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A Systematic Method to Obtain Ultimate Bounds for Perturbed Systems

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In this paper, we develop a systematic method to obtain ultimate bounds for both continuous- and discrete-time perturbed systems. The method is based on a componentwise analysis of the system in modal coordinates and thus exploits the system geometry as well as the perturbation structure without requiring calculation of a Lyapunov function for the system. The method is introduced for linear systems having constant componentwise perturbation bounds and is then extended to the case of state-dependent perturbation bounds. This extension enables the method to be applied to nonlinear systems by treating the perturbed nonlinear system as a linear system with a perturbation bounded by a nonlinear function of the state. Examples are provided where the proposed systematic method yields bounds that are tighter or at least not worse than those obtained via standard Lyapunov analysis. We also show how our method can be combined with Lyapunov analysis to improve on the bounds provided by either approach.

1 Introduction

The effect of perturbations is a common issue related to the study and analysis of dynamical systems. Perturbations could arise from modeling errors, ageing, uncertainties and disturbances, and are present in any realistic problem (Khalil, 2002).

In a typical situation, the exact value of a perturbation variable is unknown but supposed to be bounded. In the presence of *nonvanishing* perturbations, that is, perturbations that do not disappear as the state approaches an equilibrium point, asymptotic stability is in general not possible. However, under certain conditions, the *ultimate boundedness* of the trajectories can be guaranteed.

Nonvanishing perturbations can represent effects of quantization in A/D and D/A converters (Kofman, 2003), unknown disturbance signals (Rapaport and Astolfi, 2002), unmodeled dynamics (Oucheriah, 1999), data rate limitations in control systems (Bullo and Liberzon, 2006; Walsh et al., 2002; Wong and Brockett, 1999), errors in numerical methods (Kofman, 2002), etc. In all of these problems, it is important to estimate an ultimate bound as a measure of the undesirable perturbation effects. Estimation of an ultimate bound is also important in the design of practical controllers, such as in the context of emulation of continuous-time controllers (Laila et al., 2002) and sampled-data controller design via approximate discrete-time models (Nešić et al., 1999)

A standard tool for ultimate bound estimation is based on the use of Lyapunov functions (see, for example, Khalil, 2002, Section 9.2). This approach is very general and powerful although there is an inherent difficulty associated with the selection of a suitable Lyapunov function. For linear systems, however, quadratic Lyapunov functions can be easily computed and ultimate bounds can be obtained in the form of balls by using the system state 2-norm. This approach may result in conservative bounds due to the loss of the structure of the system (and also possibly of the perturbation) during a generic analysis.

A closely related approach is based on the input-to-state stability (ISS) property of systems with disturbances (Jiang and Wang, 2001; Sontag, 1989; Sontag and Wang). Systems that are ISS with respect to the disturbance input have an ultimate bound determined by the system disturbance-to-state asymptotic gain (see, for example, Khalil, 2002, Theorems 4.18 and 4.19). Huang et al. (2005) propose a computational framework based on dynamic programming for obtaining minimum ISS gains for nonlinear discrete-time systems.

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An approach that is different from those above was introduced in Kofman (2005), where an ultimate bound of a continuous-time linear time-invariant (LTI) system with a constant bound on the perturbation term was deduced based on geometrical principles. That study arrived at a simple explicit ultimate bound expression that, in the examples analyzed, provided a noticeably tighter bound as compared to what could be obtained via classical Lyapunov analysis using quadratic functions.

The current work extends the approach of Kofman (2005) in two main directions: (i) by allowing the perturbation term to be bounded by a state-dependent function and (ii) by deriving similar results for discrete-time systems. Specifically, we consider a system defined by

$$\dot{x}(t) = Ax(t) + u(t), \tag{1}$$

where $x(t) \in \mathbb{R}^n$ denotes the system state, $u(t) \in \mathbb{R}^n$ a perturbation input and $A \in \mathbb{R}^{n\times n}$ is Hurwitz. The result of Kofman (2005), which applies only when A is also diagonalizable, essentially consists in obtaining a componentwise ultimate bound on the state x, when the perturbation term $u(t)$ is componentwise bounded as

$$|u_i(t)| \leq u_{m_i}, \quad \text{for } i = 1, \ldots, n.$$

Here, we derive ultimate bounds when A is Hurwitz (not necessarily diagonalizable) and the perturbation term is componentwise bounded by a nonlinear function of the state, as follows

$$|u_i(t)| \leq \delta_i(x(t)), \quad \text{for } i = 1, \ldots, n. \tag{2}$$

We also derive a discrete-time counterpart of the method, considering a discrete-time system of the form

$$x(k+1) = Ax(k) + u(k),$$

where $A \in \mathbb{R}^{n\times n}$ has all its eigenvalues inside the unit circle and $u(k)$ is componentwise bounded by a state-dependent function.

These results are then utilized to derive ultimate bounds in perturbed nonlinear systems by regarding such a system as a linear system having a (nonlinear) state-dependent perturbation term. In all cases, we provide a systematic method for the computation of an ultimate bound and a set of initial states from which the ultimate bound obtained is guaranteed. The method is based on the iteration of a map constructed from the modal decomposition of the matrix A and the perturbation bounds (2).

As a motivation for the results in this paper, we now compare the result of Kofman (2005) with a Lyapunov analysis. To this aim, consider the system

$$\dot{x}(t) = \underbrace{\begin{bmatrix} 0 & 1 \\ -1 & -10 \end{bmatrix}}_{A} x(t) + u(t), \tag{3}$$

where $u_1(t) = 0$ and $|u_2(t)| \leq 0.1$ for all $t \geq 0$. A classical Lyapunov approach employs the quadratic function $U(x) = x^T P x$, where $P > 0$ is the solution of $A^T P + PA = -Q$, with $Q > 0$ and analyzes its time derivative using the perturbation bound $\|u(t)\| \leq 0.1$, for all $t \geq 0$. For example, this approach is used in Lemma 9.2 of Khalil (2002) to derive an ultimate bound for the 2-norm of x of the form

$$\mu = 2\frac{\lambda_{\max}(P)}{\lambda_{\min}(Q)}\sqrt{\frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}}0.1 + \epsilon, \tag{4}$$

where $\lambda_{\min}$ and $\lambda_{\max}$ denote the smallest and largest eigenvalues, respectively, of a real symmetric matrix, and $\epsilon > 0$ can be made arbitrarily small. Numerical minimisation of (4) with respect to Q yields $\mu = 1.3837+\epsilon$, whence $\|x(t)\| \leq 1.3837+\epsilon$ for all $t \geq t_f$, for some $t_f \geq 0$. We can also obtain the componentwise bounds $|x_1(t)| \leq 1.3837 + \epsilon$ and $|x_2(t)| \leq 1.3837 + \epsilon$. Note that, since A is Hurwitz, the system (3) is ISS with respect to the input u and an ISS analysis then leads to the same bounds (see, for example, Khalil, 2002, Theorems 4.18 and 4.19).

Application of the formula derived in Kofman (2005, Theorem 4) (and extended here to the general —not necessarily diagonalizable— Hurwitz case in Theorem 3.3) results in the tighter bounds $|x_1(t)| \leq 0.1021$ and $|x_2(t)| \leq 0.0204$, and $\|x(t)\| \leq 0.1041$, for all $t \geq t_f$, for some $t_f \geq 0$. In this case, we can identify two reasons why our method yields tighter bounds. First, the information on $u_1(t)$, namely $|u_1(t)| = 0$ is lost in the standard Lyapunov analysis, which requires a bound on $\|u(t)\|$. Second, obtaining ultimate bounds in the form of balls by means of quadratic Lyapunov functions seems to not be particularly well-suited to system (3). We can verify this statement by supposing that the initial information on $u(t)$ is that $\|u(t)\| \leq 0.1$ for all $t \geq 0$. Using then the componentwise bounds $|u_1(t)| \leq 0.1$ and $|u_2(t)| \leq 0.1$ and applying again the formula derived in Kofman (2005, Theorem 4) (or here in Theorem 3.3) yields the bounds $|x_1(t)| \leq 1.1023$ and $|x_2(t)| \leq 0.1225$, whence $\|x(t)\| \leq 1.1091$, for all $t \geq t_f$, for some $t_f \geq 0$. This bound is still tighter than the one obtained above via standard Lyapunov analysis.□

The selected structure, (1)–(2), permits to represent most problems where estimation of an ultimate bound is of practical importance. These problems include the presence of noise, the effect of uniform or logarithmic quantization (where the perturbation can be bounded by a constant or by a linear function of the state, respectively), systems with parametric uncertainty (where the product of an unknown matrix and the state can be modelled as a perturbation), etc. Notice that in all these cases the perturbation does not affect each component of the right-hand side of (1) in the same way and hence it may be useful to bound the perturbation componentwise as in (2). Our method can also be easily extended to systems of the form $\dot{x}(t) = Ax(t) + Bu(t)$ or $x(k+1) = Ax(k) + Bu(k)$, where $u \in \mathbb{R}^m$, with straightforward modifications of the derived expressions.

