Probabilistic Ultimate Bounds and Invariant Sets in Nonlinear Systems

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Abstract

This article extends the notions of probabilistic ultimate bounds (PUB) and probabilistic invariant sets (PIS) to nonlinear continuous-time systems providing tools for their characterization and for the usage of these concepts in control design. Two design strategies are proposed that allow finding a nonlinear control law that ensures that the closed loop system is probabilistically ultimate bounded to a desired region. These strategies are based on Lyapunov and stochastic feedback linearization, respectively.

1 Introduction

The concepts of ultimate bounds and invariant sets play a key role in control systems theory, since they replace the idea of stability of equilibrium points in the presence of non-vanishing disturbances [9]. Associated with these concepts, several problems and applications appear in the context of control theory [2], among which we can mention the characterization of these sets [4,12,8], the design of robust control to obtain ultimate bounds [6,14], and the use of these sets to detect and isolate faults in [20,23,3].

A traditional approach to determine ultimate bounds and invariant sets is based on the use of Lyapunov functions [9,4], where the sets are estimated by a level surface of a Lyapunov function on which the derivative can be guaranteed to be negative (due to the disturbance, it can not be guaranteed that it is negative in the interior of the set delimited by the said surface). Another approach uses the modal decomposition of a linear system to make a componentwise estimate of both an ultimate bounds and invariant sets [12,8]. Both approaches can also be used to design controls that guarantee desired ultimate bounds.

A problem with the concepts of ultimate bound and invariant sets is that, in the presence of unbounded perturbations (such as Gaussian white noise), these sets no longer exist because with some probability the state can reach arbitrarily large values. To overcome this difficulty, these concepts were extended in [15] defining those of *Probabilistic Ultimate Bound* (PUB) and *Probabilistic Invariant Set* (PIS). These definitions were originally conceived for discrete-time systems, but were then extended to continuous-time systems in [16].

A PUB is a set towards which the trajectories of the state converge and remain (as in the deterministic case), but the permanence occurs with a certain probability (determined by a parameter $p < 1$). That is, for sufficiently large times, the probability that the state is in the set is at least p. Similarly, if a trajectory starts inside a probabilistic invariant set, at any future time there is a probability of at least p that the state will be inside that set.

Besides defining the sets, characterization formulas were derived in [15,16] which were then used for control design in [16,1] and for fault detection purposes in [21]. However, all the aforementioned definitions, formulas and applications were limited so far to linear time invariant systems.

In this paper we extend the concepts of PUB and PIS to nonlinear cases, providing tools for their characterization using two approaches. The first one is based on a Lyapunov analysis and it exploits results related to *Noise to State Stability* (NSS) [7,18,19]. The NSS property guarantees that a region can be found such that the probability that the state leaves such region can be made arbitrarily small which in turn implies that PUB and PIS sets can be found for any probability p (i.e., NSS is stronger than the existence of PUB and PIS). The second approach uses results from linear Stochastic Differential Equations (SDEs) in order to estimate a PUB in a system where the nonlinearities are limited to the noise input term.

Then, based on these set characterization approaches, two different robust control design strategies are developed that allow, under certain conditions, to guarantee that the closed-loop system has a desired probabilistic ultimate bound. The first one follows a Lyapunov control design inspired on ideas similar to those of [6] for deterministic ultimate bounds, and the second approach is based on *stochastic feedback linearization* [17] and combines some results on linear control design developed in [16] with results on deterministic ultimate bounds for feedback linearizable systems [13].

The remainder of the paper is organized as follows: Section 2 introduces the concepts and previous results for linear SDEs on which our work is based. Then, Section 3 presents results regarding the characterization of PUB and PIS sets for classes of nonlinear systems. Section 4 develops two different approaches for robust control design, and finally, Section 5 illustrates the results with numerical examples.

2 Preliminaries

In this section we summarise the main results on PUB sets and PISs for linear systems presented in [16] and we mention some related work .

Notation 1 *For a vector x, we use x_i to denote its i-th component, and for a matrix Σ, we use $[\Sigma]_{i,j}$ to denote its (i,j)-th entry. The symbol $\preceq$ denotes elementwise inequality between two vectors, i.e., for $\alpha, \beta \in \mathbb{R}^n$, $\alpha \preceq \beta$ if and only if $\alpha_i \leq \beta_i$, for all $i = 1, \ldots, n$. For matrices M and N, $M \geq N$ ($M > N$) means that the matrix $M - N$ is positive semidefinite (positive definite), and we use M^* to denote the conjugate transpose of the matrix M.*

2.1 Definitions of PUB and PIS

We consider a continuous-time LTI system given by the following stochastic differential equation

$$\mathrm{d}x(t) = Ax(t)\mathrm{d}t + H\mathrm{d}w(t), \tag{1}$$

with $A \in \mathbb{R}^{n\times n}$, $H \in \mathbb{R}^{n\times k}$, $x(t) \in \mathbb{R}^n$ and the disturbance vector $w(t) \in \mathbb{R}^k$ being a Wiener process with incremental covariance $\mathrm{cov}[\mathrm{d}w(t)] = \mathrm{E}[\mathrm{d}w(t)\mathrm{d}w^T(t)] = I_{k\times k}\mathrm{d}t$. Note that the latter does not imply loss of generality, since the effects of a disturbance with incremental covariance $\Sigma_w dt$ can be equivalently modeled by replacing H with $\tilde{H} \triangleq H\Sigma_w^{1/2}$.

We also assume that the nominal system is asymptotically stable (i.e., A is a Hurwitz matrix).

Definition 1 (Probabilistic Ultimate Bounds) *Let $0 < p \leq 1$ and $S \subset \mathbb{R}^n$ be a compact set. We say that S is a PUB with probability p for system (1) if, for every initial state $x(t_0) = x_0 \in \mathbb{R}^n$, there exists $T = T(x_0) \in \mathbb{R}$ such that the probability*[1] $\Pr[x(t) \in S] \geq p$ *for each $t \geq t_0 + T$.*

For the definition of probabilistic invariant set, we first introduce the product of a scalar $\gamma \geq 0$ and a set S as $\gamma S \triangleq \{\gamma x : x \in S\}$. Notice that when $0 \leq \gamma \leq 1$, and provided that S is a *star–shaped set with respect to the origin,*[2] it follows that $\gamma S \subseteq S$.

Definition 2 (γ-Probabilistic Invariant Sets) *Let $0 < p \leq 1$, $0 < \gamma < 1$ and let $S \subset \mathbb{R}^n$ be a star–shaped compact set with respect to the origin. We say that S is a γ–PIS with probability p for system* (1) *if for any $x(t_0) \in \gamma S$ the probability* $\Pr[x(t) \in S] \geq p$ *for each $t > t_0$.*

The characterization of PUB and PIS for the linear system (1) was stated in [16], in particular the formula obtained for the characterization of PUBs depends on the elements of the main diagonal of the state covariance matrix Σ_x that is obtained from solving

$$A\Sigma_x + \Sigma_x A^T = -HH^T. \tag{2}$$

Based on this characterization, a linear control design strategy was proposed to guarantee that a certain closed-loop system has a desired PUB with a given probability p. The design strategy was based on the notions of *covariance assignment* (see details in [16]).

