Quantized-State Control.
A Method for Discrete Event Control of Continuous Systems.

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Abstract

This paper introduces a new method for the digital implementation of controllers designed in continuous time. Through the quantization of its state and input variables the original continuous controller is mapped into a discrete event model within the DEVS formalism framework that can be implemented in a digital device. Under certain conditions on the original continuous control system (CCS), this implementation guarantees regional convergence in finite time of system trajectories to arbitrarily small regions around the equilibrium points even in the presence of A/D and D/A quantization effects. The convergence of the new scheme to the CCS is demonstrated when the quantization width goes to zero. Further, a design algorithm for the digital controller is given, which fulfills specifications of admissible final error and convergence speed. Also discussed is the computational efficiency of the scheme, along practical implementation issues. Two numerical examples are provided illustrating some of the benefits of the new method.

Keywords: Digital Control, Discrete Event Systems, Quantized Systems, Nonlinear Systems.

1 Introduction

Practical implementations of most control systems require the use of digital devices. Due to the different nature of the signals present at the inputs and outputs of the digital controller and the continuous-time plant, the interconnection between them must be made through A/D and D/A converters. Since the number of bits used by these converters is finite, a quantization effect takes place over the related variables causing undesired consequences on the stability, steady state error and the general performance of the system.

Because of quantization problems attracting sets must be considered instead of equilibrium points, and *ultimate boundedness* of solutions instead of asymptotic stability. As shown in (Miller et al., 1988) a SISO linear continuous-time plant with a linear digital controller designed so that the closed loop system is uniformly asymptotically stable, a quantization size exists that transforms the asymptotic stability into ultimate boundedness with arbitrary small bounds. Similar studies were performed in (Miller et al., 1989) and (Farrel and Michel, 1989) with conclusions about the error introduced by the quantization in finite time. These results were extended to nonlinear plants (but with linear controllers) in (Hou et al., 1997) and then, in (Hu and Michel, 1999) a similar analysis is applied to the multirate case. In both cases only local results are obtained since Lyapunov's first method is used to analyze stability properties. Nevertheless, the unatteinability of global results is not only due to the quantization, but also to the use of fixed time-discretization. Indeed, a fixed sampling time which is adequate when the state is weakly perturbed away from an equilibrium point could provoke instability for larger state perturbations.

Instead of studying the effects of the quantization after the controller is designed, some works attempt to deal with the quantization at the design stage. In (Delchamps, 1990) the problem of stabilizing a discrete time linear system taking into account the quantization in the state measurement is studied. In (Brockett and Liberzon, 2000) there is also an study over CCS and the problem of the imposibility of convergence to the equilibrium points is solved by allowing the quantizers to change the size of the quantization intervals.

Since recently, quantization of variables is being applied for simulation purposes. In (Zeigler et al., 2000) the authors proposed that continuous time systems can be simulated through the quantization of state and input variables instead of the discretization of time. They also showed that the resulting system can be described by a discrete event model within the DEVS formalism (Zeigler, 1976).

This idea was taken in (Kofman and Junco, 2000), where the authors introduce the concept of Quantized State System (QSS), that are continuous time systems where the state variables are quantized through quantization functions with hysteresis[1]. It has been shown that QSS with piecewise constant input trajectories can be exactly represented by DEVS models. Thus, the addition of the mentioned quantization functions to a continuous model transforms it into a QSS that can be simulated in a digital device.

In this paper, based on the idea of QSS, we develop a new digital control scheme called Quantized State Control (QSC). Here the controller is a discrete event system that can be obtained from a continuous (and possibly nonlinear) previously designed controller. The mentioned discrete event model is the representation of a QSS obtained through the quantization with hysteresis of the state variables of the continuous controller. The A/D converters, working in an asynchronous way –which results in an important reduction of the computational costs of the implementation– perform the necessary quantization of the input without introducing time discretization.

Based on Lyapunov's second method, a stability study is made over the QSC scheme giving sufficient conditions to assure regional convergence of the trajectories to small regions around the equilibrium points. When the CCS satisfies some additional conditions, semiglobal convergence can be also achieved. It is important to mention that these stability properties are deduced for the general case of a nonlinear MIMO plant with a nonlinear controller. The absence of time discretization in the scheme is the main key in the proof of the mentioned properties.

The paper is organized as follows. In Section 2 the concept of Quantized State Systems is introduced and some properties of this class of systems are mentioned. In Section 3 the QSC scheme is formally defined. The study of stability and convergence is made in Section 4, and based on the stability theorem deduced there, an algorithm that allows the design of the quantized state controller achieving stability and steady state error goals is developed and then illustrated with an example.

Finally, in Section 5 the reduction of the computational costs is treated through the analysis over a simple system.

2 Quantized State Systems

Quantized State Systems (QSS) are continuous time systems where each state variable is affected by a quantization function equipped with hysteresis.

Before giving a formal definition of QSS, the concept of *quantization function with hysteresis* will be introduced.

2.1 Quantization Functions

Let $D = \{d_0, d_1, ..., d_r\}$ be a set of real numbers where $d_{i-1} < d_i$ with $1 \leq i \leq r$ and let $x \in \Omega$ be a continuous trajectory, where $x : \mathbb{R} \to \mathbb{R}$. Let $b : \Omega \times t_0 \to \Omega$ be a mapping and let $q = b(x, t_0)$ where the trajectory q for

[1] The presence of hysteresis in the quantization is necessary to assure that the resulting DEVS models are *legitimate* (Kofman and Junco, 2000). A DEVS model is said to be *legitimate* when it cannot produce an infinite number of events in a finite interval of time (Zeigler et al., 2000)

$t \geq t_0$ satisfies

$$q(t) = \begin{cases} d_m & \text{if } t = t_0 \\ d_{i+1} & \text{if } x(t) = d_{i+1} \wedge q(t^-) = d_i \wedge i < r \\ d_{i-1} & \text{if } x(t) = d_i - \varepsilon \wedge q(t^-) = d_i \wedge i > 0 \\ q(t^-) & \text{otherwise} \end{cases} \tag{1}$$

and

$$m = \begin{cases} 0 & \text{if } x(t_0) < d_0 \\ r & \text{if } x(t_0) \geq d_r \\ j & \text{if } d_j \leq x(t_0) < d_{j+1} \end{cases}$$

Then, the map b is a *Quantization Function with Hysteresis*. The width of the hysteresis window is ε. The values d_0 and d_r are the lower and upper saturation values. Figure 1 shows a typical quantization function with uniform quantization intervals.