The remainder of the paper is organized as follows. In Section 2, the notation, concepts and mathematical tools employed throughout the paper are described. Sections 3 and 4 derive the ultimate bound estimation method for continuous- and discrete-time systems, respectively. Illustrative examples are provided in Section 5 and conclusions are drawn in Section 6.

2 Mathematical Tools

This section describes the notation, concepts and mathematical tools used throughout the paper. The notation and some preliminary tools are described in Section 2.1. Since our results aim at exploiting the system and perturbation structures, we employ a nonstandard definition of ultimate boundedness that is better suited to this goal. Section 2.2 provides a standard definition of ultimate boundedness and thus derives a preliminary result that links the results obtained throughout the paper with this standard definition.

2.1 *Notation and Preliminary Tools*

If M is a matrix with (real or complex) entries $M_{i,j}$, then $|M|$ and $\mathbb{R}\mathrm{e}(M)$ denote its elementwise magnitude and real part, respectively, that is, $|M|$ is the matrix with entries $|M_{i,j}|$ and $\mathbb{R}\mathrm{e}(M)$, the one with entries $\mathbb{R}\mathrm{e}(M_{i,j})$.

If $x, y \in \mathbb{R}^n$, then $x \leq y$ and $x < y$ denote the sets of componentwise inequalities $x_i \leq y_i$ and $x_i < y_i$, respectively, for $i = 1, \ldots, n$, and similarly for $x \geq y$ and $x > y$. The expression '$x \not\leq y$' is used as equivalent to '$x \leq y$ is not true'. Thus, $x \not\leq y$ does not necessarily imply that $x > y$.

According to these definitions, it is easy to show that

$$|x + y| \leq |x| + |y|, \quad |M\,x| \leq |M| \cdot |x|, \tag{5}$$

$$|x| \leq |y| \Rightarrow |M| \cdot |x| \leq |M| \cdot |y|, \tag{6}$$

whenever $x, y \in \mathbb{C}^n$ and $M \in \mathbb{C}^{m \times n}$.

$\mathbb{R}^n_+$ and $\mathbb{R}^n_{+,0}$ denote the sets of vectors in $\mathbb{R}^n$ with positive and nonnegative components, respectively. Consequently, if $x \in \mathbb{R}^n$ then $x \in \mathbb{R}^n_+ \Leftrightarrow x > 0$ and $x \in \mathbb{R}^n_{+,0} \Leftrightarrow x \geq 0$.

$\mathbf{1}_n$ denotes the vector in $\mathbb{R}^n$ all of whose components are equal to 1.

2.2 *Ultimate Boundedness*

We now provide a standard definition of ultimate boundedness, adapted from Khalil (2002), and derive a preliminary result that links the results obtained later in the paper to this definition.

Definition 2.1 The solutions of $\dot{x} = f(t,x)$ are said to be uniformly ultimately bounded if there exist a vector norm $\|\cdot\|$ and positive constants d and c such that for every $\alpha \in (0,c)$ there is a positive constant $\mathcal{T} = \mathcal{T}(\alpha)$ such that

$$\|x(t_0)\| < \alpha \Rightarrow \|x(t)\| \leq d, \quad \forall t \geq t_0 + \mathcal{T}. \tag{7}$$

In essence, the results that we provide in Sections 3 and 4 guarantee that if $|V^{-1}x(t_0)| \leq \beta$, then the following implicit ultimate bound holds:

$$|V^{-1}x(t)| \leq c, \quad \text{for all } t \geq t_0 + \mathcal{T}, \tag{8}$$

where $\beta, c \in \mathbb{R}^n_+$, $V \in \mathbb{C}^{n\times n}$ is a nonsingular matrix and $\mathcal{T} \in \mathbb{R}_{+,0}$. Since (8) implies the componentwise bound

$$|x(t)| \leq |V| \cdot |V^{-1}x(t)| \leq |V|c, \quad \text{for all } t \geq t_0 + \mathcal{T},$$

then the following Lemma shows that such results lead to ultimate bounds in the sense of Definition 2.1.

LEMMA 2.2 *Consider the system $\dot{x} = f(t,x)$, where $x(t) \in \mathbb{R}^n$, and suppose that there exist $\beta, b \in \mathbb{R}^n_+$, $\mathcal{T} \in \mathbb{R}_{+,0}$ and a nonsingular matrix $V \in \mathbb{C}^{n\times n}$ such that*

$$|V^{-1}x(t_0)| \leq \beta \Rightarrow |x(t)| \leq b, \quad \forall t \geq t_0 + \mathcal{T}.$$

Then, the solutions of $\dot{x} = f(t,x)$ are uniformly ultimately bounded.

Proof Let $\beta_{\min} \triangleq \min_i \beta_i$, $b_{\max} \triangleq \|b\|_\infty$, where $\|\cdot\|_\infty$ denotes the infinity norm of a vector and the corresponding induced norm for a matrix. Note that $\beta_{\min} > 0$. Then, for any $\alpha \in (0, \beta_{\min}/\|V^{-1}\|_\infty)$, we have

$$\begin{aligned}\|x(t_0)\|_\infty < \alpha &\Rightarrow \|V^{-1}x(t_0)\|_\infty < \|V^{-1}\|_\infty\,\alpha\\ &\Rightarrow \|V^{-1}x(t_0)\|_\infty < \beta_{\min}\\ &\Rightarrow |V^{-1}x(t_0)| < \beta\\ &\Rightarrow |x(t)| \leq b \Rightarrow \|x(t)\|_\infty \leq b_{\max},\end{aligned}$$

for all $t \geq t_0 + \mathcal{T}$. This concludes the proof. □

The discrete-time counterparts to Definition 2.1 and Lemma 2.2 are straightforward.

3 Ultimate Bounds for Continuous-time Systems

In this section, we develop a systematic method to obtain ultimate bounds for perturbed continuous-time systems which is based on a componentwise analysis of the system in modal coordinates. In Section 3.1, we derive ultimate bound expressions when the perturbation input is componentwise bounded by a constant.

This result is used as an intermediate tool to derive ultimate bound expressions at the beginning of Section 3.2, where the perturbation input is bounded by a state-dependent function. Section 3.2 then proceeds to develop the aforementioned systematic method. In Section 3.3, we show how the results of Section 3.2 may be applied to nonlinear systems.

3.1 *Constant Perturbation Bounds*

In this section, we present ultimate bounds of a linear system when the perturbation bound is constant. This result is presented in Theorem 3.3, which requires the following two lemmas. Lemma 3.1 contains a preliminary result for a perturbed scalar system and Lemma 3.2 a similar result for a system whose evolution matrix consists of a single Jordan block.