2.2 Other Related Works

The closest concept to PUB and PIS is that of NSS [7,18,19], but it does not provide tools to characterize sets with a given probability $p < 1$. Similarly, the concept of practical stability applied to SDEs [5] is connected to our work, but again the probability is always $p \to 1$.

[1] In this work, the expression $\Pr[x(t) \in S \subset \mathbb{R}^n]$ denotes the probability that the solution $x(t)$, at time t, is within the set $S \subset \mathbb{R}^n$. Thus, $\Pr[\cdot]$ is the probability measure on $\mathbb{R}^n$ induced by the stochastic process $\{w(\tau)|t_0 \leq \tau \leq t\}$, via the solution, at time t, of the stochastic differential equation (1), with initial condition $x(t_0)$.

[2] A set $S \subset R^n$ is star shaped, or a star domain, with respect to the origin if $x \in S \Rightarrow \gamma x \in S$ for all $0 \leq \gamma \leq 1$

The notions of barrier certificates [22] are also related, but they not consider the possibility that the state leaves and returns to the considered set.

3 PUB in nonlinear SDEs

In this section we develop two approaches to characterize PUB for nonlinear SDEs. The first one is more general and relies on the use of a Lyapunov function. The second one is restricted to systems that are linear in the state and nonlinear in the perturbation, which can be the result of performing stochastic feedback linearization.

3.1 Nonlinear SDE

We consider a system of the form

$$\mathrm{d}x(t) = f(x)\mathrm{d}t + h(x)\mathrm{d}w(t) \tag{3}$$

where $f : \mathbb{R}^n \to \mathbb{R}^n$ is a smooth vector field, $h : \mathbb{R}^n \to \mathbb{R}^{n\times k}$ is a smooth matrix field, and the disturbance $w(t)$ is a Wiener process with incremental covariance $\mathrm{cov}[dw(t)] = I_{k\times k}\mathrm{d}t$.

The definitions of PUB and PIS given for linear systems in Section 2.1 are readily applicable to the general nonlinear system given above. Below, we propose two alternative approaches to characterize PUB sets.

3.2 Lyapunov-Based Characterization of PUB

The characterization developed below uses the following operator:

Definition 3 (SDE Generator) *Let $V : \mathbb{R}^n \to \mathbb{R}$ be positive definite and twice continuously differentiable. Then, the generator of the SDE* (3) *acting on function V is the map $\mathcal{L}[V] : \mathbb{R}^n \to \mathbb{R}$ defined by*

$$\mathcal{L}[V](x) \triangleq \frac{\partial V}{\partial x} f(x) + \frac{1}{2}\mathrm{tr}\left(h^T(x)\frac{\partial^2 V}{\partial x^2}h(x)\right). \tag{4}$$

The following theorem provides a characterisation of PUB sets for the system of Eq. (3):

Theorem 4 (PUB characterisation) *Let $0 < p < 1$ and $V : \mathbb{R}^n \to \mathbb{R}$ be positive definite and twice continuously differentiable and radially unbounded. Let $\alpha \in \mathcal{K}_\infty$ [3] be convex and let $\beta > 0$ such that, for every $x \in \mathbb{R}^n$,*

$$\mathcal{L}[V](x) \leq -\alpha(V(x)) + \beta. \tag{5}$$

Then, for any $\varepsilon > 0$, the set

$$S = \left\{x \in \mathbb{R}^n : V(x) \leq \frac{\alpha^{-1}(\beta) + \varepsilon}{1-p}\right\}$$

is a PUB with probability p for the system of Eq. (3).

PROOF. S is a PUB with probability p provided that for sufficiently large values of t

$$\Pr\left[V(x(t)) \geq \frac{\alpha^{-1}(\beta) + \varepsilon}{1-p}\right] \leq 1-p$$

In order to verify this inequality, we will first show that there exists $t_1 \geq t_0$ such that for $t \geq t_1$

$$\mathrm{E}[V(x(t))] \leq \alpha^{-1}(\beta) + \varepsilon \tag{6}$$

and then we will use Markov's inequality to conclude the proof.

Using the Ito rule (see [10, Th. 61]) on Eq. (3) for $V(x(t))$, we obtain

$$\mathrm{d}V(x(t)) = \mathcal{L}[V](x)\mathrm{d}t + \frac{\partial V}{\partial x}h(x)\mathrm{d}w(t)$$

then,

$$\begin{aligned} V(x(t)) =& V(x(t_0)) + \int_{t_0}^{t} \mathcal{L}[V](x(s))\mathrm{d}s + \\ &+ \int_{t_0}^{t} \frac{\partial V}{\partial x}(x(s))h(x(s))\mathrm{d}w(s). \end{aligned}$$

According to Dynkin's formula ([10, p. 152, Eq.(6.14)]) and using Eq. (5), it results

$$\begin{aligned} &\mathrm{E}[V(x(t))] = \mathrm{E}[V(x(t_0))] + \mathrm{E}\left[\int_{t_0}^{t} \mathcal{L}[V](x(s))\mathrm{d}s\right] \\ &\leq \mathrm{E}[V(x(t_0))] + \mathrm{E}\left[-\int_{t_0}^{t} \alpha(V(x(s)))\mathrm{d}s + \int_{t_0}^{t} \beta\ \mathrm{d}s\right] \\ &= \mathrm{E}[V(x(t_0))] - \int_{t_0}^{t} \mathrm{E}[\alpha(V(x(s)))]\mathrm{d}s + \int_{t_0}^{t} \beta\ \mathrm{d}s. \end{aligned}$$

In the last equation we used the fact that the expected value and integration can be exchanged for a non-negative process (result derived from Fubini's Theorem, see details in [10]). Since α is convex, from Jensen's inequality, it results that

$$\mathrm{E}[V(x(t))] \leq \mathrm{E}[V(x(t_0))] + \int_{t_0}^{t} (-\alpha(\mathrm{E}[V(x(s))]) + \beta)\mathrm{d}s. \tag{7}$$

[3] A scalar continuous function $\alpha(r)$, defined for all $r \geq 0$ is said to belong to class $\mathcal{K}_\infty$ if it is strictly increasing with $\alpha(0) = 0$ and $\alpha(r) \to \infty$ as $r \to \infty$.

In order to prove Eq.(6) we will proceed by contradiction. Suppose that $\mathrm{E}[V(x(t))] > \alpha^{-1}(\beta) + \varepsilon$, for all $t \geq t_0$. Then, since α is a continuous and strictly increasing function, there exists $\tilde{\varepsilon} > 0$ such that $\alpha(\alpha^{-1}(\beta)+\varepsilon) = \beta + \tilde{\varepsilon}$, and then $\alpha(\mathrm{E}[V(x(t))]) \geq \beta + \tilde{\varepsilon}$ for all $t \geq t_0$.