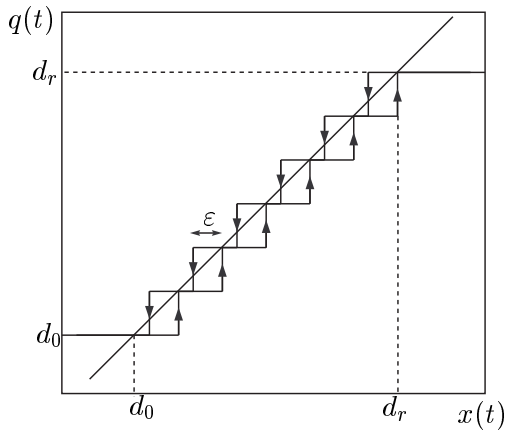

Figure 1: Quantization Function with hysteresis

A fundamental property of a Quantized Function with hysteresis when $t \geq t_0$ is given by the following inequality

$$d_0 \leq x \leq d_r \Rightarrow |q(t) - x(t)| \leq \max_{1 \leq i \leq r} (d_i - d_{i-1}, \varepsilon) \tag{2}$$

2.2 QSS related to a State Equation System

Consider the State Equation System given by:

$$\begin{cases} \dot{x}(t) & = f(x(t), u(t)) \\ y(t) & = g(x(t), u(t)) \end{cases} \tag{3}$$

Related to this system, an associated QSS is defined as follows:

$$\begin{cases} \dot{x}(t) & = f(q(t), u(t)) \\ y(t) & = g(q(t), u(t)) \end{cases} \tag{4}$$

where $q(t)$ and $x(t)$ are related (componentwise) by quantization functions with hysteresis. The components of the vector $q(t)$ are called *quantized variables*. Figure 2 shows a block diagram of a QSS.

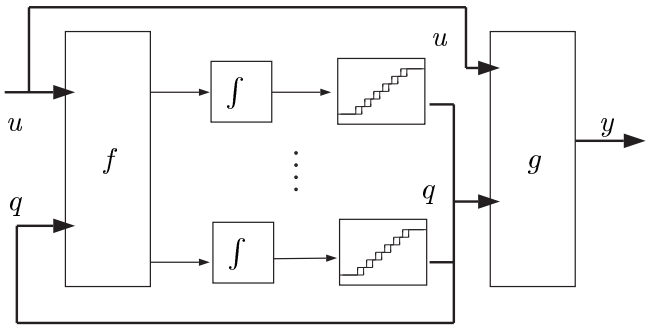

Figure 2: Block Diagram of the QSS

2.3 Properties of QSS

The most significant properties of the QSS are related to the form of the trajectories. Provided that the inputs have piecewise constant trajectories and the function f is continuous and bounded in any bounded domain, the following properties are satisfied:

- The quantized variables have piecewise constant trajectories
- The state variable derivatives have also piecewise constant trajectories
- The state variables have continuous piecewise linear trajectories

As a consequence of these properties the QSS can be exactly simulated by a discrete event model within the DEVS formalism framework. The DEVS model related to a generic QSS, the proof of the mentioned properties and an extended explanation about the necessity of using hysteresis can be found in (Kofman and Junco, 2000).

That DEVS model shows formaly the way of exactly simulating a QSS. Here, instead of presenting that DEVS model, the principles that rule the discrete event simulation of the QSS will be explained.

2.4 Discrete Event Simulation of QSS

The basic idea for the discrete event simulation of the QSS is the representation by events of the changes in the quantized variables and state derivatives (it is possible because these variables have piecewise constant trajectories).

Consider a *quantized integrator*, that is, a simple integrator and a quantizer connected to its output so that the output of the system is the quantized variable $q(t)$ as Figure 3 shows.

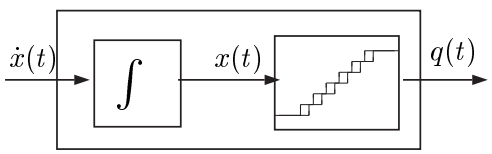

Figure 3: A Quantized Integrator

According to the properties explained above the trajectories of q and $\dot{x}(t)$ will be piecewise constant while the state variable x will have a piecewise linear trajectory. Figure 4 shows typical trajectories in a quantized integrator.

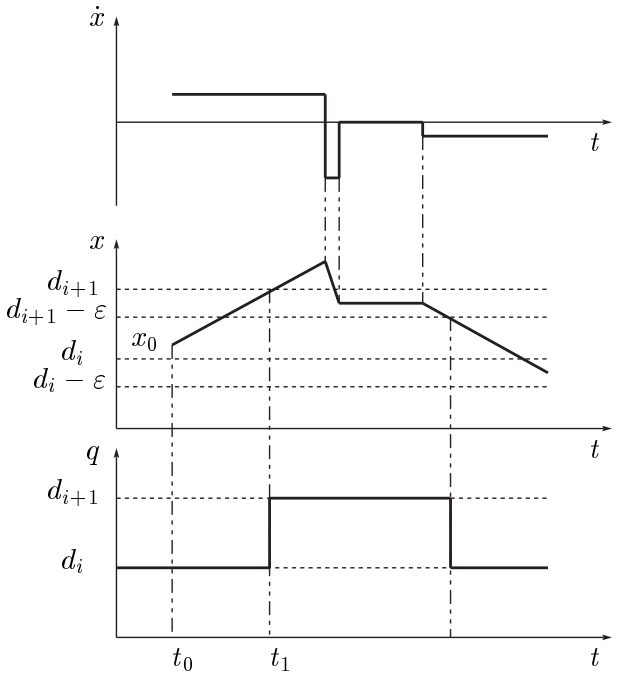

Figure 4: Trajectories in a quantized integrator

Performing a simulation of this behaviour is equivalent to predict when the variable q will change and what value it will adopt after that change. These calculations require the knowledge of the initial condition of the state variable (x_0). Due to the presence of hysteresis it is also necessary to know the initial value of the quantized variable ($q(t_0) = d_i$).

Two different cases must be taken into account in order to predict when the next change of q will occur. The first case is when there is no modification of the input value before the change of q. In this case, considering that the initial time of the simulation is t_0, the time of the next change can be calculated as:

$$t_1 = \begin{cases} t_0 + \dfrac{d_{i+1} - x_0}{\dot{x}} & \text{if } \dot{x} > 0 \\ t_0 + \dfrac{x_0 - (d_i - \varepsilon)}{\dot{x}} & \text{if } \dot{x} < 0 \\ \infty & \text{if } \dot{x} = 0 \end{cases} \tag{5}$$

where d_i and d_{i+1} are the actual and the next quantized values and ε is the width of the hysteresis window. The value of q after the change will be either d_{i+1} or d_{i-1} depending on the sign of $\dot{x}$. The new initial condition for x can be calculated as d_{i+1} or $d_i - \varepsilon$ also according to the sign of $\dot{x}$.

If there is a modification of the input value before the change of q, the situation can be treated as a change in the initial condition x_0 and a change in the initial time t_0. The increment in x_0 can be calculated as:

$$\Delta x_0 = \dot{x}(t_0) \cdot e \tag{6}$$

where e is the elapsed time from t_0 to the time of the change in the value of u.

The alternate use of equations (5) and (6) allows the exact simulation of the quantized integrator. In terms of the DEVS formalism, the rules explained for the first case (changes in the value of q) determine the *internal transition function* while the second case determines the *external transition function*. Equation (5) will be related to the *time advance function*. The state of the DEVS model will save the last values of x_0, d_i and $\dot{x}$.