LEMMA 3.1 *Consider the scalar system*

$$\dot{z}(t) = \lambda z(t) + v(t) \tag{9}$$

where $\lambda, z(t), v(t) \in \mathbb{C}$ *and* $\mathbb{R}\mathrm{e}(\lambda) < 0$*. Suppose that* $|v(t)| \le v_m$ *for all* $0 \le t \le \tau$*, where* $v_m \in \mathbb{R}$*. If* $|z(0)| \le \left|[\mathbb{R}\mathrm{e}(\lambda)]^{-1}\right| v_m$*, then* $|z(t)| \le \left|[\mathbb{R}\mathrm{e}(\lambda)]^{-1}\right| v_m$ *for all* $0 \le t \le \tau$*.*

Proof Express $z(t)$ in polar form as $z(t) = \rho(t)\, e^{j\theta(t)}$, where $\rho(t) \in \mathbb{R}_{+,0}$ and $\theta(t) \in \mathbb{R}$. Substituting into (9) and multiplying by $e^{-j\theta(t)}$ yields

$$\dot{\rho}(t) + j\rho(t)\,\dot{\theta}(t) = \lambda\rho(t) + v(t)\, e^{-j\theta(t)}.$$

Taking real part and using the bound on $v(t)$, we have

$$\dot{\rho}(t) = \mathbb{R}\mathrm{e}(\lambda)\rho(t) + \mathbb{R}\mathrm{e}\left(v(t)\, e^{-j\theta(t)}\right) \le \mathbb{R}\mathrm{e}(\lambda)\rho(t) + v_m, \tag{10}$$

where the inequality is valid for $0 \le t \le \tau$. Define the auxiliary system

$$\dot{y}(t) = \mathbb{R}\mathrm{e}(\lambda) y(t) + v_m, \tag{11}$$

with initial condition $y(0) \triangleq \rho(0) = |z(0)|$. This linear differential equation can be solved as

$$y(t) = |z(0)| e^{\mathbb{R}\mathrm{e}(\lambda)t} + \frac{v_m}{|\mathbb{R}\mathrm{e}(\lambda)|}\left(1 - e^{\mathbb{R}\mathrm{e}(\lambda)t}\right), \tag{12}$$

where we have used the fact that $\mathbb{R}\mathrm{e}(\lambda) < 0$. Using the assumption $|z(0)| \le \left|[\mathbb{R}\mathrm{e}(\lambda)]^{-1}\right| v_m$ in (12), it follows that $y(t) \le \left|[\mathbb{R}\mathrm{e}(\lambda)]^{-1}\right| v_m$ for all $t \ge 0$. Applying the Comparison Lemma to (10) and (11) (see, for example, Khalil, 2002, p.102), we conclude that $|z(t)| \le y(t)$, for all $0 \le t \le \tau$, whence the result follows. □

LEMMA 3.2 *Consider the system*

$$\dot{z}(t) = \Lambda z(t) + v(t) \tag{13}$$

where $z(t), v(t) \in \mathbb{C}^r$ *and* $\Lambda \in \mathbb{C}^{r\times r}$ *is a Jordan block with eigenvalue* λ *satisfying* $\mathbb{R}\mathrm{e}(\lambda) < 0$*. Suppose that* $|v(t)| \le v_m$ *for all* $0 \le t \le \tau$*, where* $v_m \in \mathbb{R}^r$*. If* $|z(0)| \le \left|[\mathbb{R}\mathrm{e}(\Lambda)]^{-1}\right| v_m$*, then* $|z(t)| \le \left|[\mathbb{R}\mathrm{e}(\Lambda)]^{-1}\right| v_m$ *for all* $0 \le t \le \tau$*.*

Proof First, note that the matrix $|[\mathbb{R}\mathrm{e}(\Lambda)]^{-1}|$ satisfies

$$|[\mathbb{R}\mathrm{e}(\Lambda)]^{-1}| = \begin{bmatrix} |\mathbb{R}\mathrm{e}(\lambda)^{-1}| & |\mathbb{R}\mathrm{e}(\lambda)^{-2}| & \dots & |\mathbb{R}\mathrm{e}(\lambda)^{-r}| \\ 0 & |\mathbb{R}\mathrm{e}(\lambda)^{-1}| & \dots & |\mathbb{R}\mathrm{e}(\lambda)^{-(r-1)}| \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & |\mathbb{R}\mathrm{e}(\lambda)^{-1}| \end{bmatrix}.$$

Define $a \triangleq \left|[\mathbb{R}\mathrm{e}(\Lambda)]^{-1}\right| v_m$ and let a_i denote the i-th component of a. Then, we can write

$$a_i = \sum_{j=i}^{r} \left|[\mathbb{R}\mathrm{e}(\lambda)]^{-(j-i+1)}\right| v_{m_j}, \tag{14}$$

for $i = 1, \dots, r$. Let $z_i(t)$ denote the i-th component of $z(t)$. We will prove by induction that

$$|z_i(t)| \le a_i, \quad \text{for } 0 \le t \le \tau, \tag{15}$$

for $i = 1, \dots, r$. By assumption, $|z(0)| \le \left|[\mathbb{R}\mathrm{e}(\Lambda)]^{-1}\right| v_m$, and using a as defined above, we have

$$|z_i(0)| \le a_i, \quad \text{for } i = 1, \dots, r. \tag{16}$$

In particular, $|z_r(0)| \le a_r = \left|[\mathbb{R}\mathrm{e}(\lambda)]^{-1}\right| v_{m_r}$. From (13) and the Jordan form of Λ, it follows that $\dot{z}_r(t) = \lambda z_r(t) + v_r(t)$, with $|v_r(t)| \le v_{m_r}$ for $0 \le t \le \tau$. Applying Lemma 3.1 yields $|z_r(t)| \le |\mathbb{R}\mathrm{e}(\lambda)^{-1}| v_{m_r} = a_r$ for $0 \le t \le \tau$, proving (15) for $i = r$.

We now prove that if z_{i+1} satisfies (15), then z_i also does. Thus, suppose that z_{i+1} satisfies (15). This implies

$$|z_{i+1}(t) + v_i(t)| \le a_{i+1} + v_{m_i}, \quad \text{for } 0 \le t \le \tau. \tag{17}$$

Using (14), the bound on the right-hand side of (17) satisfies

$$\begin{aligned} \left|[\mathbb{R}\mathrm{e}(\lambda)]^{-1}\right| (a_{i+1} + v_{m_i}) &= \left|[\mathbb{R}\mathrm{e}(\lambda)]^{-1}\right| \left(\sum_{j=i+1}^{r} \left|[\mathbb{R}\mathrm{e}(\lambda)]^{-(j-i)}\right| v_{m_j} + v_{m_i} \right) \\ &= \left|[\mathbb{R}\mathrm{e}(\lambda)]^{-1}\right| \sum_{j=i}^{r} \left|[\mathbb{R}\mathrm{e}(\lambda)]^{-(j-i)}\right| v_{m_j} \\ &= \sum_{j=i}^{r} \left|[\mathbb{R}\mathrm{e}(\lambda)]^{-(j-i+1)}\right| v_{m_j} = a_i. \end{aligned} \tag{18}$$

From (13) and the Jordan form of Λ, we have $\dot{z}_i(t) = \lambda z_i(t) + z_{i+1}(t) + v_i(t)$, for $i = 1, \dots, r-1$, where the last two terms satisfy (17). From (16) and (18), we have $|z_i(0)| \le a_i = \left|[\mathbb{R}\mathrm{e}(\lambda)]^{-1}\right| (a_{i+1} + v_{m_i})$. Applying Lemma 3.1 then yields

$$|z_i(t)| \le \left|[\mathbb{R}\mathrm{e}(\lambda)]^{-1}\right| (a_{i+1} + v_{m_i}) = a_i, \text{ for } 0 \le t \le \tau,$$

proving that z_i also satisfies (15).

Since we have already shown that z_r satisfies (15), it follows that (15) is satisfied for $i = 1, \dots, r$ and the proof is complete. □

The following theorem provides ultimate bounds for linear systems with constant perturbation bounds. This theorem extends the result of Kofman (2005) to the case where the system's evolution matrix is required to be only Hurwitz (not necessarily diagonalizable). The main feature of this result is that it does not require the calculation of a Lyapunov function for the system and may yield tighter bounds than those obtained via standard Lyapunov analysis using quadratic functions, as shown in the introductory example in Section 1. Part i) of the theorem characterizes a bounded invariant region in the state space, that is, a region with the property that trajectories originating in that region remain in the region while the perturbation remains bounded. Part ii) shows that, if the perturbation is bounded for all $t \geq 0$, then the trajectories converge to the bounded invariant region from any initial condition.

THEOREM 3.3 *Consider the system*

$$\dot{x}(t) = Ax(t) + u(t) \tag{19}$$

where $x(t), u(t) \in \mathbb{R}^n$, *and* $A \in \mathbb{R}^{n\times n}$ *is a Hurwitz matrix with Jordan canonical form* $\Lambda = V^{-1}AV$. *Suppose that* $|u(t)| \leq u_m$ *for all* $0 \leq t \leq \tau$ *and define*

$$S \triangleq \left|[\mathbb{R}\mathrm{e}(\Lambda)]^{-1}\right| \cdot |V^{-1}|. \tag{20}$$

Then,

i) Invariance. If $|V^{-1}x(0)| \leq Su_m$, *then for all* $0 \leq t \leq \tau$,
  a) $|V^{-1}x(t)| \leq Su_m$.
  b) $|x(t)| \leq |V|Su_m$.

ii) Convergence. If $\tau = \infty$, *then given any initial condition* $x(0) \in \mathbb{R}^n$ *and positive vector* $\epsilon \in \mathbb{R}^n_+$, *a finite time* $t_f = t_f(\epsilon, x(0))$ *exists so that for all* $t \geq t_f$,
  a) $|V^{-1}x(t)| \leq Su_m + \epsilon$.
  b) $|x(t)| \leq |V|Su_m + |V|\,\epsilon$.