According to inequality (7), it results that, for all $t \geq t_1$,

$$\begin{aligned}\mathrm{E}[V(x(t))] &\leq \mathrm{E}[V(x(t_0))] + \int_{t_0}^{t} (-(\beta+\tilde{\varepsilon}) + \beta)\mathrm{d}s \\ &= \mathrm{E}[V(x(t_0))] - \tilde{\varepsilon}(t - t_0), \quad \forall\, t \geq t_0.\end{aligned}$$

The right hand side of this inequality , for t large enough, becomes smaller than $\alpha^{-1}(\beta)+\varepsilon$, contradicting Eq.(6) and showing that the condition $E[V(x(t))] \leq \alpha^{-1}(\beta)+\varepsilon$ is eventually achieved. Therefore, there exists $t_1 > t_0$ for which $E[V(x(t_1))] \leq \alpha^{-1}(\beta) + \varepsilon$.

It remains to show that, Eq.(6) holds for all $t \geq t_1$. We shall proceed again by contradiction supposing that for certain $\varepsilon_1 > 0$, there exists some $t_3 > t_1$ for which $\mathrm{E}[V(x(t_3))] \geq \alpha^{-1}(\beta) + \varepsilon + \varepsilon_1$. Let t_2 be the last exit time from $\mathrm{E}[V(x(t))] \leq \alpha^{-1}(\beta) + \varepsilon$, i.e.,

$$t_2 = \sup\{t < t_3 : \mathrm{E}[V(x(t))] \leq \alpha^{-1}(\beta) + \varepsilon\}$$

Then, in the period (t_2, t_3) we have $\mathrm{E}[V(x(t))] \geq \alpha^{-1}(\beta)+\varepsilon$, and using Eq. (7) and a reasoning analogous to the previous one, it results

$$\begin{aligned}\mathrm{E}[V(x(t))] &\leq \mathrm{E}[V(x(t_2))] + \int_{t_2}^{t} (-\alpha(\mathrm{E}[V(x(s))]) + \beta)\mathrm{d}s \\ &\leq \mathrm{E}[V(x(t_2))] - \tilde{\varepsilon}(t - t_2)\end{aligned}$$

contradicting the assumption that $\mathrm{E}[V(x(t_3))] \geq \alpha^{-1}(\beta) + \varepsilon + \varepsilon_1$.

We conclude that $\mathrm{E}[V(x(t))] \leq \alpha^{-1}(\beta)+\varepsilon$ for all $t \geq t_1$. Using Markov's inequality, we get

$$\Pr\left[V(x) \geq \frac{\alpha^{-1}(\beta)+\varepsilon}{1-p}\right] \leq \frac{\mathrm{E}[V(x)]}{\dfrac{\alpha^{-1}(\beta)+\varepsilon}{1-p}} \leq 1 - p$$

Then,

$$\Pr[x(t) \in S] = \Pr\left[V(x) \leq \frac{\alpha^{-1}(\beta)+\varepsilon}{1-p}\right] \geq p$$

completing the proof.

Theorem 4 allows one to compute a PUB set S for a given probability p. Notice that it could be also used to compute a probability p given a set S containing the origin.

Corollary 5 (PIS Characterization) *Let $0 < p < 1$, take $0 < \gamma < 1 - p$ and let $V : \mathbb{R}^n \to \mathbb{R}$ be a twice continuously differentiable positive definite function for which*

$$V\left(\frac{x}{\gamma}\right) \geq \frac{V(x)}{\gamma}, \quad \forall x \in \mathbb{R}^n \tag{8}$$

Let $\alpha \in \mathcal{K}_\infty$ be a convex function and let $\beta > 0$ be such that Eq.(5) is verified for every $x \in \mathbb{R}^n$. Then, for any $\varepsilon > 0$, the set

$$S = \left\{x \in \mathbb{R}^n : V(x) \leq \frac{\alpha^{-1}(\beta)+\varepsilon}{1-p} \triangleq \frac{V_0}{1-p}\right\} \tag{9}$$

is a γ–PIS with probability p for the system of Eq. (3).

PROOF. Notice first that Eq.(8) implies that $V(x)$ is radially unbounded and star shaped. Let us suppose that $x(t_0) = x_0 \in \gamma S$. Then, from Eqs.(8) and (9),

$$\frac{V_0}{1-p} \geq V\left(\frac{x_0}{\gamma}\right) \geq \frac{V(x_0)}{\gamma} \geq \frac{V(x_0)}{1-p} \implies V(x_0) \leq V_0$$

From Eq. (7), with $\mathrm{E}[V(x_0)] \leq V_0$, using the fact that α is strictly increasing and $V_0 > \alpha^{-1}(\beta)$ (or equivalently $\alpha(V_0) > \beta$) and following a reasoning analogous to the proof of the previous theorem, it results $\mathrm{E}[V(x(t))] \leq V_0$ for all $t \geq t_0$. Then, using Markov's inequality,

$$\Pr\left[V(x) \geq \frac{V_0}{1-p}\right] \leq \frac{\mathrm{E}[V(x(t))]}{\dfrac{V_0}{1-p}} \leq 1 - p$$

Then, the proof concludes by noting that

$$\Pr[x(t) \in S] = 1 - \Pr[x(t) \notin S] \geq p$$

3.3 Linearly Bounded Characterization of PUBs

A particular case of Eq. (3) is given by the following nonlinear SDE.

$$\mathrm{d}x(t) = Ax\mathrm{d}t + h(x)\mathrm{d}w(t) \tag{10}$$

where we assume that the matrix $A \in \mathbb{R}^{n\times n}$ is Hurwitz and the function h is bounded as follows

$$h(x)h^T(x) \leq HH^T, \text{ for all } x \in \mathbb{R}^n, \tag{11}$$

with $H \in \mathbb{R}^{n\times k}$.

The PUB characterization for the nonlinear system (10) is based on bounding its state covariance by that of the linear system (1) as stated by the following lemma.