Since DEVS allows the coupling of different models, a general QSS can be exactly simulated by a coupled DEVS structure. The model of the integrators can be built using the ideas explained above. The rest of the system, composed only by static functions, has a straightforward representation in DEVS.

The possibility of representing a QSS by a DEVS model and the fact that DEVS models can be simulated in real time by a digital device[2] (Zeigler and Kim, 1993) allow the use of QSS as digital controllers.

3 Quantized State Control

Consider the CCS consisting of plant and controller, Eqs. (7) and (8) respectively, and their (ideal) interconnection, Eq. (9).

$$\begin{cases} \dot{x}_p(t) &= f_p(x_p(t), u_p(t)) \\ y_p(t) &= g_p(x_p(t)) \end{cases} \tag{7}$$

$$\begin{cases} \dot{x}_c(t) &= f_c(x_c(t), u_c(t)) \\ y_c(t) &= g_c(x_c(t), u_c(t)) \end{cases} \tag{8}$$

$$u_p(t) = y_c(t), \quad u_c(t) = y_p(t) \tag{9}$$

With this representation, general problems of regulation with linear and nonlinear plants and controllers can be treated.

Definition 1. *The QSS associated to a continuous controller* (8) *is called Quantized State Controller (QSC controller).*

The connections between the plant outputs and the controller inputs require the use of A/D and D/A converters. Since QSC controllers avoid time discretization, it would be desirable that the converters do the same. Thus, the A/D conversions will be performed only when the analog input and the digital output of the converters differ in a quantity corresponding to a quantization interval[3]. Similarly, the D/A conversions will be only performed when the digital outputs of the controller change.

Definition 2. *A QSC system is defined as a control scheme composed by a continuous plant and a QSC controller connected through asynchronous A/D and D/A converters.*

Figure 5 shows a block diagram representation of a QSC system. Here, according to (7), there is a strictly proper plant[4].

The QSC implementation of the controller transforms (8) into the new set of equations:

$$\begin{cases} \dot{x}_c(t) &= f_c(q_c(t), u_c(t)) \\ y_c(t) &= g_c(q_c(t), u_c(t)) \end{cases} \tag{10}$$

where the difference between q_c and x_c is bounded according to (2).

Taking into account the way they work, the asynchronous A/D converters can be seen as quantization functions with hysteresis where the quantization intervals and the hysteresis windows have the same size. In a similar way, the D/A converters can be represented by quantization functions without hysteresis ($\varepsilon = 0$). Thus, the presence of the asynchronous A/D and D/A converters transforms (9) into:

$$u_p(t) = y_{c_q}(t), \quad u_c(t) = y_{p_q}(t) \tag{11}$$

[2]The DEVS representation of QSS is exact. However, the real time simulation of DEVS has errors related to the temporal resolution and the round-off introduced by the digital device

[3]This is an asynchronous sampling technique that is used in some works to obtain fast A/D conversions with a good tradeoff between speed and resolution (Sayiner et al., 1993).

[4]It is also possible to use QSC with plants having relative degree equal to 0. However, in this case, the relative degree of the continuous controller must be at least one to avoid an infinite number of conversions in a finite interval of time

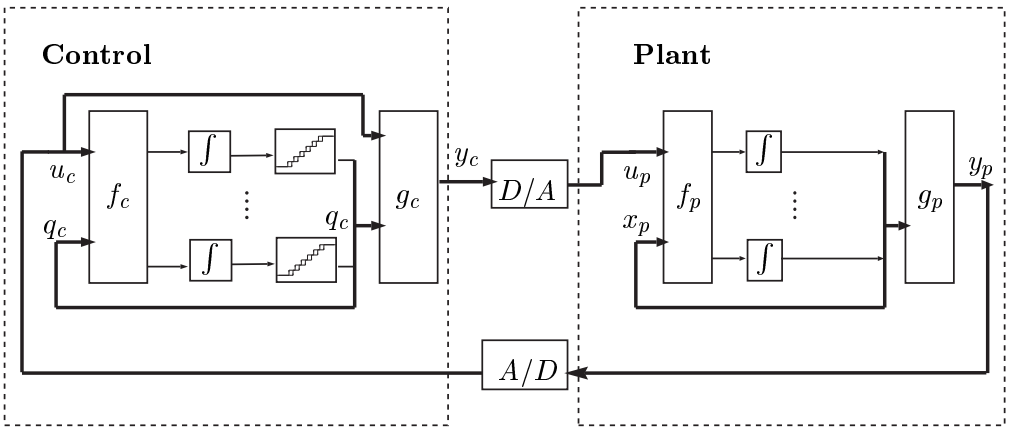

Figure 5: Block Diagram of the QSC system

where the variables $y_{c_q}(t)$ and $y_{p_q}(t)$ are the quantized versions of the plant and the controller output variables, which differ from the continuous y_c and y_p in a quantity bounded by (2).

The presence of quantization with hysteresis in the A/D converters also guarantees that the controller inputs have piecewise constant trajectories. Since the QSC controller is a QSS, it can be exactly represented by a DEVS model and implemented in a digital device.

4 Stability and Convergence of QSC

One of the most important properties of QSC is the conservation of the stability properties of the original CCS even in presence of A/D and D/A quantization effects. The substitution of time-discretization by the quantization of variables is the key element yielding this property.

4.1 A stability theorem for QSC

The following theorem shows that, when the CCS has a regional stable equilibrium point a quantization exists such that the QSC system also assures regional convergence of the trajectories to arbitrary regions around that point.

Theorem 1. *Consider that the origin of the closed loop CCS* (7)-(9) *is a regionally stable equilibrium point. Suppose that the functions* f_p, g_p, f_c *and* g_c *are continuously differentiable. Further assume that a Lyapunov function* V *is known, defined in an open region* D *containing the origin. Then, a QSC system associated to the original CCS can be found, such that all initial conditions lying in an arbitrary interior region of* D *are attracted in finite time to another arbitrary interior region of the former one. Both interior regions must be limited by level surfaces of* V.

Proof. From equations (7) and (9) the following closed loop equations of the continuous system can be obtained:

$$\left\{ \begin{array}{rcl} \dot{x}_p & = & f_p(x_p, g_c(x_c, g_p(x_p))) \\ \dot{x}_c & = & f_c(x_c, g_p(x_p)) \end{array} \right. \tag{12}$$