Proof Let $x(t) = Vz(t)$ and $v(t) \triangleq V^{-1}u(t)$. Then, using (19) we have

$$\dot{z}(t) = \Lambda z(t) + v(t), \tag{21}$$

where $v(t)$ satisfies

$$|v(t)| \leq v_m \triangleq |V^{-1}|\,u_m, \quad \text{for all } 0 \leq t \leq \tau. \tag{22}$$

Note that (21) constitutes a set of k ($k \leq n$) uncoupled differential equations of the form

$$\dot{z}_i(t) = \Lambda_i z_i(t) + v_i(t), \quad \text{for } 1 \leq i \leq k, \tag{23}$$

where $z_i, v_i \in \mathbb{C}^{r_i}$ and $\Lambda_i \in \mathbb{C}^{r_i\times r_i}$ is a Jordan block (r_i is the multiplicity of the eigenvalue of the i–th block). From (22), $|v_i(t)| \leq v_{m_i}$ for all $0 \leq t \leq \tau$.

i) By assumption, $|z(0)| = |V^{-1}x(0)| \leq Su_m$. Using (20) and (22), then $|z(0)| \leq \left|[\mathbb{R}\mathrm{e}(\Lambda)]^{-1}\right| v_m$ and hence $|z_i(0)| \leq \left|[\mathbb{R}\mathrm{e}(\Lambda_i)]^{-1}\right| v_{m_i}$, for $i = 1, \ldots, k$. Applying Lemma 3.2 to (23), we obtain

$$|z_i(t)| \leq \left|[\mathbb{R}\mathrm{e}(\Lambda_i)]^{-1}\right| v_{m_i}, \quad \text{for } i = 1, \ldots, k, \tag{24}$$

for all $0 \leq t \leq \tau$. A compact expression for (24) is

$$|z(t)| \leq \left|[\mathbb{R}\mathrm{e}(\Lambda)]^{-1}\right| v_m, \text{ for all } 0 \leq t \leq \tau, \tag{25}$$

and the proof of i) a) follows by recalling that $z(t) = V^{-1}x(t)$, $v_m = |V^{-1}|\,u_m$ and (20). To prove i) b) note that $|x(t)| = |Vz(t)| \leq |V| \cdot |z(t)|$ and use (25). This completes the proof of i).

ii) Consider again system (21) with initial condition $z(0)$ and with the perturbation term bounded by (22) for all $t \geq 0$, since by assumption $\tau = \infty$. Let $\tilde{z}(t)$ satisfy $\dot{\tilde{z}}(t) = \Lambda\tilde{z}(t)$ and $\tilde{z}(0) = z(0)$.

Notice that $\lim_{t\to\infty} \tilde{z}(t) = 0$ since Λ is the Jordan form of A, which is Hurwitz. Then, given any positive vector ϵ, a finite time $t_f = \tilde{t}_f(\epsilon, \tilde{z}(0)) = t_f^z(\epsilon, z(0)) = t_f(\epsilon, x(0))$ can be found so that

$$|\tilde{z}(t)| \le \epsilon \text{ for all } t \ge t_f. \tag{26}$$

Define $\hat{z}(t) \triangleq z(t) - \tilde{z}(t)$. Then, $\hat{z}(t)$ verifies (21) and (22). Note also that $|\hat{z}(0)| = 0 \le \left|[\mathbb{R}\mathrm{e}(\Lambda)]^{-1}\right| v_m$. Thus, applying the result of part i), we conclude that $|\hat{z}(t)| \le \left|[\mathbb{R}\mathrm{e}(\Lambda)]^{-1}\right| v_m$ for all $t \ge 0$. Then, using the definition of $\hat{z}$ and (26), we obtain

$$|z(t)| \le |\hat{z}(t)| + |\tilde{z}(t)| \le \left|[\mathbb{R}\mathrm{e}(\Lambda)]^{-1}\right| v_m + \epsilon \tag{27}$$

for all $t \ge t_f$, and the proof of ii) a) follows by recalling that $z(t) = V^{-1}x(t)$, $v_m = |V^{-1}|\, u_m$ and (20). To prove ii) b) note that $|x(t)| = |Vz(t)| \le |V| \cdot |z(t)|$ and use (27). This completes the proof of the theorem. □

Theorem 3.3 gives both implicit and componentwise ultimate bound estimations (see Lemma 2.2) of LTI systems when the perturbation bound is constant, that is, when it does not depend on the state. The regions of the state space defined by the implicit bounds given in Theorem 3.3 ii) a) are contained in the corresponding axis-aligned sets given in Theorem 3.3 ii) b). The latter provides componentwise ultimate bounds on the state.

3.2 *State-dependent Perturbation Bounds*

In this section, we present the main contribution of the paper for continuous-time systems. We provide ultimate bound expressions for linear systems with state-dependent perturbation bounds that satisfy a monotonicity condition [see (29) and (30) below]. These bounds are derived in Theorem 3.4, which requires the existence of a point (x_m) satisfying a certain condition. We thus subsequently provide an algorithm to test whether this condition is satisfied and a proof of its convergence. All these results provide a systematic method to obtain ultimate bounds for continuous-time systems. As we will see in the examples, the bounds provided by this systematic method may be tighter than those obtained via standard Lyapunov analysis using quadratic functions, and can also be combined with the latter methodology to improve on the bounds provided by either approach.

Theorem 3.4 *Consider the system*

$$\dot{x}(t) = Ax(t) + u(t), \tag{28}$$

where $x(t), u(t) \in \mathbb{R}^n$, *and* $A \in \mathbb{R}^{n\times n}$ *is Hurwitz with Jordan canonical form* $\Lambda = V^{-1}AV$. *Suppose that*

$$|u(t)| \le \delta(x(t)) \quad \forall t \ge 0 \tag{29}$$

where $\delta : \mathbb{R}^n \to \mathbb{R}^n_{+,0}$ *is a continuous map verifying*

$$|x_1| \le |x_2| \Rightarrow \delta(x_1) \le \delta(x_2). \tag{30}$$

Consider the map $T : \mathbb{R}^n \to \mathbb{R}^n_{+,0}$ *defined by*

$$T(x) \triangleq |V| S\delta(x), \tag{31}$$

with S *as defined in (20). Suppose that there exists* $x_m \in \mathbb{R}^n$ *satisfying,* $T(x_m) < x_m$. *Then,*

i) $b \triangleq \lim_{k\to\infty} T^k(x_m)$ *exists and satisfies* $0 \le b < x_m$.

ii) If $|V^{-1}x(0)| \le S\delta(x_m)$ *then, given any positive vector* $\epsilon \in \mathbb{R}^n_+$, *a finite time* $t_f = t_f(\epsilon, x_m)$ *exists so that for all* $t \ge t_f$,

a) $|V^{-1}x(t)| \leq S\delta(b) + \epsilon$.
b) $|x(t)| \leq b + |V|\,\epsilon$.