Lemma 6 (State Covariance Bound) *For the system (10), under the assumption* (11),

$$\lim_{t\to\infty} \mathrm{E}[x(t)] = 0, \tag{12}$$
$$\lim_{t\to\infty} \Sigma_x(t) \le \Sigma, \tag{13}$$

where Σ *is the unique positive definite solution of*

$$A\Sigma + \Sigma A^T = -HH^T. \tag{14}$$

PROOF. For a given initial state x_0 at $t = 0$, we have

$$x(t) = e^{At}x_0 + \int_0^t e^{A(t-s)} h\left(x(s)\right) \mathrm{d}w(s).$$

Let

$$M(t,s) = e^{A(t-s)} h\left(x(s)\right),$$

so that

$$x(t) = e^{At}x_0 + \int_0^t M(t,s)\mathrm{d}w(s).$$

Using [24, p. 60, Eq. (5.17)] and the fact that w is a Wiener process (zero mean), we have

$$\begin{aligned}[\mathrm{E}[x(t)]]_i &= \left[e^{At}x_0\right]_i + \sum_{l=1}^{k} \mathrm{E}\left[\int_0^t [M(t,s)]_{i,l}\, \mathrm{d}\,[w(s)]_l\right] \\ &= \left[e^{At}x_0\right]_i.\end{aligned}$$

Hence

$$\mathrm{E}[x(t)] = e^{At}x_0,$$

and (12) follows.

On the other hand, using [24, Eq. (5.10) and (6.9)], and the fact that x_0 is deterministic, it results that

$$\begin{aligned}\Sigma_x(t) &= \int_0^t \mathrm{E}\left[M(t,s)M^T(t,s)\right] \mathrm{d}s \\ &= \int_0^t e^{A(t-s)}\mathrm{E}\left[h\left(x(s)\right) h^T\left(x(s)\right) \left(e^{A(t-s)}\right)^T\right] \mathrm{d}s \\ &\le \int_0^t e^{A(t-s)} HH^T \left(e^{A(t-s)}\right)^T \mathrm{d}s \triangleq \Sigma(t).\end{aligned}$$

This last inequality arises from property: $h(x)h^T(x) \le HH^T \Rightarrow \mathbf{\Phi} h(x)h^T(x)\mathbf{\Phi}^T \le \mathbf{\Phi} HH^T\mathbf{\Phi}^T$, for any matrix $\mathbf{\Phi}$.

Note, also, that $\Sigma(t)$ is the covariance matrix of $\bar{x}(t)$ defined by

$$\mathrm{d}\bar{x} = A\bar{x}\mathrm{d}t + H\mathrm{d}\bar{w}, \tag{15}$$

where $\bar{w}$ is a Wiener process with identity covariance matrix.

The proof concludes by observing that the covariance of $\bar{x}(t)$ in Eq. (15) with $t \to \infty$ can be computed from the Lyapunov equation (14).

Based on the previous lemma, the following result characterizes a PUB for the system (10).

Theorem 7 *Let* $0 < p < 1$, *and* $\tilde{p} \in \mathbb{R}^n$ *with* $0 < \tilde{p}_i$, $i = 1, \ldots, n$, *satisfying*

$$\sum_{i=1}^{n} \tilde{p}_i = 1 - p. \tag{16}$$

Then, the set $\mathcal{S} = \{x : |x_i| < b_i + \varepsilon,\ i = 1, \ldots, n\}$, *with* $\varepsilon > 0$ *and*

$$b_i = \sqrt{\frac{[\Sigma]_{i,i}}{\tilde{p}_i}},\ i = 1, \ldots, n, \tag{17}$$

and Σ *being the solution of* (14), *is a PUB with probability* p *for* (10) *under the assumption* (11).

PROOF. ¿From Chebyshev's inequality [11, Th. 5.11], it follows that

$$\Pr\left[|[x(t) - \mathrm{E}[x(t)]]_i| \ge b_i + \frac{\varepsilon}{2}\right] \le \frac{[\Sigma_x(t)]_{i,i}}{\left(b_i + \frac{\varepsilon}{2}\right)^2}.$$

Then, from (13), there exists $T_{\varepsilon,i} \in \mathbb{R}$ such that, for all $t > T_{\varepsilon,i}$,

$$\Pr\left[|[x(t) - \mathrm{E}[x(t)]]_i| > b_i + \frac{\varepsilon}{2}\right] \le \frac{[\Sigma]_{i,i}}{b_i^2} = \tilde{p}_i \tag{18}$$

Also, from (12), there exists $\tilde{T}_{\varepsilon,i} \in \mathbb{R}$ such that, for all $t > \tilde{T}_{\varepsilon,i}$

$$|\mathrm{E}[x(t)]_i| < \frac{\varepsilon}{2}. \tag{19}$$

Then, from (18) and (19), for all $t > \max\left\{T_{\varepsilon,i}, \tilde{T}_{\varepsilon,i}\right\}$,

$$\begin{aligned}\Pr\left[|[x(t)]_i| > b_i + \varepsilon\right] &= \Pr\left[|[x(t)]_i| - \frac{\varepsilon}{2} > b_i + \frac{\varepsilon}{2}\right] \\ &\le \Pr\left[|[x(t)]_i| - |[\mathrm{E}[x(t)]]_i| > b_i + \frac{\varepsilon}{2}\right] \\ &\le \Pr\left[|[x(t)]_i - [\mathrm{E}[x(t)]]_i| > b_i + \frac{\varepsilon}{2}\right] \\ &\le \tilde{p}_i. \end{aligned} \tag{20}$$

Then, for all $t > T_\varepsilon \triangleq \max\left\{T_{\varepsilon,i}, \tilde{T}_{\varepsilon,i} : i = 1, \cdots, n\right\}$, we have,

$$\begin{aligned}\Pr\left[x(t) \notin \mathcal{S}\right] &\leq \sum_{i=1}^{n} \Pr\left[\left|[x(t)]_i\right| > b_i + \varepsilon\right] \\ &\leq \sum_{i=1}^{n} \tilde{p}_i = 1 - p,\end{aligned}$$

o equivalently,

$$\Pr\left[x(t) \in \mathcal{S}\right] \geq p.$$

and the result follows.

4 Control Design with Guaranteed PUB

In this section, we propose two alternative approaches for designing a nonlinear control law such that a closed loop system has a desired ultimate bound.

4.1 Problem Formulation

We consider a nonlinear stochastic system of the form

$$\mathrm{d}x(t) = f(x)\mathrm{d}t + g(x)u(t)\mathrm{d}t + h(x)\mathrm{d}w(t) \qquad (21)$$

where $f : \mathbb{R}^n \to \mathbb{R}^n$ is a smooth vector field, $g : \mathbb{R}^n \to \mathbb{R}^{n\times m}$ and $h : \mathbb{R}^n \to \mathbb{R}^{n\times k}$ are smooth matrix fields, $u \in \mathbb{R}^m$ is the control input, and the disturbance $w(t)$ is a Wiener process with incremental covariance $cov[\mathrm{d}w(t)] = I_{k\times k}\mathrm{d}t$. The initial state is $x(0) = x_0 \in \mathbb{R}^n$.

Then, given a certain set S and a probability $0 < p < 1$, the goal is to find a control law $u(t) = \kappa(x)$ such that S is a PUB for Eq. (21).

4.2 Lyapunov Based Design

The following theorem utilizes the PUB characterization result of Theorem 4 to obtain a nonlinear control law for system (21). It relies on the existence of a stabilizing control law $u(t) = \kappa_1(x(t))$ and a Lyapunov function $V(x)$ for the unperturbed nonlinear system $\dot{x} = f(x) + g(x)u(t)$.