The implementation of the corresponding QSC system –Eqs. (10) and (9)– transforms (12) into:

$$\left\{ \begin{array}{l} \dot{x}_p = f_p(x_p, g_c(x_c + \Delta x_c, g_p(x_p) + \Delta y_p) + \Delta y_c) \\ \dot{x}_c = f_c(x_c + \Delta x_c, g_p(x_p) + \Delta y_p) \end{array} \right. \tag{13}$$

where $\Delta x_c = q_c - x_c$, $\Delta y_p = u_c - y_p$ and $\Delta y_c = u_p - y_c$. Define:

$$\begin{aligned}\alpha(x, \Delta x_c, \Delta y_p, \Delta y_c) = & \\ & \frac{\partial V}{\partial x_p}(x) \cdot f_p(x_p, g_c(x_c + \Delta x_c, g_p(x_p) + \Delta y_p) + \Delta y_c) + \\ & \frac{\partial V}{\partial x_c}(x) \cdot f_c(x_c + \Delta x_c, g_p(x_p) + \Delta y_p)\end{aligned} \tag{14}$$

with:

$$\begin{aligned}\frac{\partial V}{\partial x_p}(x) &= \left[\frac{\partial V}{\partial x_{p_1}} \cdots \frac{\partial V}{\partial x_{p_n}}\right](x) \\ \frac{\partial V}{\partial x_c}(x) &= \left[\frac{\partial V}{\partial x_{c_1}} \cdots \frac{\partial V}{\partial x_{c_k}}\right](x)\end{aligned}$$

with n and k being the order of the plant and the controller respectively and $V(x) = V(x_p, x_c)$ is the Lyapunov function of the closed loop system defined in (12). From equation (14) it can be easily seen that:

$$\alpha(x, 0, 0, 0) = \dot{V}(x)\Big|_{(12)} \tag{15}$$

Let D_1 be an interior region of D ($D \subset \mathbb{R}^{n+k}$) limited by some level surface of V. Let D_2 be an interior region of D_1 also limited by a level surface of V. Let D_3 be the region defined by $D_3 = D_1 - D_2$.

Since $\dot{V}(x)$ is negative definite, it is possible to find a positive number s so that:

$$\dot{V}(x) < -s, \forall x \in D_3 \tag{16}$$

Let α_M be the function defined by:

$$\alpha_M(\Delta x_c, \Delta y_p, \Delta y_c) = \sup_{x \in D_3}(\alpha(x, \Delta x_c, \Delta y_p, \Delta y_c)) \tag{17}$$

From (15) and (16), it follows that:

$$\alpha_M(0, 0, 0) < -s \tag{18}$$

Since the function α is continuous, the function α_M is also continuous. From this property and (18), given an arbitrary number s_1 ($s > s_1 > 0$), it is possible to find a positive number r so that the condition:

$$\|(\Delta x_c, \Delta y_p, \Delta y_c)\| < r \tag{19}$$

implies that:

$$\alpha_M(\Delta x_c, \Delta y_p, \Delta y_c) < -s_1 \tag{20}$$

Taking into account (2), the condition given in (19) can be satisfied with the choice of an adequate quantization.[5] Observe that saturation must be outside of region D_1.

[5]For instance, considering the same uniform quantization for all the quantized variables the mentioned condition can be achieved by taking:

$$\max(\Delta q, \varepsilon) < \frac{r}{\sqrt{k + m + p}}$$

where Δq and ε are the quantization interval and the hysteresis window size respectively, k is the controller order (i.e. is the size of Δx_c), p is the number of output variables of the plant (size of Δy_p) and m is the number of input variables of the plant (size of Δy_c).

Let $x(t)$ be a solution of equation (13) for the initial condition $x(t=0) = x_0 \in D_3$. Consider that the quantization was done in order to satisfy the condition given by (19). From (13) and (14) it follows that:

$$\begin{aligned} \alpha(x, \Delta x_c, \Delta y_p, \Delta y_c) &= \frac{\partial V}{\partial x_p}(x) \cdot \dot{x_p} + \frac{\partial V}{\partial x_c}(x) \cdot \dot{x_c} = \\ &= \frac{\partial V}{\partial x}(x) \cdot \dot{x} \end{aligned}$$

Using (17) and (20) in the equation above, we have:

$$\frac{\partial V}{\partial x}(x) \cdot \dot{x} < -s_1 \tag{21}$$

This condition will be satisfied at least during certain time while $x(t)$ remains inside D_3 (this is guaranteed by the continuity of $x(t)$). After integrating both sides of the inequality (21), we have:

$$\begin{aligned} \int_0^t \frac{\partial V}{\partial x}(x) \cdot \dot{x} \cdot dt &< \int_0^t -s_1 \cdot dt \\ V(x(t)) - V(x(0)) &< -s_1 \cdot t \\ V(x(t)) &< V(x_0) - s_1 \cdot t \end{aligned}$$

This implies that V evaluated along the solution is bounded by a strictly decreasing function while that solution remains inside D_3. Since the value $V(x_0)$ is smaller than the value that V takes in the bound of D_1, it is clear that the trajectory will never leave D_1.

Let V_1 be the value that V takes in the bound of region D_2. Then, it can be easily seen that the trajectory will reach the region D_2 in a finite time t_1 with:

$$t_1 < \frac{V(x_0) - V_1}{s_1}$$

which completes the proof. □

Corollary 1. *When the Lyapunov function derivative is negative definite in all the state space, the QSC implementation can assure semiglobal ultimately boundedness.*

The proof of this corollary is straightforward. Achieving semiglobal stability requires enlarging the region D_1. Unfortunately, it also implies enlarging the saturation region and then, global stabilization cannot be assured in general cases.

4.2 Convergence of the QSC scheme

It was shown that the QSC implementation can approximate the stability properties of the original controller designed in continuous time. The following theorem shows that the trajectories of the QSC system go to the trajectories of the CCS system when the quantization goes to zero. Thus, any performance measure achieved by the original continuous controller can be approximately achieved by the QSC controller with the choice of sufficiently small quantization intervals.

Theorem 2. *Consider the CCS* (12) *and the associated QSC implementation* (13). *Let D_{x_c}, D_{y_c} and D_{y_p} be the non-saturation regions of the QSC controller, the D/A converters and the A/D converters respectively. D_{x_c} is defined as $D_{x_c} = \{x = (x_1, ..., x_k)/d_{0_i} < x_i < d_{r_i}\}$, while D_{y_c} and D_{y_p} are defined in a similar way. Let D_{x_p} be a bounded region in $\mathbb{R}^n$ and let D be a non-saturation region of the QSC system defined by*

$$\begin{aligned} D = \{(x_p, x_c) | & x_p \in D_{x_p}, x_c \in D_{x_c}, \\ & g_p(x_p) \in D_{y_p}, g_c(x_c, g_p(x_p)) \in D_{y_c}\} \end{aligned}$$

Assume that the functions f_c and g_c are Lipschitz on $D_{x_c} \times D_{y_p}$, the function f_p is Lipschitz on $D_{x_p} \times D_{y_c}$ and the function g_p is Lipschitz on D_{x_p}. Let $\phi(t)$ be the solution of (12) from the initial condition $x(0) = x_0$ and let $\phi_1(t)$ be a solution of (13) starting in the same initial condition x_0. Assume that $\phi(t) \in D_1$ $\forall t$ being D_1 a closed interior region of D. Then, $\phi_1(t) \to \phi(t)$ when the quantization intervals go to 0.