Proof By (30), (31), (20) and (6), then T satisfies

$$|x_1| \leq |x_2| \Rightarrow T(x_1) \leq T(x_2). \tag{32}$$

i) Note that $0 \leq T(x_m) < x_m$ and hence $|T(x_m)| < |x_m|$. By (32), then $T(T(x_m)) \leq T(x_m)$ and applying T repeatedly we obtain $0 \leq T^k(x_m) \leq T^{k-1}(x_m) < x_m$, for all $k \geq 2$. The sequence $T^k(x_m)$ is thus nonincreasing and lower bounded by 0, and hence it must converge to some point $b = \lim_{k\to\infty} T^k(x_m)$ that satisfies $0 \leq b < x_m$.

ii) We now prove that $|x(t)| \leq x_m$ for all $t \geq 0$. For a contradiction, suppose that $|x(t_d)| \not\leq x_m$, where $0 \leq t_d < \infty$. Define

$$t_c \triangleq \inf t, \quad \text{subject to } t \geq 0 \text{ and } |x(t)| \not\leq x_m. \tag{33}$$

Note that $|x(0)| \leq |V| \cdot |V^{-1}x(0)|$ and by assumption and (31), then

$$|x(0)| \leq |V| S\delta(x_m) = T(x_m) < x_m.$$

Hence, $0 < t_c \leq t_d$ and since $x(t)$ is continuous we have $|x(t)| \leq x_m$, for all $0 \leq t \leq t_c$. By (29) and (30) then $|u(t)| \leq \delta(x_m)$, for all $0 \leq t \leq t_c$. Applying Theorem 3.3, i) b) and using (31), then $|x(t)| \leq T(x_m) < x_m$, for all $0 \leq t \leq t_c$. Since $x(t)$ is continuous, then there exists a positive real constant $a > 0$ such that $|x(t)| \leq x_m$ for all $0 \leq t \leq t_c + a$, contradicting (33) and proving that $|x(t)| \leq x_m$ for all $t \geq 0$.

Therefore, using (29) and (30) it follows that $|u(t)| \leq \delta(x_m)$ for all $t \geq 0$. Using Theorem 3.3, ii) a), then, given a positive vector $\gamma \in \mathbb{R}^n_+$, a finite time t_1 exists so that

$$|V^{-1}x(t)| \leq S\delta(x_m) + \gamma$$

for all $t \geq t_1$. Then,

$$|x(t)| \leq |V| \cdot |V^{-1}x(t)| \leq T(x_m) + |V|\,\gamma,$$

where we have used (31). Therefore, using (29) and (30), it follows that $|u(t)| \leq \delta(T(x_m) + |V|\,\gamma)$ for all $t \geq t_1$. Applying again Theorem 3.3, ii) a), we conclude that a positive time t_2 exists so that

$$|V^{-1}x(t)| \leq S\delta(T(x_m) + |V|\,\gamma) + \gamma,$$

for all $t \geq t_1 + t_2$. Defining $T_\gamma(x) \triangleq T(x) + |V|\,\gamma$, the recursive use of this procedure yields

$$|V^{-1}x(t)| \leq S\delta(T^k_\gamma(x_m)) + \gamma, \tag{34}$$

for all $t \geq \sum_{i=1}^{k+1} t_i$. Since $T_\gamma(x_m) = T(x_m) + |V|\,\gamma$ and $T(x_m) < x_m$, we may choose γ small enough so that

$$T_\gamma(x_m) < x_m. \tag{35}$$

Note that, since $\gamma \in \mathbb{R}^n_+$, then $|V|\,\gamma \in \mathbb{R}^n_{+,0}$ and thus $T_\gamma(x) = |T_\gamma(x)| \geq 0$ for all $x \in \mathbb{R}^n$. Then, from (32) and (35), it follows that

$$T^2_\gamma(x_m) \leq T_\gamma(x_m),$$

and applying T_γ recursively yields

$$T_\gamma^k(x_m) \leq T_\gamma^{k-1}(x_m).$$

Clearly, the sequence $T_\gamma^k(x_m)$ is nonincreasing. Being nonincreasing and lower bounded (by 0), $T_\gamma^k(x_m)$ must converge to some point b_γ. Note that

$$\lim_{\gamma\to 0^+} b_\gamma = b,$$

since $\lim_{\gamma\to 0^+} T_\gamma(x) = T(x)$. Then, given $\epsilon \in \mathbb{R}_+^n$, for any $\epsilon_1 \in \mathbb{R}_+^n$ we can select $\gamma = \gamma(\epsilon_1) \in \mathbb{R}_+^n$ so that $\gamma \leq \epsilon/2$ and $b_\gamma \leq b + \epsilon_1$. In addition, as

$$\lim_{k\to\infty} T_\gamma^k(x_m) = b_\gamma$$

for any $\epsilon_2 \in \mathbb{R}_+^n$, a positive integer $N = N(\epsilon_2)$ can be found so that

$$T_\gamma^k(x_m) \leq b_\gamma + \epsilon_2 \quad \text{for all } k \geq N.$$

Then, using (34), (30) and $b_\gamma \leq b + \epsilon_1$, it follows that

$$\begin{aligned} |V^{-1}x(t)| &\leq S\delta(T_\gamma^N(x_m)) + \gamma \\ &\leq S\delta(b + \epsilon_1 + \epsilon_2) + \gamma, \end{aligned} \tag{36}$$

for all $t \geq \sum_{i=1}^{N+1} t_i$. By (30), $\delta(b) \leq \delta(b + \epsilon_1 + \epsilon_2)$ and hence given $\epsilon \in \mathbb{R}_+^n$, we can select $\epsilon_1, \epsilon_2 \in \mathbb{R}_+^n$ so that

$$S\delta(b + \epsilon_1 + \epsilon_2) \leq S\delta(b) + \epsilon/2. \tag{37}$$

Adding $\epsilon/2$ to both sides of (37) and recalling that $\gamma = \gamma(\epsilon_1)$ is selected so that $\gamma \leq \epsilon/2$, then

$$S\delta(b + \epsilon_1 + \epsilon_2) + \gamma \leq S\delta(b) + \epsilon$$

and from (36), it follows that

$$|V^{-1}x(t)| \leq S\delta(b) + \epsilon,$$

for all $t \geq \sum_{i=1}^{N+1} t_i$, and the proof of ii) a) concludes by setting $t_f = \sum_{i=1}^{N+1} t_i$. To prove ii) b), note that $|x(t)| \leq |V| \cdot |V^{-1}x(t)|$ and using ii) a) yields

$$|x(t)| \leq |V| S\delta(b) + |V|\,\epsilon.$$

The proof is concluded by recalling (31) and the fact that $T(b) = b$, which follows from i). □

Theorem 3.4 provides a simple ultimate bound expression and shows that the set $\{x \in \mathbb{R}^n : |V^{-1}x| \leq S\delta(x_m)\}$ is an estimation of the region of attraction to the ultimate bound. The theorem relies on finding a point x_m so that $T(x_m) < x_m$. Although checking this condition analytically might be possible, this cannot be ensured in all cases. Therefore, we provide the following numerical algorithm, and then analyze its convergence.

Algorithm 1 Numerical Obtention of x_m Consider a map $T : \mathbb{R}^n \to \mathbb{R}_{+,0}^n$.

1. Choose a scalar $c > 0$.

2. Define the map $T_c(x) \triangleq T(x) + c\mathbf{1}_n$ and iterate it from $x = 0$.

THEOREM 3.5 *Suppose that a map $T : \mathbb{R}^n \to \mathbb{R}^n_{+,0}$ satisfies (32).*

a) If, choosing $c = \psi > 0$, Algorithm 1 converges to a point $\bar{x}^\psi_m$, then $T(\bar{x}^\psi_m) < \bar{x}^\psi_m$. Also, if $0 < \phi < \psi$, choosing $c = \phi$ Algorithm 1 converges to $\bar{x}^\phi_m$, where $\bar{x}^\phi_m < \bar{x}^\psi_m$.
b) If x_m exists such that $T(x_m) < x_m$ and if $c > 0$ is chosen small enough in Step 1 of Algorithm 1, then the algorithm converges to a point $\bar{x}_m$ satisfying $\lim_{k\to\infty} T^k(\bar{x}_m) \leq \lim_{k\to\infty} T^k(x_m)$.