Theorem 8 *Consider the system* (21) *and let* $V : \mathbb{R}^n \to \mathbb{R}$ *be a twice continuously differentiable positive definite function that verifies*

$$\frac{\partial V}{\partial x}\left[f(x) + g(x)\kappa_1(x)\right] \leq -\alpha(V(x)). \qquad (22)$$

where $\kappa_1 : \mathbb{R}^n \to \mathbb{R}^m$ *and* α *is a class* $\mathcal{K}_\infty$ *function. Assume that, for a given* $c > 0$ *and* $0 < p < 1$, *the condition*

$$\frac{\partial V}{\partial x}g(x) = 0 \qquad (23)$$

implies that [4]

$$\alpha(c(1-p)) > \left|\frac{1}{2}\mathrm{tr}\left(h^T(x)\frac{\partial^2 V}{\partial x^2}(x)h(x)\right)\right|. \qquad (24)$$

Then, there exists a constant $\rho > 0$ *such that the set* $S = \{x \in \mathbb{R}^n : V(x) \leq c\}$ *is a PUB of* (21) *with probability* p *under the control law*

$$u(t) = \kappa(x(t)) \triangleq \kappa_1(x(t)) + \kappa_2(x(t)) \qquad (25)$$

with

$$\kappa_2(x) = \begin{cases} \dfrac{-g^T(x)\frac{\partial V}{\partial x}^T(x)r(x)}{\left\|\frac{\partial V}{\partial x}g(x)\right\|^2} & \text{if } \left\|\frac{\partial V}{\partial x}g(x)\right\| > \rho, \\[2ex] \dfrac{-g^T(x)\frac{\partial V}{\partial x}^T(x)r(x)}{\rho^2} & \text{if } \left\|\frac{\partial V}{\partial x}g(x)\right\| \leq \rho. \end{cases} \qquad (26)$$

PROOF. Define

$$r(x) \triangleq \frac{1}{2}\mathrm{tr}\left(h^T(x)\frac{\partial^2 V}{\partial x^2}(x)h(x)\right)$$

and let $\varepsilon > 0$ be arbitrarily small such that

$$|r(x)| < \alpha(c(1-p) - \varepsilon) \triangleq \beta \quad \forall x : \frac{\partial V}{\partial x}g(x) = 0. \qquad (27)$$

Note that such an ε exists due to the continuity of α and the inequality (24) subject to the condition (23). Then, we can take $\rho > 0$ such that

$$\left\|\frac{\partial V}{\partial x}g(x)\right\| \leq \rho \Rightarrow |r(x)| \leq \beta. \qquad (28)$$

Note that the existence of ρ is guaranteed by Eq. (27) and the continuity of all involved functions.

Then, using the control law of Eq. (25) and computing $\mathcal{L}$ for V, it results

$$\mathcal{L}\left[V\right](x) = \frac{\partial V}{\partial x}\left[f(x) + g(x)\kappa_1(x) + g(x)\kappa_2(x)\right] + r(x).$$

Taking into account Eq. (22), we obtain

$$\mathcal{L}\left[V\right](x) \leq -\alpha(V(x)) + \frac{\partial V}{\partial x}g(x)\kappa_2(x) + r(x)$$

[4] This conditions tells that in the regions where the input function $g(x)$ is orthogonal to the level sets of $V(x)$, the disturbance action must be bounded. The reason for this restriction is that the control action through the product $g(x)\cdot u$ cannot drive the state inside the level sets of $V(x)$.

and using (26) for $\left\|\frac{\partial V}{\partial x}g(x)\right\| > \rho$, it results

$$\mathcal{L}[V](x) \le -\alpha(V(x)) - \left\|\frac{\partial V}{\partial x}g(x)\right\|^2 \frac{r(x)}{\left\|\frac{\partial V}{\partial x}g(x)\right\|^2} + r(x)$$
$$= -\alpha(V(x)) \le -\alpha(V(x)) + \beta.$$

On the other hand, if $\left\|\frac{\partial V}{\partial x}g(x)\right\| \le \rho$, it results

$$\mathcal{L}[V](x) \le -\alpha(V(x)) - \left\|\frac{\partial V}{\partial x}g(x)\right\|^2 \frac{r(x)}{\rho^2} + r(x)$$

$$\le -\alpha(V(x)) + \left[1 - \frac{\left\|\frac{\partial V}{\partial x}g(x)\right\|^2}{\rho^2}\right] r(x)$$
$$\le -\alpha(V(x)) + |r(x)| \le -\alpha(V(x)) + \beta,$$

where we have used the condition (28) in the last inequality.

Thus, we obtain, for all $x \in \mathbb{R}^n$,

$$\mathcal{L}[V](x) \le -\alpha(V(x)) + \beta.$$

Note that $\alpha^{-1}(\beta) = c(1-p) - \varepsilon$ implies that

$$c = \frac{\alpha^{-1}(\beta) + \varepsilon}{1-p},$$

and then Theorem 4 guarantees that

$$S = \{x \in \mathbb{R}^n : V(x) \le c\}$$

is a PUB with probability p of the system (21) with the feedback law of Eq.(25), concluding the proof.

4.3 Stochastic Feedback Linearization

The alternative design strategy uses stochastic feedback linearization [17], i.e., using a controller $\kappa(x)$ and a change of coordinates that converts the system of Eq. (21) into a closed-loop linear system with a nonlinear perturbation of the form (10). This strategy can be applied provided that the system is *Stochastic Feedback Linearizable*:

Definition 9 *[Stochastic Feedback Linearizable System] The system* (21) *is exact stochastic feedback linearizable when there exist a twice continuously differentiable diffeomorphism $\phi : \mathbb{R}^n \to \mathbb{R}^n$, a vector field $\alpha : \mathbb{R}^n \to \mathbb{R}^m$ and a matrix field $\beta : \mathbb{R}^n \to \mathbb{R}^{m\times m}$, with $\beta(x)$ nonsingular for all $x \in \mathbb{R}^n$, such that, if*

$$u(t) = \alpha(x(t)) + \beta(x(t))v(t)$$

then the change of variables $z = \phi(x)$ transforms the system (21) *into the form*

$$\mathrm{d}z = A_0 z \mathrm{d}t + Bv\mathrm{d}t + h_z(z)\mathrm{d}w(t), \tag{29}$$

with $z(0) = z_0 \triangleq \phi(x_0)$. Also, $A_0 = \operatorname{diag}(A_1, \cdots, A_m)$ and $B = \operatorname{diag}(B_1, \cdots, B_m)$, with $A_i \in \mathbb{R}^{d_i\times d_i}$ and $B_i \in \mathbb{R}^{d_i}$, $i = 1, \cdots, m$, given by

$$A_i = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix} \text{ and } B_i = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix},$$

where d_i, $i = 1, \cdots, m$, are the controllability indices of A_0 satisfying $\sum_{i=1}^m d_i = n$.