Proof. Let $S = \mathbb{R}^{n+k} - D$ and let

$$D_2 = D_{x_p} \times D_{x_c} \times D_{y_p} \times D_{y_c}$$

let F be

$$F = \sup_{(x_p, x_c, y_p, y_c) \in D_2} \left\| \begin{array}{c} f_p(x_p, y_c) \\ f_c(x_c, y_p) \end{array} \right\| \tag{22}$$

Let d be defined by

$$d = \inf_{x \in S} (\inf_{t \in [0, \infty]} \|\phi(t) - x\|) \tag{23}$$

Since d is positive and F is finite, a positive number t_1 can be taken satisfying

$$t_1 < \frac{d}{F} \tag{24}$$

From (22),(23) and (24) it can be seen that $\phi_1(t)$ does not leave the region D during the interval $[0, t_1]$.

In this time interval, in the system (13) the following conditions are fulfilled

$$\begin{array}{rcl} \|\Delta x_c\| & \leq & \Delta_{x_c} \\ \|\Delta y_p\| & \leq & \Delta_{y_p} \\ \|\Delta y_c\| & \leq & \Delta_{y_c} \end{array} \tag{25}$$

being $\Delta_{x_c}, \Delta_{y_p}$ and Δ_{y_c} constants defined by the quantization intervals in the QSC controller, A/D converters and D/A converters respectively.

It follows from (12), (13) and the fact that $\phi_1(0) = \phi(0)$ that:

$$\phi_1(t) - \phi(t) = \left[\begin{array}{c} \phi_{1_p}(t) - \phi_p(t) \\ \phi_{1_c}(t) - \phi_c(t) \end{array} \right] = \int_0^t \left[\begin{array}{c} \psi_{1_p}(\tau) - \psi_p(\tau) \\ \psi_{1_c}(\tau) - \psi_c(\tau) \end{array} \right] d\tau \tag{26}$$

where

$$\begin{array}{rcll} \psi_{1_p}(\tau) & = & f_p(\phi_{1_p}, g_c(\phi_{1_c} + \Delta x_c, g_p(\phi_{1_p}) + \Delta y_p) + \Delta y_c) & (27) \\ \psi_p(\tau) & = & f_p(\phi_p, g_c(\phi_c, g_p(\phi_p))) & (28) \\ \psi_{1_c}(\tau) & = & f_c(\phi_{1_c} + \Delta x_c, g_p(\phi_{1_p}) + \Delta y_p) & (29) \\ \psi_c(\tau) & = & f_c(\phi_c, g_p(\phi_p)) & \end{array}$$

Then, applying the Euclidean norm to the first row in (26),

$$\begin{array}{rcll} \|\phi_{1_p}(t) - \phi_p(t)\| & = & \| \displaystyle\int_0^t \psi_{1_p}(\tau) - \psi_p(\tau) d\tau \| & \\ & \leq & \displaystyle\int_0^t \|\psi_{1_p}(\tau) - \psi_p(\tau)\| d\tau & (30) \end{array}$$

In a similar way, for the second row, we have

$$\|\phi_{1_c}(t) - \phi_c(t)\| \leq \int_0^t \|\psi_{1_c}(\tau) - \psi_c(\tau)\| d\tau \tag{31}$$

From (30) and (31) it follows that

$$\|\phi_{1_p} - \phi_p\| + \|\phi_{1_c} - \phi_c\| \leq \int_0^t (\|\psi_{1_p}(\tau) - \psi_p(\tau)\| + \|\psi_{1_c}(\tau) - \psi_c(\tau)\|)d\tau \tag{32}$$

Taking into account that $\phi(t)$ and $\phi_1(t)$ are inside region D, and taking Δx_c, Δy_c and Δy_p enough small we can assure that the arguments of the functions f_p, f_c, g_c and g_p in (27) and (29) are inside the regions where these functions satisfy the Lipschitz conditions. Using these conditions and the definitions (27) and (28), we have

$$\begin{aligned}
\|\psi_{1_p}(\tau) - \psi_p(\tau)\| &\leq M_{f_p}\|(\phi_{1_p}, g_c(\phi_{1_c} + \Delta x_c, g_p(\phi_{1_p}) + \Delta y_p) + \Delta y_c) \\
&\quad -(\phi_p, g_c(\phi_c, g_p(\phi_p)))\| \\
&\leq M_{f_p}(\|\phi_{1_p} - \phi_p\| + \|\Delta y_c\| + M_{g_c} \\
&\quad \|(\phi_{1_c} + \Delta x_c, g_p(\phi_{1_p}) + \Delta y_p) - (\phi_c, g_p(\phi_p))\|) \\
&\leq M_{f_p}(\|\phi_{1_p} - \phi_p\| + \|\Delta y_c\| + M_{g_c} \\
&\quad (\|\phi_{1_c} - \phi_c\| + \|\Delta x_c\| + \|\Delta y_p\| + M_{g_p}\|\phi_{1_p} - \phi_p\|)) \\
&\leq M_1(\|\phi_{1_p} - \phi_p\| + \|\phi_{1_c} - \phi_c\| + \Delta)
\end{aligned} \tag{33}$$

where M_{f_p}, M_{g_c} and M_{g_p} are the Lipschitz constants of the corresponding functions. $M_1 = \max(M_{f_p}(1 + M_{g_c}M_{g_p}), M_{f_p}M_{g_c})$, and $\Delta = \Delta_{x_c} + \Delta_{y_c} + \Delta_{y_p}$.

In a similar way, we can obtain that

$$\|\psi_{1_c}(\tau) - \psi_c(\tau)\| \leq M_2(\|\phi_{1_p} - \phi_p\| + \|\phi_{1_c} - \phi_c\| + \Delta) \tag{34}$$

with $M_2 = \max(M_{f_c}, M_{f_c}M_{g_p})$. Let $M = M_1 + M_2$. From (33) and (34) it follows that

$$\begin{aligned}
\|\psi_{1_p}(\tau) - \psi_p(\tau)\| \quad + \quad \|\psi_{1_c}(\tau) - \psi_c(\tau)\| \leq \\
M(\|\phi_{1_p} - \phi_p\| + \|\phi_{1_c} - \phi_c\| + \Delta)
\end{aligned} \tag{35}$$

Defining $\Delta_\phi(t) = \|\phi_{1_p} - \phi_p\| + \|\phi_{1_c} - \phi_c\|$ and using (35) into (32) we have

$$\Delta_\phi(t) \leq \int_0^t (M\Delta_\phi(\tau) + M\Delta)d\tau$$

and finally

$$\Delta_\phi(t) \leq \int_0^t M\Delta_\phi(\tau)d\tau + Mt\Delta$$

It can be easily seen that the function Δ_ϕ is continuous. Being also M positive, it is possible to apply the *Gronwall-Bellman Inequality*, resulting in

$$\begin{aligned}
\Delta_\phi(t) &\leq Mt\Delta + \int_0^t M^2 s\Delta e^{\int_s^t M d\tau} ds \\
\Rightarrow \Delta_\phi(t) &\leq (e^{Mt} - 1)\Delta
\end{aligned}$$

Then, for any $t \in [0, t_1]$ it results that

$$\lim_{\Delta \to 0} \|\phi_{1_p}(t) - \phi_p(t)\| + |\phi_{1_c}(t) - \phi_c(t)\| = 0$$

and then

$$\lim_{\Delta \to 0} \|\phi_1(t) - \phi(t)\| = 0 \tag{36}$$

From (23) we have

$$d \leq \inf_{x \in S}(\|\phi(t_1) - x\|) \tag{37}$$

Taking into account (24), (36) and (37) an enough small quantization interval can be found such that

$$t_1 < \frac{\inf_{x \in S} \|\phi_1(t_1) - x\|}{F}$$

This inequality implies that ϕ_1 does not leave the region D during the interval $[t_1, 2t_1]$. Thus, all the analysis done from (25) is also valid for the interval $[0, 2t_1]$. Repeating this argument the result of (36) holds for all t. □

4.3 A QSC implementation procedure

Based on Theorem 1 the following procedure can be given in order to design QSC controllers satisfying some conditions on convergence.