Proof a). Convergence of Algorithm *1* to a point $\bar{x}^\psi_m$ implies that $\bar{x}^\psi_m = T_\psi(\bar{x}^\psi_m)$ and by definition of T_ψ and the fact that $\psi > 0$, then $T(\bar{x}^\psi_m) < T(\bar{x}^\psi_m) + \psi\mathbf{1}_n = \bar{x}^\psi_m$. Since $\phi < \psi$, we have

$$T_\phi(0) = T(0) + \phi\mathbf{1}_n < T(0) + \psi\mathbf{1}_n = T_\psi(0).$$

Since T satisfies (32), applying T to the inequality above yields $T(T_\phi(0)) \leq T(T_\psi(0))$. Then, $T(T_\phi(0)) + \phi\mathbf{1}_n < T(T_\psi(0)) + \psi\mathbf{1}_n$ whence $T^2_\phi(0) < T^2_\psi(0)$. Repeating this procedure we can obtain

$$T^k_\phi(0) < T^k_\psi(0), \text{ for all } k > 0. \tag{38}$$

Also, $0 \leq T_\phi(0)$, whence $T(0) \leq T(T_\phi(0))$ and $T(0) + \phi\mathbf{1}_n = T_\phi(0) \leq T(T_\phi(0)) + \phi\mathbf{1}_n = T^2_\phi(0)$. Repeating this procedure we can obtain

$$T^k_\phi(0) \leq T^{k+1}_\phi(0), \text{ for all } k > 0. \tag{39}$$

From (39), the sequence $T^k_\phi(0)$ is nondecreasing and from (38) it is bounded above by the converging sequence $T^k_\psi(0)$. Therefore, $T^k_\phi(0)$ must converge to some point $\bar{x}^\phi_m$. From (38) it follows that $\bar{x}^\phi_m \leq \bar{x}^\psi_m$. Using (32) then $T(\bar{x}^\phi_m) \leq T(\bar{x}^\psi_m)$, whence $T_\phi(\bar{x}^\phi_m) = T(\bar{x}^\phi_m) + \phi\mathbf{1}_n < T(\bar{x}^\psi_m) + \psi\mathbf{1}_n = T_\psi(\bar{x}^\psi_m)$. Hence, $\bar{x}^\phi_m = T_\phi(\bar{x}^\phi_m) < T_\psi(\bar{x}^\psi_m) = \bar{x}^\psi_m$. This concludes the proof of a).

b). Since $T(x_m) < x_m$, then by Theorem 3.4 i) the limit $b \triangleq \lim_{k\to\infty} T^k(x_m)$ exists and satisfies $b < x_m$. This implies that $b = T(b) < T(b) + c\mathbf{1}_n = T_c(b) \leq T_c(x_m)$, where the first inequality follows from $c > 0$ and the second one from the facts that T satisfies (32) and $b < x_m$. Also, by choosing $c > 0$ small enough, we can guarantee that $T_c(x_m) < x_m$. Applying T_c iteratively we arrive to

$$b < T^k_c(x_m) \leq T^{k-1}_c(x_m) < x_m.$$

Then, the sequence $T^k_c(x_m)$ is nonincreasing and lower bounded, which implies that it converges to some point b_c satisfying

$$b \leq b_c < x_m. \tag{40}$$

Consider now the sequence $T^k_c(0)$. Notice that since $0 \leq b_c$ and T_c satisfies (32), then $0 \leq T_c(0) \leq b_c$, and applying T_c iteratively yields

$$T^{k-1}_c(0) \leq T^k_c(0) \leq b_c.$$

This implies that $T^k_c(0)$ is nondecreasing and upper bounded by b_c, which shows that Algorithm *1* must converge to some point $\bar{x}_m$ satisfying

$$\bar{x}_m \leq b_c. \tag{41}$$

By a), then $T(\bar{x}_m) < \bar{x}_m$ and by assumption, $T(x_m) < x_m$. Thus, Theorem 3.4 i) proves that $\lim_{k\to\infty} T^k(\bar{x}_m)$ and $\lim_{k\to\infty} T^k(x_m)$ both exist. Also, (40) and (41) imply that $\bar{x}_m < x_m$. Since T satisfies

(32), applying T iteratively yields $T^k(\bar{x}_m) \leq T^k(x_m)$, whence the result follows straightforwardly. □

Remark 1 If Algorithm *1* converges, then by Theorem 3.5 a) the resulting $\bar{x}_m$ satisfies $T(\bar{x}_m) < \bar{x}_m$ and thus the hypotheses of Theorem 3.4 are satisfied. We emphasize that this holds irrespective of how large or small the chosen scalar c is (provided Algorithm *1* converges). On the other hand, the scalar c may need to be small enough to ensure the convergence of Algorithm *1*. The use of different values of c for which Algorithm *1* converges yields different points $\bar{x}_m$. Higher values of c for which Algorithm *1* converges are more desirable since they provide larger $\bar{x}_m$, hence resulting in a larger region of attraction to the ultimate bound. In some cases, iteration of the map T from different $\bar{x}_m$ provided by Algorithm *1* may converge to different points, corresponding to different ultimate bounds. In addition, if c is small enough, then iteration of the map T from the point $\bar{x}_m$ provided by Algorithm *1* leads to the smallest ultimate bound that can be obtained via application of Theorem 3.4.

Remark 2 Theorem 3.4, Algorithm *1* and Theorem 3.5 provide a *systematic* method to obtain ultimate bounds for continuous-time linear systems with perturbations bounded componentwise by state-dependent functions.

3.3 *Application to Nonlinear Systems*

Consider a nonlinear system

$$\dot{x}(t) = f(x(t), u(t)), \tag{42}$$

where $f(0,0) = 0$ and $A \triangleq \left.\frac{\partial f}{\partial x}\right|_{(0,0)}$ is Hurwitz. Rewrite system (42) as

$$\dot{x}(t) = Ax(t) + [f(x(t), u(t)) - Ax(t)].$$

If we can find a continuous function $\delta : \mathbb{R}^n \to \mathbb{R}^n_{+,0}$ so that

$$|f(x(t), u(t)) - Ax(t)| \leq \delta(x(t)), \quad \text{for all } t \geq 0,$$

and (30) is satisfied, then we can analyze the map given by (31) and expect to be able to use Theorem 3.4 to estimate an ultimate bound and a region of attraction. In Section 5.1 we illustrate this procedure with an example.

4 Ultimate Bounds for Discrete-time Systems

In this section, we develop a systematic method to obtain ultimate bounds for perturbed discrete-time systems which is based on a componentwise analysis of the system in modal coordinates. In this case, the ultimate bound expressions can be obtained in a more straightforward manner, via a procedure that is different from the one developed in the continuous-time case. In particular, the result for constant perturbation bounds is not needed as an intermediate tool to obtain ultimate bounds for state-dependent perturbation bounds. We therefore directly obtain ultimate bounds for this latter case in Section 4.1 where we also develop the aforementioned systematic method. We then show how to apply this result to nonlinear systems in Section 4.2.

4.1 *State-dependent Perturbation Bounds*

As we did for continuous-time systems in Section 3.2, we now provide ultimate bound expressions for linear systems with state-dependent perturbation bounds that satisfy a monotonicity condition [see (45) and (46) below] and then develop a corresponding systematic method for the discrete-time case.

Theorem 4.1 *Consider the system*

$$x(k+1) = Ax(k) + u(k), \tag{43}$$

where $x(k), u(k) \in \mathbb{R}^n$ and $A \in \mathbb{R}^{n\times n}$ has all its eigenvalues strictly inside the unit circle and Jordan canonical form

$$\Lambda = V^{-1}AV. \tag{44}$$

Suppose that

$$|u(k)| \leq \delta(|x(k)|), \quad \textit{for all } k \geq 0, \tag{45}$$

where $\delta : \mathbb{R}^n_{+,0} \to \mathbb{R}^n_{+,0}$ is a continuous map verifying

$$|x_1| \leq |x_2| \Rightarrow \delta(|x_1|) \leq \delta(|x_2|). \tag{46}$$

Consider the map $T : \mathbb{R}^n_{+,0} \to \mathbb{R}^n_{+,0}$ defined by

$$T(y) \triangleq |\Lambda|\, y + |V^{-1}|\, \delta(|V|y). \tag{47}$$

Suppose that a point b satisfying $b = T(b)$ exists. Let $x_m \in \mathbb{R}^n$ denote any point satisfying $\lim_{k\to\infty} T^k(|V^{-1}x_m|) = b$ (note that Vb is one such point). If the initial condition $x(0)$ satisfies $|V^{-1}x(0)| \leq |V^{-1}x_m|$, then for any $\epsilon \in \mathbb{R}^n_+$ there exists $\ell = \ell(\epsilon, x_m) \geq 0$, such that for all $k \geq \ell$

a) $|V^{-1}x(k)| \leq b + \epsilon$.
b) $|x(k)| \leq |V|\, b + |V|\, \epsilon$.