The following auxiliary results are necessary before introducing the control design strategy.

Proposition 10 *Assume that the system of Eq.* (21) *is stochastic feedback linearizable and it verifies the matching condition $h(x) = g(x)\gamma(x)$. Then, the matrix field $h_z(z)$ in Eq.* (29) *verifies the matching condition $h_z(z) = Bg_z(z)$ where $g_z(z) \triangleq [\beta(\phi^{-1}(z))]^{-1}\gamma(\phi^{-1}(z))$.*

PROOF. Differentiating both sides of $z = \phi(x)$, applying the Ito rule, and using Eqs. (21) and (29), it can be derived that

$$B = \left[\frac{\partial \phi}{\partial x}(g(x)\beta(x))\right],\ h_z(z) = \frac{\partial \phi}{\partial x}h(x)$$

with $x = \phi^{-1}(z)$. Then, using the matching condition and the last two equations, it results that

$$h_z(z) = \frac{\partial \phi}{\partial x}h(x) = \frac{\partial \phi}{\partial x}g(x)\gamma(x) =$$
$$= B[\beta(x)]^{-1}\gamma(x) = Bg_z(z)$$

concluding the proof.

Lemma 11 *Consider system* (29)*. Let $b \in R^n$ verifying $b \succ 0$. Then, there exists a vector $\tilde{p} \in \mathbb{R}^n$ with $\tilde{p}_i > 0$, $i = 1, \dots, n$ satisfying* (16)*, such that*

$$\Sigma_{\tilde{p},b} = \operatorname{diag}(\Sigma_1, \cdots, \Sigma_m) \tag{30}$$

is a positive definite block diagonal Xiao matrix[5] *, with*

$$\Sigma_i = \mathcal{X}\left(\tilde{p}_j b_j^2 : j = \sigma_i - d_i + 1, \ldots, \sigma_i\right), \qquad (31)$$

for $i = 1, \cdots, m$*, and where* d_i *are the controllability indices of* A_0 *and* $\sigma_i = \sum_{j=1}^{i} d_j$.

PROOF. According to [16, Lemma 22, p. 7] , given $b^j \succ 0$ and $p_j \in (0,1)$, a vector $\tilde{p}^j$ subject to $\sum_{i=1}^{d_j} \tilde{p}_i^j = 1 - p_j$ can be found such that $\mathcal{X}(y^j) > 0$ with $y_i^j = \frac{[b_i^j]^2}{\eta(\tilde{p}_i^j)^2}$ and where $\eta(\tilde{p}_i^j)$ is a strictly monotonically decreasing function with image in $[a, \infty)$ for some constant $a \geq 0$.

Then, defining $\eta(\tilde{p}_i^j) = \frac{1}{\sqrt{\tilde{p}_i^j}}$ (which is strictly monotonically decreasing) we can choose $p_j \in (0,1)$ for $j = 1, \cdots, m$ such that $\sum_{j=1}^{m}(1 - p_j) = 1 - p$. We can then consider $b^j = [b_{\sigma_j - d_j + 1}, \cdots, b_{\sigma_j}]^T$ and we find m vectors $\tilde{p}^j$ such that $\mathcal{X}(y^j)$ is positive definite for all j.

Then, taking $\tilde{p} = [\tilde{p}^{1^T}, \cdots, \tilde{p}^{m^T}]^T$, it results that $\sum_{i=1}^{n} \tilde{p}_i = 1 - p$ and the fact that $\mathcal{X}(y^j) > 0$ for $j = 1, \cdots, m$ implies that the resulting block diagonal matrix $\Sigma_{\tilde{p},b}$ is positive definite, completing the proof.

The design goal is to find a feedback law $v(t) = Kz(t)$ for (29), or equivalently

$$u(t) = \alpha(x) + \beta(x)K\phi(x) \triangleq \kappa(x(t)) \qquad (32)$$

in (21), such that the closed-loop system has a desired PUB set $\mathcal{S}_x$. The following theorem establishes the existence and provides the expression for this control law.

Theorem 12 (Control Design) *Consider the feedback linearizable system* (21) *where* $h(x) = g(x)\gamma(x)$ *and* $g_z(z) \triangleq [\beta(\phi^{-1}(z))]^{-1}\gamma(\phi^{-1}(z))$ *is bounded by* $g_z(z)g_z(z)^T \leq GG^T$ *for some constant matrix* $G \in \mathbb{R}^{m \times k}$*. Suppose that a compact set* $\mathcal{S}_x \subset \mathbb{R}^n$ *with* $0 \in \operatorname{int}(\mathcal{S}_x)$ *and a scalar* $p \in (0,1)$ *are given. Then,*

(1) For any $\varepsilon > 0$*, a vector* $b \in \mathbb{R}^n$*,* $b \succ 0$*, can be found such that the set* $\mathcal{S}_z = \{z \in \mathbb{R}^n : |z_i| < b_i + \varepsilon\}$ *verifies* $\phi^{-1}(\mathcal{S}_z) \subset \mathcal{S}_x$.
(2) A vector $\tilde{p} \in \mathbb{R}^n$ *with* $\tilde{p} \succ 0$ *satisfying* (16) *can be found such that* $\Sigma_{\tilde{p},b} > 0$ *in Eqs.*(30)–(31).
(3) The control law

$$u(x) = \kappa(x) \triangleq \alpha(x) + \beta(x)K_{\tilde{p},b}\phi(x), \qquad (33)$$

with

$$K_{\tilde{p},b} = -B^{\dagger}\Psi\left(I - \frac{1}{2}BB^{\dagger}\right)\Sigma_{\tilde{p},b}^{-1} \qquad (34)$$

where $\Psi \triangleq A_0\Sigma_{\tilde{p},b} + \Sigma_{\tilde{p},b}A_0^T + BGG^TB^T$*, makes* $\mathcal{S}_x$ *a PUB with probability* p *for system* (21).

PROOF. Item 1 is a direct consequence of ϕ being a diffeomorphism and $\mathcal{S}_x$ being compact and containing the origin in its interior. Item 2 is established by Lemma 11.