1. Design a continuous controller that allows finding an appropriate Lyapunov function for the closed loop system.
2. Identify the region D where the derivative of the Lyapunov function is negative
3. Define the region $D_1 \subset D$ from which the trajectories should converge.
4. Choose the region $D_2 \subset D_1$ according to the desired steady state error.(D_1 and D_2 must be limited by level surfaces of the Lapunov function).
5. Calculate the function α using equation (14).
6. Obtain the function α_M with equation (18).
7. Find the maximum s satisfying (16) and choose the lower bound of convergence speed s_1 so that $0 < s_1 < s$.
8. Estimate the maximum value of r so that (19) implies (20).
9. Choose the quantization intervals in order to satisfy (19).
10. Choose the saturation bounds of the quantization functions outside the region D_1

It can be seen that this procedure for the choice of the quantization intervals guarantees the convergence of the system trajectories to the region D_2.

4.4 An example of QSC design

The following example illustrates the use of the procedure developed in the previous section.

Consider the plant:

$$\begin{cases} \dot{x}_p(t) &= x_p^2 + u_p \\ y_p(t) &= x_p(t) \end{cases} \tag{38}$$

We will suppose that the goal is stabilizing the plant around the origin. The first step in the algorithm is the design of a continuous controller. For instance, the following controller can achieve the mentioned goal.

$$\begin{cases} \dot{x}_c(t) &= -x_c - u_c \\ y_c(t) &= x_c(t) - u_c(t) - u_c^2(t) \end{cases} \tag{39}$$

The resulting closed loop equations are:

$$\begin{cases} \dot{x}_p(t) &= -x_p + x_c \\ \dot{x}_c(t) &= -x_p - x_c \end{cases}$$

It can be easily verified that the origin is the equilibrium point, and it is asymptotically and globally stable. By taking the Lyapunov function

$$V(x) = \frac{1}{2}x_p^2 + \frac{1}{2}x_c^2 \tag{40}$$

it follows that:

$$\dot{V}(x) = -x_p^2 - x_c^2$$

Since the stability is global ($D = \mathbb{R}^2$) the definition of the region D_1 will be only neccessary for the choice of the saturation bounds. Suposse also that the goal is assuring the convergence of the trajectories to the region $D_2 = \{x / \|x\| < 1\}$. ($\|\cdot\|$ stands for the euclidean norm).

It follows from (14), (38), (39) and (40) that:

$$\begin{aligned} &\alpha(x, \Delta x_c, \Delta y_p, \Delta y_c) = \\ &\quad -x_p^2 - x_c^2 + x_p(\Delta x_c - \Delta y_p - \Delta y_p^2 - 2x_p\Delta y_p + \\ &\quad +\Delta y_c) + x_c(-\Delta x_c - \Delta y_p) \end{aligned} \tag{41}$$

The calculation of α_M according to the definition in (17) is quite difficult. However, a bound of this function can be easily obtained. It follows from (41) that:

$$\begin{aligned} \alpha(x, \Delta x_c, \Delta y_p, \Delta y_c) \leq &-\|x\|^2 + \|x\|(|\Delta x_c| + |\Delta y_p|) + \\ &+\|x\|(|\Delta x_c| + |\Delta y_p| + |\Delta y_p^2| + 2\|x\|\ |\Delta y_p| + |\Delta y_c|) \end{aligned}$$

Then, it results from (17) and the inequality above that:

$$\begin{aligned} \alpha_M(\Delta x_c, \Delta y_p, \Delta y_c) \leq \sup_{\|x\| \geq 1} &[-\|x\|^2 + \\ &+\|x\|(2|\Delta x_c| + 2(\|x\| + 1)|\Delta y_p| + |\Delta y_p^2| + |\Delta y_c|)] \end{aligned}$$

Since outside the region D_2 the condition $\dot{V}(x) < -1$ is satisfied, the convergence speed s_1 (seventh step) can be chosen to be bounded between 0 and 1. Suppose that the choice is $s_1 = 0.5$. Then, the quantization must be chosen in order to satisfy $\alpha_M < -0.5$, condition than is verifyied using quantization intervals $\Delta q = \varepsilon = 0.07$ for all the variables.

If the restriction about the convergence speed is not taken into account and the goal is just assuring stability, that quantization interval of $\Delta q = 0.07$ is sufficiently small to guarantee convergence to the region given by $\|x\| < 0.4127$.

The simulation was done for an initial condition $x_p = 10$ and the results are shown in figures 6 to 11. The number of conversions performed by the A/D converter were 178 for 40 seconds of simulation time. The minimun time between two successive conversions was 5.6 milliseconds (at the begining of the simulation) while the maximum was greater than 2 seconds.

The trajectory of Fig.6 seems to be quite similar to the one obtained with the continuous controller except for the final oscillations in Figure 8. However, in Fig.11 the difference with the CCS behaviour is more evident. There, the phenomena of ultimately boundedness due to quantization and trajectory crossing due to hysteresis can be observed.

It is interesting to mention that the simulation was performed using the technique of QSS simulation (Kofman and Junco, 2000), that is, adding quantizers to the continuous plant and then generating the equivalent DEVS model for the entire system. Since the controller is itself a QSS, it was not neccessary to introduce any quantization in that part of the system.