Proof Let $x(k) = Vz(k)$ and substitute into (43) to obtain

$$z(k+1) = \Lambda z(k) + V^{-1}u(k).$$

Taking magnitudes and using (45) yields

$$\begin{aligned} |z(k+1)| &\leq |\Lambda| \cdot |z(k)| + |V^{-1}|\delta(|Vz(k)|) \\ &\leq |\Lambda| \cdot |z(k)| + |V^{-1}|\delta(|V| \cdot |z(k)|), \end{aligned}$$

where in the last line we have used (46). Define the auxiliary system

$$y(k+1) = |\Lambda| y(k) + |V^{-1}|\delta(|V|\, y(k)) = T(y(k)). \tag{48}$$

Note that by (46), (47), (5) and (6), T satisfies

$$|p| \leq |q| \Rightarrow T(|p|) \leq T(|q|), \tag{49}$$

and also $|z(k)| \leq y(k)$ for all $k \geq 0$ whenever the initial condition $y(0)$ satisfies $|z(0)| \leq y(0)$. By assumption, $|z(0)| = |V^{-1}x(0)| \leq |V^{-1}x_m|$ and hence set the initial condition $y(0) = |V^{-1}x_m|$. Then, by assumption, $\lim_{k\to\infty} T^k(y(0)) = b$. Thus, iteration of (48) converges to the point $b = T(b) = \lim_{k\to\infty} y(k)$. Therefore, given any $\epsilon \in \mathbb{R}^n_+$, there exists $\ell = \ell(\epsilon, x_m) \geq 0$ such that $y(k) \leq b + \epsilon$, for all $k \geq \ell$. The proof of a) then follows by recalling that $|V^{-1}x(k)| = |z(k)| \leq y(k)$. To prove b) note that $|x(k)| \leq |V| \cdot |V^{-1}x(k)|$ and use a). □

The hypotheses of Theorem 4.1 are weaker than those of its continuous-time counterpart (Theorem 3.4). In Theorem 3.4, it is required that a point x_m exist so that $T(x_m) < x_m$. As shown in Theorem 3.4 i), this assumption is sufficient for T to have a fixed point. However, in the discrete-time case, we need only assume the latter, that is, that T has a fixed point. To find such a point, we may iterate T starting from the origin, as the following theorem proves.

THEOREM 4.2 *Let $T : \mathbb{R}^n_{+,0} \to \mathbb{R}^n_{+,0}$ be a continuous map satisfying (49) and suppose that there exists b satisfying $b = T(b)$. Then, $\lim_{k\to\infty} T^k(0) = \bar{b}$, $\bar{b} = T(\bar{b})$ and $\bar{b} \leq b$.*

Proof Since $b = T(b)$, then $b \geq 0$. Therefore, using (49), we have $b = T(b) \geq T(0) \geq 0$ and applying T iteratively yields $b \geq T^k(0) \geq T^{k-1}(0)$. Thus, the sequence $T^k(0)$ is nondecreasing and upper bounded by b and hence it converges to some point $\bar{b}$ satisfying $T(\bar{b}) = \bar{b}$ and $\bar{b} \leq b$. □

Remark 1 Theorems 4.1 and 4.2 provide a *systematic* method to obtain ultimate bounds for discrete-time linear systems with perturbations bounded componentwise by a state-dependent function.

4.2 *Application to Nonlinear Systems*

Consider a nonlinear system

$$x(k+1) = f(x(k), u(k)), \tag{50}$$

where $f(0,0) = 0$ and $A \triangleq \left.\frac{\partial f}{\partial x}\right|_{(0,0)}$ has all its eigenvalues inside the unit circle. Rewrite system (50) as

$$x(k+1) = Ax(k) + [f(x(k), u(k)) - Ax(k)] \,.$$

If we can find a continuous function $\delta : \mathbb{R}^n_{+,0} \to \mathbb{R}^n_{+,0}$ so that

$$|f(x(k), u(k)) - Ax(k)| \leq \delta(|x(k)|), \quad \text{for all } k \geq 0,$$

and (46) is satisfied, then we can analyze the map given by (47) and expect to be able to use Theorem 4.1 to estimate an ultimate bound and a region of attraction. In Section 5.2 we illustrate this procedure with an example.

5 Examples

To illustrate the proposed methodology, we next consider continuous- and discrete-time systems with unknown disturbance signals. Application of the method to the study of quantization effects in sampled-data systems can be found in Haimovich et al. (2006)

5.1 *Continuous Time*

The system

$$\dot{x} = \underbrace{\begin{bmatrix} 0 & 1 \\ -1 & -10 \end{bmatrix}}_{A} x + \underbrace{\begin{bmatrix} 0 \\ x_1 - \sin(x_1) + \tau(t) \end{bmatrix}}_{u(t)}, \tag{51}$$

represents the dynamics of a pendulum with friction, where $x = [x_1\, x_2]^T$ and $\tau(t)$ represents a perturbation torque that is bounded by $|\tau(t)| \leq 0.1$. This system has been written as suggested in Section 3.3. The matrix A is Hurwitz with Jordan canonical form $\Lambda = V^{-1}AV$, where

$$V = \begin{bmatrix} 0.9949 & -0.1005 \\ -0.1005 & 0.9949 \end{bmatrix}, \quad \Lambda = \begin{bmatrix} -0.1010 & 0 \\ 0 & -9.8990 \end{bmatrix}.$$

The term $u(t)$ in (51) can be bounded by

$$|u(t)| \leq \delta(x) \triangleq \begin{bmatrix} 0 \\ \frac{|x_1|^3}{6} + 0.1 \end{bmatrix}.$$

Note that δ satisfies (30). The map T, from (31), is $T(x) = |V|S\delta(x)$, where $S \triangleq \left|[\mathbb{R}\mathrm{e}(\Lambda)]^{-1}\right| \cdot |V^{-1}|$.

Choosing $c = 0.5$, Algorithm *1* converges to the point $[0.6484\ 0.5297]^T$, from which iteration of the map T converges to $b = [0.1023\ \ 0.0205]^T$. Hence, Theorem 3.4 concludes that if $|V^{-1}x(0)| \leq S\delta([0.6484\ 0.5297]^T) = [0.1477\ 0.0149]^T$ then given any $\epsilon \in \mathbb{R}^n_+$, a finite time t_f exists so that for all $t \geq t_f$,

$$|V^{-1}x(t)| \leq \begin{bmatrix} 0.1017 \\ 0.0103 \end{bmatrix} + \epsilon, \quad \text{and} \tag{52}$$

$$|x(t)| \leq \begin{bmatrix} 0.1023 \\ 0.0205 \end{bmatrix} + |V|\,\epsilon. \tag{53}$$

These bounds yield the parallelogram and the axis-aligned rectangle shown in Figure 1. We have also checked that using $c = 10^{-6}$ in Algorithm *1* yields a point from which iteration of the map T also converges to $b = [0.1023\ 0.0205]^T$. The use of a higher value of c in Algorithm *1* provides a larger region of attraction to the ultimate bound.

We now compare the ultimate bounds obtained above with the results from Lyapunov analysis. Extension of a systematic Lyapunov analysis, such as the one described in the introduction, to this nonlinear perturbation case is not straightforward. We tried analyzing this system along the lines in Khalil (2002, Examples 9.2 and 9.5), using a quadratic function $U(x) = x^T P x$, with P to be determined. We performed this analysis bounding $u(t)$ by $\|u(t)\| \leq \frac{|x_1|^3}{6} + 0.1$. Note at this point that the perturbation structure is already lost, since the fact that the first component of $u(t)$ is zero is not taken into account. On the other hand, taking this structure into account makes the analysis case-dependent and difficult to systematically generalize. We next proceed similarly to Khalil (2002, Examples 9.2 and 9.5), and bound the term $|x_1|^3$ by $\alpha d\|x\|$, where αd is the maximum value of $|x_1|^2$ on the level surface $U(x) = d$. Pursuing the analysis in this way, we concluded that, for the values of the parameters in this example, such a method did not yield useful information since the different constraints involved could not be satisfied.

Having found this procedure uninformative for this example, we proceed in a nonsystematic way by employing the function $U(x)$ used in the introductory example in Section 1 and analyzing the exact possible values of $\dot{U}(x)$ (for all values of $\tau(t)$) on the level surfaces of $U(x)$. Note that the matrix A in (51) is the same as that in (3) and that the function $U(x)$ used in the introductory example minimizes the Lyapunov-based formula (4) for the 2-norm ultimate bound on the state in the case of a constant perturbation bound. After performing this tedious numerical evaluation, we find that convergence of the system's trajectories to the set enclosed by the level surface $U(x) = 0.0378$ (see Figure 1) is guaranteed. For any x in this set, we have $|x_1| \leq 0.1076$ and $|x_2| \leq 0.1181$. These bounds are more conservative than the ones given in (53).