For the last point, using the control law (33) and the transformation $z = \phi(x)$, taking also into account Proposition 10, the system (21) becomes

$$\begin{aligned} \mathrm{d}z &= (A_0 + BK_{\tilde{p},b})z\mathrm{d}t + Bg_z(z)\mathrm{d}w(t) \\ &= (A_0 + BK_{\tilde{p},b})z\mathrm{d}t + h_z(z)\mathrm{d}w(t), \end{aligned} \qquad (35)$$

¿From [16, Lemma 21, p. 6] , the use of the matrix gain $K_{\tilde{p},b}$ defined in Eq. (34) implies that $A = A_0 + BK_{\tilde{p},b}$ is Hurwitz and that the solution Σ of the Lyapunov equation

$$(A_0 + BK_{\tilde{p},b})\Sigma + \Sigma(A_0 + BK_{\tilde{p},b})^T + BGG^TB^T = 0 \quad (36)$$

is given by $\Sigma_{\tilde{p},b}$, verifying Eqs. (30)–(31). In particular,

$$[\Sigma]_{i,i} = \tilde{p}_i b_i^2.$$

Note also that the condition $g_z(z)g_z(z)^T \leq GG^T$ implies that $h_z(z)h_z(z)^T \leq HH^T$ with $H \triangleq BG$. Then, Theorem 7 ensures that the set $\mathcal{S}_z$ is a PUB for system (35) and the fact that $\phi^{-1}(\mathcal{S}_z) \subset \mathcal{S}_x$ implies that $\mathcal{S}_x$ is a PUB for the closed-loop system (21), (33), concluding the proof.

Theorem 12 provides an expression for the control law that ensures that the nonlinear system (21) has a desired PUB $\mathcal{S}_x$. The following algorithm summarizes the design procedure.

Algorithm 1 (Nonlinear Design Procedure)

(1) Find the change of coordinates $z = \phi(x)$*, jointly with the feedback functions* $\alpha(x)$ *and* $\beta(x)$*, that brings the system* (21) *into* (29).
(2) Take a small $\varepsilon > 0$ *and a vector* $b \succ 0$*, such that the set* $\mathcal{S}_z = \{z \in \mathbb{R}^n : |z_i| < b_i + \varepsilon\}$ *verifies* $\phi^{-1}(\mathcal{S}_z) \subset \mathcal{S}_x$.
(3) Choose m *probabilities* $p_j \in (0,1)$ *such that*[6] $\sum_{j=1}^{m}(1 - p_j) = 1 - p$.

[5] see definition 20 in [16].

[6] A reasonable choice would be $p_j = 1 - (1-p)d_j/n$, which assigns an exit probability proportional to the dimension of each controllability subspace.

(4) For each j in $1,\ldots,m$, use Algorithm for PUB design [16, p. 8] to find d_j constants $\tilde{p}_i^j$ verifying $\sum_{i=1}^{d_j} \tilde{p}_i^j = 1 - p_j$ such that the Xiao matrix with diagonal entries $[\Sigma_j]_{i,i} = b^2_{\sigma_j - d_j + i}\tilde{p}_i^j$ is positive definite.
(5) Form the block diagonal covariance matrix $\Sigma_{\tilde{p},b} = \mathrm{diag}(\Sigma_1,\ldots,\Sigma_m)$.
(6) Compute the control law $\kappa(x)$ using Eqs.(33)–(34).

5 Design Examples

This section provides examples for both design strategies.

5.1 Lyapunov Based Design

We consider a system described by

$$\mathrm{d}x = \begin{bmatrix} -x_1 - x_1x_2 \\ x_1x_2 - x_2 \end{bmatrix} \mathrm{d}t + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u\mathrm{d}t + \begin{bmatrix} x_1 & 0.5 \\ 0 & 0 \end{bmatrix} \mathrm{d}w, \quad (37)$$

where $w(t)$ is a 2-dimensional Wiener process with identity covariance matrix.

The nominal system can be stabilized using a feedback law $\kappa_1(x) = x_1x_2 - x_2^2$, under which the Lyapunov candidate $V(x) = \frac{1}{2}(x_1^2 + x_2^2)$ with $\alpha(V) = 2V$ verifies the inequality (22) in Theorem 8, since

$$\frac{\partial V}{\partial x}\left[f(x) + g(x)\kappa_1(x)\right] = -x_1^2 - x_2^2 \leq -\alpha(V(x)).$$

We want the set $S = \left\{x \in \mathbb{R}^2 : V(x) \leq c\right\}$, with $c = 1$, to be a PUB of this system with probability $p = 0.9$.

In orden to verify condition (24) we calculate

$$\frac{\partial V}{\partial x} g(x) = x_1$$

and

$$r(x) = \frac{1}{2}\mathrm{tr}\left(h^T(x)\frac{\partial^2 V}{\partial x^2}(x)h(x)\right) = \frac{x_1^2 + 0.25}{2},$$

and then, for $\frac{\partial V}{\partial x}g(x) = x_1 = 0$ it results that $r(x) = 0.125 < \alpha(c(1-p)) = 0.2$.

Note that taking $\rho < \sqrt{0.15}$, there exists $\varepsilon > 0$ such that $\beta \triangleq \alpha(c(1-p) - \varepsilon)$ satisfies Eq. (28), and it results that the control law $u(t) = \mathcal{K}(x) = \kappa_1(x) + \kappa_2(x)$ with

$$\kappa_2(x) = \begin{cases} \dfrac{-x_1(x_1^2 + 0.25)}{2x_1^2} & \text{if } |x_1| > \rho, \\[2ex] \dfrac{-x_1(x_1^2 + 0.25)}{2\rho^2} & \text{if } |x_1| \leq \rho. \end{cases}$$

ensures that the closed-loop system has the desired PUB with probability $p = 0.9$.

To verify the results, we perform simulations of the closed-loop system with the control law calculated considering $\rho = 0.3872$ (just less than $\sqrt{0.15}$) from different initial conditions. Figure 1 shows two of the trajectories obtained in the state space together with the desired PUB. On the other hand, in Figure 2 the graph of the first component of one of the trajectories of the simulated system is shown, from where can be seen that the trajectory leaves the PUB, $S = \left\{x \in \mathbb{R}^2 : x_1^2 + x_2^2 \leq 2\right\}$, during several instants of time (notice that $|x_1| > \sqrt{2} \Rightarrow (x_1, x_2) \notin S$).

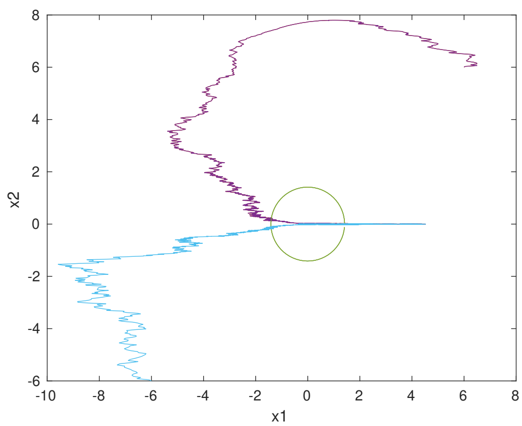

Fig. 1. Probabilistic Ultimate Bound region and closed-loop trajectories.

The fact that the trajectories are allowed to leave the set S constitute the main difference between PUBs and other probabilistic approaches such as that of stochastic barrier certificates [22].