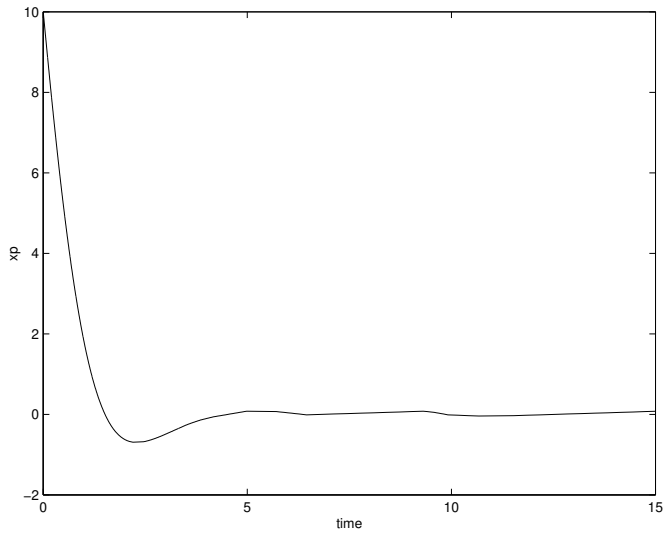

Figure 6: Plot of x_p for the plant with QSC

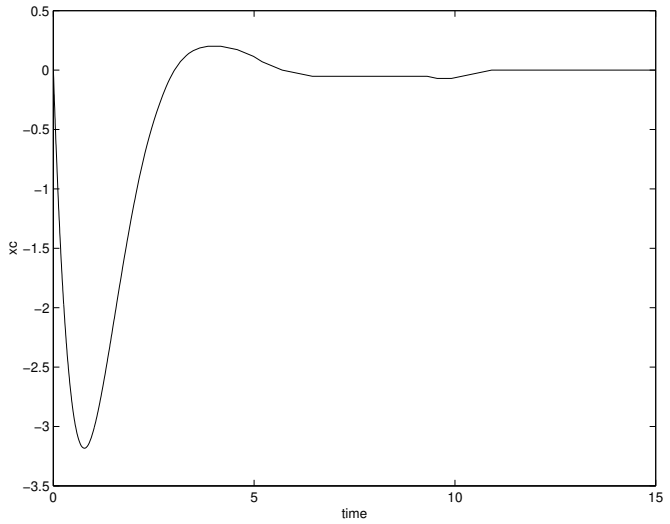

Figure 7: Plot of x_c in the QSC

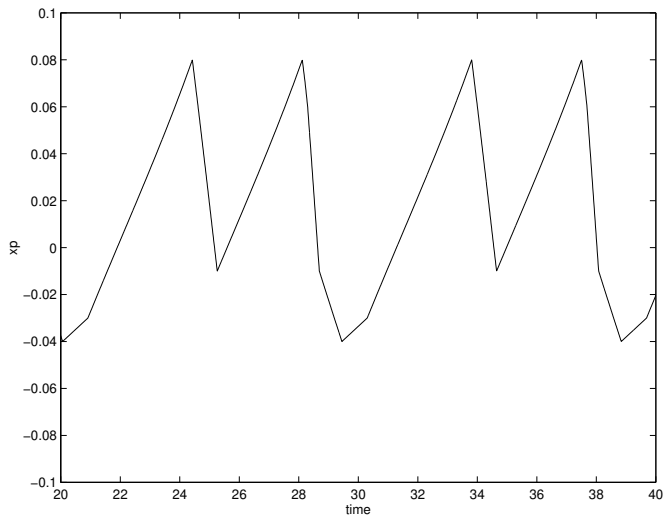

Figure 8: Final oscillations in x_p

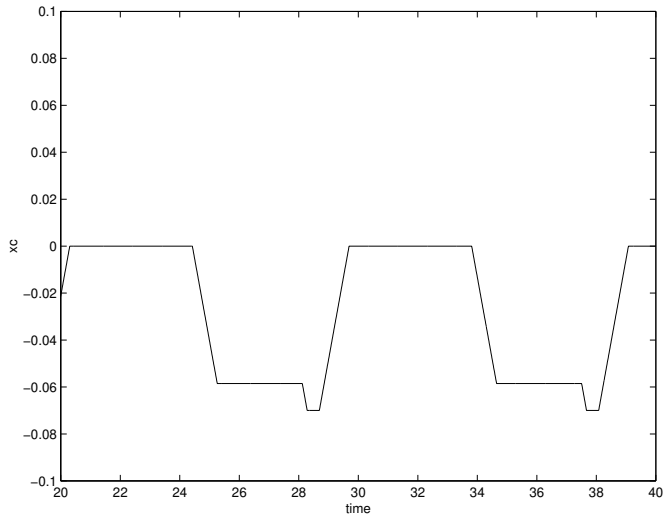

Figure 9: Final oscillations in x_c

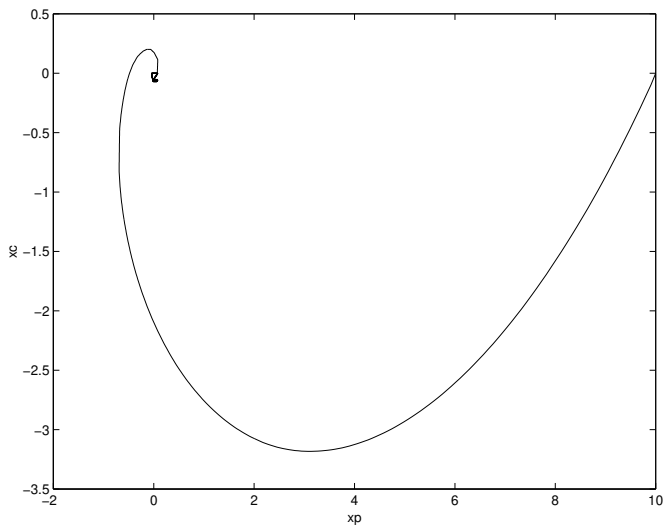

Figure 10: Phase portrait x_p *vs.* x_c

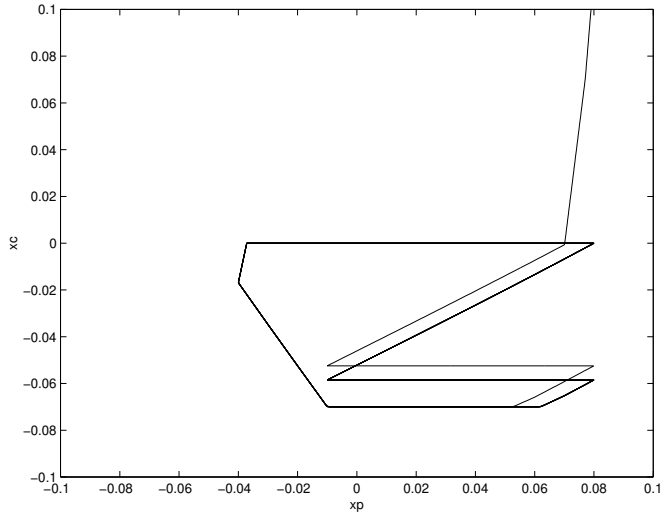

Figure 11: Final oscillations in the phase portrait

5 Further Advantages of QSC

While traditional discrete time controllers perform calculations at regular intervals, QSC controllers only do it when a variable becomes greater (or less) than some threshold. For instance, in the example of Section IV, when the trajectory arrives near the origin the controller performs about one calculation per second. Any discrete time controller using that sampling rate (and even a rate 10 times faster) will diverge for the same initial condition of $x_p = 10$ due to problems of *finite escape time*[6].

The following example illustrates better the reduction of the computational costs in the QSC scheme.

Consider the first order system

$$\dot{x} = \text{sgn}(x) + u \tag{42}$$

Suppose that the controller can measure the variable x, but the A/D converter only produces even numbers (-2, 0, 2, 4, ...) giving the nearest to its analog input. When x is in the shaded region of Figure 12 (between -1 and 1) the control system sees the value 0.