In an attempt to obtain a tighter bound on x_2, we find, via trial and error, the Lyapunov function $U_1(x) = x_1^2 + 5x_2^2 + x_1x_2$, which ensures convergence to the set enclosed by the level surface $U_1(x) = 0.0205$, also shown in Figure 1. For any x in this set, we have $|x_2| \leq 0.0657$, still larger than the bound given in (53).

This example illustrates that the Lyapunov approach using quadratic functions may be more conservative, and that finding an appropriate Lyapunov function may be a difficult task. In addition, if the systematic approach of minimizing (4) is overly conservative, one is obliged to resort to the tedious and complicated procedure of evaluating the time derivatives of the Lyapunov function along its level surfaces. On the other hand, the approach we propose provides a systematic method to obtain ultimate bounds that can be easily computer coded. Moreover, we have shown in this example that our approach may lead to tighter bounds.

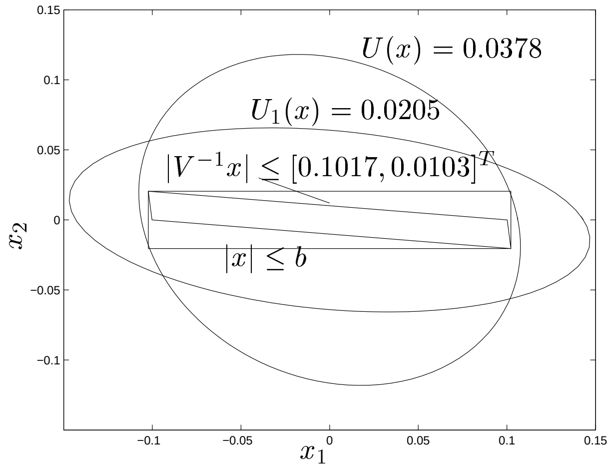

Figure 1. Different ultimate bounds in the pendulum system

5.2 *Discrete Time*

Eq. (54) represents the Euler discretization of a controlled inverted pendulum with a perturbation $w(k)$ that satisfies $|w(k)| \leq 0.01$ and where $x = [x_1\ x_2]^T$.

$$x(k+1) = \begin{bmatrix} 1 & 0.1 \\ -0.9 & 0 \end{bmatrix} x(k) + \begin{bmatrix} 0 \\ 0.1[\sin(x_1(k)) - x_1(k)] - w(k) \end{bmatrix}. \tag{54}$$

The system has been written as suggested in Section 4.2 and has the form $x(k+1) = Ax(k) + u(k)$. The term $u(k)$ can be bounded by $|u(k)| \leq \delta(|x(k)|)$, where the function $\delta : \mathbb{R}^2_{+,0} \to \mathbb{R}^2_{+,0}$ is given by

$$\delta(z) \triangleq \begin{bmatrix} 0 \\ \frac{z_1^3}{60} + 0.01 \end{bmatrix}$$

and can be easily shown to satisfy (46). The matrices V and Λ in the Jordan canonical form of the matrix A, $\Lambda = V^{-1}AV$, are

$$V = \begin{bmatrix} 0.7071 & -0.1104 \\ -0.7071 & 0.9939 \end{bmatrix}, \quad \Lambda = \begin{bmatrix} 0.9 & 0 \\ 0 & 0.1 \end{bmatrix}.$$

The map $T : \mathbb{R}^2_{+,0} \to \mathbb{R}^2_{+,0}$, defined as

$$T(y) \triangleq |\Lambda|\, y + |V^{-1}|\delta(|V|y)$$

has a fixed point at $b = [0.0177\ \ 0.0126]^T$. Also, $x_m = [5\ \ 5]^T$ satisfies $\lim_{k\to\infty} T^k(|V^{-1}x_m|) = b$. Then, Theorem 4.1 states that if $|V^{-1}x(0)| \leq |V^{-1}x_m|$, then for any $\epsilon \in \mathbb{R}^n_+$ there exists $\ell \geq 0$ such that for all $k \geq \ell$, $|V^{-1}x(k)| \leq b + \epsilon$ and $|x(k)| \leq |V|b + |V|\epsilon$. These bounds yield the parallelogram and the axis-aligned rectangle, respectively, shown in Figure 2.

To compare with a Lyapunov approach, the perturbed system was analyzed using the quadratic function $U(x) \triangleq x^T Px$, where P is the solution to $A^T PA - P = -I$. After analyzing the increment $\Delta U(x(k)) \triangleq U(x(k+1)) - U(x(k))$ on the level surfaces of U, ΔU satisfied $\Delta U(x) > 0$ at some point x for which $U(x) = 0.00147$ and then, using the function U we cannot insure an ultimate bound smaller than this level surface. This surface is shown in Figure 2. We stress that the exact value of the nonlinear function ΔU was numerically analyzed, without bounding any term. If the Lyapunov analysis had been performed —as is usually done— by bounding some expressions (like $\sin(x)$ for instance), the resulting ultimate bound would have been significantly more conservative. Note that in the case of higher order systems, the numerical analysis of the exact value of the Lyapunov function increment is computationally intractable and thus the usual approach of bounding terms is the only resort in this case.

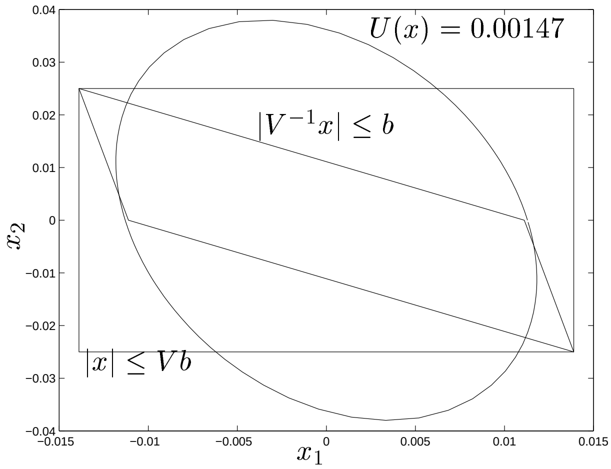

Figure 2. Different ultimate bounds in the discretized inverted pendulum

In this example, the ultimate bound obtained with the suggested method cannot be said to be tighter than the one obtained via Lyapunov analysis. However, one can combine the results obtained by both methodologies and compute an ultimate bound given by the intersection of the parallelogram and the ellipse shown in Figure 2. Moreover, for this example, the Lyapunov analysis shows global convergence to the ultimate bound given by the ellipse. Thus, since our method guarantees convergence to the parallelogram shown in Figure 2 from the set $\{x \in \mathbb{R}^2 : |V^{-1}x| \leq |V^{-1}x_m|\}$ with $x_m = [5 \quad 5]^T$, and this set contains the ellipse shown in Figure 2, then global convergence to the ultimate bound given by the intersection of the parallelogram and the ellipse is ensured. This example then illustrates how the strengths of both methodologies can be combined to obtain tighter bounds and larger regions of attraction.

6 Conclusions

We have presented a systematic method to obtain ultimate bounds for both continuous- and discrete-time systems. The method is based on a componentwise analysis of the system in modal coordinates and thus exploits the system geometry as well as the perturbation structure without requiring calculation of a Lyapunov function. We have developed the method for linear perturbed systems with componentwise state-dependent perturbation bounds and then shown that the method may be applied to nonlinear systems by treating nonlinear terms as perturbations. The resulting ultimate bounds are given as simple expressions in terms of the solution of a fixed point problem which can be solved analytically or numerically. The method also provides an estimation of the region of attraction to the ultimate bound. We have shown by means of examples that the method proposed may offer a simple alternative to the classical Lyapunov-based analysis and may sometimes yield tighter bounds. In addition, the strengths of both methodologies can be combined to obtain even tighter bounds and/or larger regions of attraction.

Given the practical importance of obtaining tight ultimate bounds, future work should aim at deriving conditions under which our proposed methodology is guaranteed to yield tighter bounds than those obtained via quadratic Lyapunov functions. Having such conditions could greatly simplify the task of deriving ultimate bounds by providing a means to decide, in advance, which method should be employed. Future work should also extend the method by deriving conditions for the ultimate bounds to be guaranteed from any initial condition (global ultimate bounds). Work along these lines, jointly with the application of the method to quantized sampled-data control systems can be found in Haimovich et al. (2006).

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