As an additional verification, we performed 1,000 different simulations of the system and for each value of t of the form $t_k = 0.001k$ with k between 0 and 20,000 we calculated the *exit ratio*. This value is calculated as the number of sample-times where $x(t_k)$ lies outside the PUB divided by the total number of simulations (1,000). The results are shown in Figure 3.

In this case we can see that from $t \approx 2.5$ the computed exit rate is less than the escape probability of the PUB, which can be calculated as $p_{\mathrm{esc}} = 1 - p = 0.1$. Moreover,

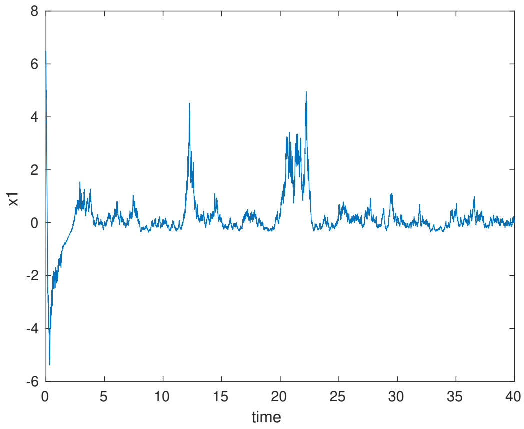

Fig. 2. First component of state versus time.

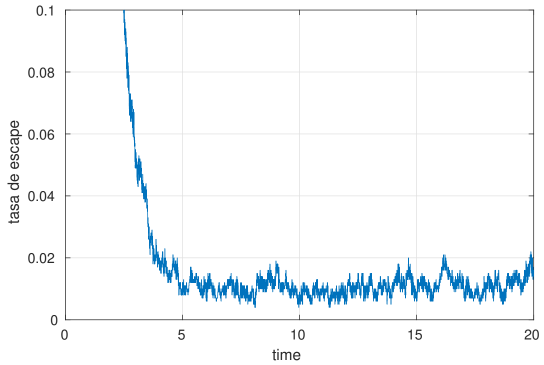

Fig. 3. Exit ratio.

starting at $t \approx 5$ the exit rate always remains lower than 0.022, showing that the control design was somewhat conservative.

There are several reasons why the design can be conservative. One of the most important reasons is that the characterisation of the ultimate bound provided by Theorem 4 is based on Markov's inequality, which only takes into account the expected value of the process $V(x(t))$ without using information about variance or higher order moments.

5.2 *Stochastic Feedback Linearization*

Consider a nonlinear stochastic system whose dynamics are given by

$$\mathrm{d}x = \begin{bmatrix} \dfrac{x_2}{1+x_1^2} \\ \\ \dfrac{2x_1x_2^2}{(1+x_1^2)^2} \end{bmatrix} \mathrm{d}t + \begin{bmatrix} 0 \\ \\ 1 \end{bmatrix} u\mathrm{d}t + \begin{bmatrix} 0 \\ \\ a \end{bmatrix} \mathrm{d}w$$

with $w(t)$ a Wiener process with identity covariance and $a = \sqrt{0.001}$.

We want the set $\mathcal{S}_x = \{x : |x| \preceq b_x\}$, with $b_x = \begin{bmatrix} 0.01 & 0.1 \end{bmatrix}^T$, to be a PUB of this system with probability $p = 0.9$.

The map

$$z = \phi(x) = [x_1 \quad \frac{x_2}{1+x_1^2}]^T,$$

jointly with the feedback functions

$$\alpha(x) = \frac{x_2}{1+x_1^2} \quad \text{and} \quad \beta(x) = 1 + x_1^2$$

transform the nonlinear system to a linear system of the form (29), where $A_0 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 1 \end{bmatrix}^T$, $h_z(z) = \begin{bmatrix} 0 & \dfrac{a}{1+z_1^2} \end{bmatrix}^T = Bg_z(z)$, with $g_z(z) = \frac{a}{1+z_1^2}$, completing step 1 of the design procedure.

In step 2, b can chosen in many ways. A possible selection is $b = \begin{bmatrix} 0.01 & 0.09 \end{bmatrix}^T$.

Now, since $m = 1$, from step 6 it results $p_1 = p = 0.9$. Then, from step 4, we obtain the escape probabilities $\tilde{p}_1 = \tilde{p}_2 = 0.05$, from which the following assignable covariance matrix results (step 5):

$$\Sigma_{\tilde{p},b} = \begin{bmatrix} \tilde{p}_1 b_1^2 & 0 \\ 0 & \tilde{p}_2 b_2^2 \end{bmatrix} = 10^{-3} \begin{bmatrix} 0.005 & 0 \\ 0 & 0.405 \end{bmatrix}.$$

Then, in the last step the controller gain is computed as $K_{\tilde{p},b} = \begin{bmatrix} -81 & -1.2346 \end{bmatrix}$ and (33) gives the final control law as:

$$\kappa(x) = \frac{(x_2 - 81x_1)(1+x_1^2) - 1.2346x_2}{(1+x_2^2)^2}$$

Figure 1 shows the region $\mathcal{S}_x$ defined in the state space and simulation results for the trajectory of the closed-loop system starting from a certain initial condition, corroborating that the region $\mathcal{S}_x$ is a PUB. This simulation result also shows that the design is conservative since it is not observed that the trajectory escapes from that region with a frequency commesurate with $p_{esc} = 1 - p = 0.1 = 10\%$. The cause of the latter conservatism can be attributed in part to the fact that the ultimate bound set is obtained from the Chebyshev's inequality, which is conservative since it only takes into account the expected value and the covariance of the process.

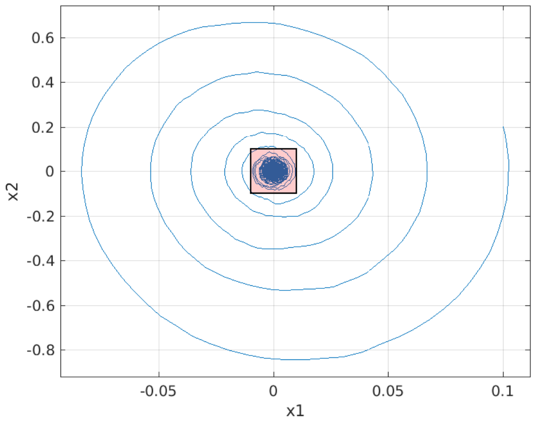

Fig. 4. Probabilistic Ultimate Bound region and closed-loop trajectory.

6 Conclusions

We have extended the notions of probabilistic ultimate bound and probabilistic invariant set to the nonlinear continuous-time domain, providing formulas for their calculation. Then, in order to guarantee a desired probabilistic ultimate bound having a given probability p, two different control design strategies were proposed. One based on the use of Lyapunov functions and the other taking into account stochastic feedback linearization. Both designs are somewhat conservative, mainly due to the fact that the ultimate bounds are obtained using conservative inequalities. Future work will extend these tools to nonlinear discrete-time systems.

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