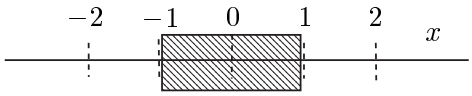

Figure 12: *Invisible* zone due to the quantization

Consider that the goal is keeping the value of x between -2 and 2. The time to go from 1 to 2 (or from -1 to -2) with $u = 0$ is $t_1 = 1$. This implies that it is impossible to find a discrete time controller achieving the proposed goal with a sampling period greater than t_1. The reason of this is that the controller cannot distinguish if the value of x is positive or negative when it is in the shaded region and then, the sign of u could be the same as the sign of x and the trajectory will abandon the desired region before the time t_1.

However, using QSC, the time between samplings can be done arbitrarily big. For instance, consider the following static control law

$$y_c = -u_c(1 + a) \tag{43}$$

where a is a positive constant. When x goes away of the shaded region the controller immediately detects the change and it inverts the sign of the derivative. The new speed on x is a. Then, x enters again the shaded region and the time to reach the origin is $1/a$. After that, x goes to the other bound of the shaded region, but with a new speed of $2 + a$. When x leaves the shaded region again the controller inverts the sign of the derivative and we obtain a cyclic behaviour where x oscillates between 1 and -1. The time between successive A/D conversions is

$$t_2 = \frac{1}{a} + \frac{1}{2 + a} \tag{44}$$

It is clear that taking small values for the parameter a, t_2 can be done arbitrarily big. Then, with this implementation the number of calculations in the controller and the size of the oscillation can be considerably reduced.

Unfortunately, the QSC implementation is not exact. In fact, there are delays related to the presence of converters and the digital processor. In this last example, these delays must be smaller than the minimum sampling period $t_1 = 1$ in order to obtain the proposed goal of keeping x between 2 and -2.

One could think that if it is possible to implement a QSC achieving such minimum delay, then it would be possible to implement a discrete time controller using that sampling period. However, the delay in the QSC system is only the time required to detect the error and to perform the D/A conversion because the calculations can be

[6]The phrase "finite escape time" is used to describe the phenomenon that a trajectory escapes to infinity at a finite time (Khalil, 1996)

done before the error is detected. This is possible because the QSC controller knows that its next input value can only adopt two different values[7].

A classic discrete time controller during each sampling period must perform the A/D conversion, calculate the next state and output of the controller and then perform the D/A conversion. The time required to finish all these tasks is much greater than the delay in the QSC scheme.

6 Remarks and Practical Topics

6.1 Making Use of the State Quantization in the D/A Conversion

The difference between the continuous designed control and the discrete event implementation is due to the effects of the quantization introduced by the state quantization and the presence of A/D and D/A converters. However, in some cases, the effects of quantization in the D/A converters can be eliminated.

The input of the D/A converters is given by

$$y_c = g_c(q_c, u_c) \tag{45}$$

Since q_c and u_c are quantized variables, g_c has a finite domain and then y_c can only take values in a finite set. This set is determined by the quantization values in q_c, u_c and by the function g_c. If that quantization values are chosen so that the mentioned set is a subset of the quantization values of the D/A converters, these converters will not introduce any error.

One of the cases where D/A errors can be avoided is when the output variables of the controller coincide with state variables of that system. In this case the mentioned error can be eliminated by choosing the same quantization in these state variables and in the corresponding D/A converters.

6.2 Practical Implementation of the A/D Converters

As it was mentioned, the A/D converters in the QSC scheme do not work in the classical way. They only perform conversions when the difference between de analogical and digital variables is greater than the quantization interval.

A way of achieving this behavior is placing a counter, a D/A converter and a comparer instead of the A/D converter. The output of the counter should be connected to the D/A and the D/A output must be compared with the output of the system. When the difference is greater than some threshold (the quantization interval), the value in the counter is increased (or decreased according to the sign of the difference). The value saved on this counter corresponds to the digital output of the asynchronous A/D converter.

6.3 Delays Introduced by the Real Devices

One of the most important features of the QSC implementation is related to the fact that it takes into account the effects of quantization in the converters. It seems to be a methodology that considers what really happens in a real implementation. However, it was implicitly assumed that the A/D and D/A converters as well as the digital device work in an ideal way without introducing any delay. It is clear that all real devices need some time to calculate and give their output values. The mentioned delays will produce that the performance of a real implementation of a QSC differs from the predicted one.

For instance, the exact implementation of the controller designed in the example (see Eq.(39)) would require an infinity processing speed since it has a direct path from the input to the output. However, as it is explained in the example of Section 5, the delays can be minimized by taking into account some properties of the scheme.

[7] In the example, if the last detected value of x was 0, then 2 and -2 are the only two values that the new input can adopt. Thus, the controller has two possible output values that could be calculated before the detection of the new input value. For a dynamic controller similar ideas can be applied.

7 Conclusions and future research

QSC constitutes a new class of digital controllers that can be designed taking into account the effects of D/A and A/D converters obtaining regional convergence of the trajectories to small regions around the equilibrium points and reducing the computational costs of the implementation.

A/D conversions are performed in a way that allows the controller having as much information as possible but without taking useless data. Similarly, D/A conversions are performed only when the corresponding variables change. In applications in which the optimization of conversions could be important (for instance in distributed control systems in which each conversion implies some transmission over a channel) the use of QSC might be highly convenient.

In large complex hybrid systems, the supervisory controllers for the higher hierarchical layers can be implemented (and even designed) using DEVS models (Zeigler et al., 1995). The implementation of the different controllers for the continuous parts in the lower layers via QSC allows the programming of a unique DEVS model to control the entire system.

Although the principles of QSC implementation seems to be quite difficult (at least for people who is not familiar with the DEVS formalism) its programming is rather simple. Another advantage is the fact that for QSC design, classic design techniques for continuous controllers can be applied (for instance, in the example of Section IV, the original continuous controller was designed via *exact linearization* (Khalil, 1996)). Then, according to the design algorithm, an standard Lyapunov analysis to determine the regions of attraction and the ultimately bounds is performed. This kind of analysis is very common in nonlinear control systems design.

Future research in the line opened by this paper is pointed to the particularization of the methodology of QSC implementation for linear systems and for some classic control methods. Dealing with particular cases we expect to obtain closed formulas to calculate the quantization intervals. It is also important to obtain results related to closed loop performance measures and it would be interesting arriving to less conservative design conditions.

As it was said, the QSC scheme can guarantee semiglobal stabilization. The presence of saturation does not allow assuring global stabilization for general cases. However, under some conditions, global stabilization could be guaranteed even in presence of saturation. Some results about global stabilization in presence of saturation ((Sussman et al., 1994) for instance) might be useful at least for the cases of linear plants.

Some work must be also done in order to optimize the methods for real time simultion of DEVS taking into account the properties of QSC, since the minimization of the delays in the implementation is extremely important. The remarks of Section V about the fact that the controllers have some information about the next input event and the possibilities of precalculating the next output consitute a starting point in this direction.

Finally we should mention that the QSS represent just a small class of the systems that can be represented within the DEVS framework. In fact, any discrete time system (as standard discrete time controllers) can be also represented by a DEVS model. The possibility of finding new classes of DEVS models having nice properties for the control of continuous system seems to be more than promising.

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