunov analysis and the resulting design conditions are based on expressions like (8). Thus, the usage of these new approach could lead to less conservative design conditions.

Future work should generalize the results for the non-diagonalizable case, and it should formalize the nonlinear analysis. The ideas which conduced to (23) are a starting point in this direction.

As suggested by a reviewer, the problem presented can be also studied using the infinite induced norm of an equivalent system arriving to an exact bound. However, the obtention of that norm requires numerical calculations –the goal here was to obtain a closed formula– and it cannot be extended to nonlinear systems. Anyway, the idea might effectively improve the estimation given by (15) if practical and non conservative estimations of the infinite norm of the system are provided.

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