Quantized-State Control.
A Method for Discrete Event Control of Continuous Systems. Part I

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Abstract

This paper introduces a new method for the digital implementation of controllers designed in continuous time. Based on the quantization of the state and input variables of the original continuous controller, the method transforms it into a discrete event specification that can be implemented in a digital device. The method makes use of the presence of the A/D converters placed between the plant and the controller to perform the quantization of the input trajectories. Provided that certain conditions on the original continuous design of the control are fulfilled, this implementation guarantees the regional convergence in finite time of the system trajectories to an arbitrarily small region around the equilibrium points even in the presence of A/D and D/A quantization effects.

1 Introduction

Practical implementations of control systems require the use of digital devices. Due to the different nature of the signals present at the inputs and outputs of the digital controller and the continuous-time plant, the interconnection between them should be made through A/D and D/A converters. Since the number of bits used by these converters is finite, a quantization effect takes place over the related variables causing undesired consequences on the stability, steady state error and the general performance of the system.

One of the consequences of the quantization effects has to do with the fact that in general the equilibrium points are replaced by attracting sets and the concept of asymptotic stability is replaced by *ultimate boundedness* of the solutions. In [1] the authors show that in a SISO linear continuous-time plant with a linear digital controller designed so that the closed loop system is uniformly asymptotically stable, it is possible to find a quantization size that transforms the asymptotic stability into ultimate boundedness with arbitrary small bounds. Similar studies are performed in [2] and [3] with conclusions about the error introduced by the quantization in finite time.

These results are extended to nonlinear plants (but with linear controllers) in [4]. In [5] the analysis is also extended to the multirate case. In both cases, the stability properties are studied based on Lyapunov's first method. As a consequence of this, the works arrive to local results and the properties deduced are only guaranteed in a small region around the equilibrium points.

The mentioned works start from an ideal system, without quantization, and then study the consequences of the presence of quantizers. A different approach can be found in [6] where the problem of stabilization of discrete time linear systems under the effect of quantization in the state measurement is analized without starting from that ideal system.

The problem of the imposibility of convergence to the equilibrium points is treated in [7] where the authors allow the quantizer to change the size of the quantization intervals achieving asymptotic stability. The study is also made over continuous time linear system and there are only some remarks regarding nonlinear systems. Although we do not allow such changes in the quantization size and we deal with output feedback instead of state feedback, that work and our paper have some common aspects since both deal with continuous time systems.

The aim of this paper is to introduce a discrete event based control scheme for continuous systems. One of the main advantages of the method is related to the mentioned quantization effects.

The developed method is based on the DEVS formalism [8] which constitutes the most general tool for representing discrete event systems. Some recent works ([9] and [10]) show the possibilities and some advantages of using DEVS models as approximation of continuous systems for simulation purposes. These approximations are made through the quantization of the state and input variables instead of the time discretization performed in classical numerical methods. In [10] the authors introduce the concept of Quantized State System (QSS), that are continuous time systems where the state variables are quantized through quantization functions with hysteresis. It is shown that when a QSS has piecewise constant input trajectories, it can be exactly represented by a DEVS model. Thus, the addition of the mentioned quantization functions to a continuous model transforms it into a QSS that can be simulated in a digital device.

Based on the idea of QSS, we developed a new digital control scheme, called Quantized State Control (QSC), in which the controller is a discrete event system that can be obtained from a continuous (and possibly nonlinear) previously designed controller. The mentioned discrete event model is the representation of a QSS obtained through the quantization with hysteresis of the state variables of the original continuous controller. The A/D converters, working in an asynchronous way, perform the necessary quantization of the inputs and then, in some sense, the control scheme can make use of the quantization in the converters instead of seeing them just as undesirable effects.

Based on Lyapunov's second method, the stability study made over the QSC scheme gives sufficient conditions to assure regional convergence of the trajectories to small regions around the equilibrium points. When the continuous controller satisfies some conditions, semiglobal convergence can be also achieved. It is important to mention that these stability properties are deduced for the general case of a nonlinear MIMO plant with a nonlinear controller.

The paper is organized as follows. In Section 2 the concept of Quantized State Systems [10] is introduced and some properties of this class of systems are mentioned. In Section 3 the QSC scheme is formally defined. The study of Lyapunov's stability is made in Section 4, and based on the stability theorem deduced there, an algorithm that allows the design of the quantized state controller achieving stability and steady state error goals is developed in Section 5.

Finally, a theorem that shows the convergence of the solutions of the system with the QSC controller to the solutions of the system with the continuous control when the quantization goes to 0 can be found in Section 6.

2 Quantized State Systems

Quantized State Systems (QSS) are continuous time systems where each state variable is affected by a quantization function equipped with hysteresis.

Before giving a formal definition of QSS, the concept of *quantization function with hysteresis* will be introduced.

2.1 Quantization Functions

Let $D = \{d_0, d_1, ..., d_r\}$ be a set of real numbers where $d_{i-1} < d_i$ with $1 \leq i \leq r$ and let $x \in \Omega$ be a continuous trajectory, where $x : \mathbb{R} \to \mathbb{R}$. Let $b : \Omega \to \Omega$ be a mapping and let $q = b(x)$ where the trajectory q satisfies:

$$q(t) = \begin{cases} d_m & \text{if } t = -\infty \\ d_{i+1} & \text{if } x(t) = d_{i+1} \wedge q(t^-) = d_i \wedge i < r \\ d_{i-1} & \text{if } x(t) = d_i - \varepsilon \wedge q(t^-) = d_i \wedge i > 0 \\ q(t^-) & \text{otherwise} \end{cases} \tag{1}$$

and

$$m = \begin{cases} 0 & \text{if } x(-\infty) < d_0 \\ r & \text{if } x(-\infty) \geq d_r \\ j & \text{if } d_j \leq x(-\infty) < d_{j+1} \end{cases}$$

Then, the map b is a *Quantization Function with Hysteresis*. The width of the hysteresis window is ε. The values d_0 and d_r are the lower and upper saturation values. Figure 1 shows a typical quantization function with uniform quantization intervals.

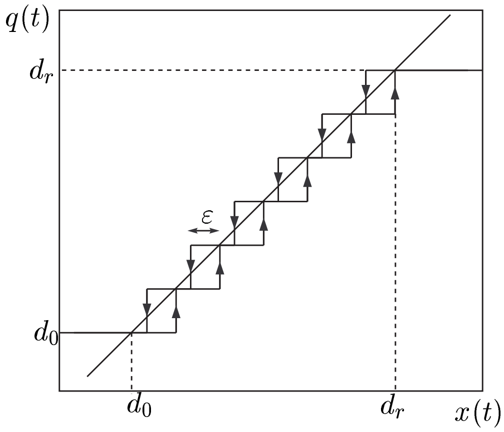

Figure 1: Quantization Function with hysteresis

A fundamental property of a Quantized Function with hysteresis is given by the following inequality.

$$d_0 \leq x(t) \leq d_r \Rightarrow |q(t) - x(t)| = |b(x(t)) - x(t)| \leq \max_{1 \leq i \leq r} (d_i - d_{i-1}, \varepsilon) \tag{2}$$

2.2 QSS related to a State Equation System

Consider the State Equation System given by:

$$\begin{cases} \dot{x}(t) & = f(x(t), u(t)) \\ y(t) & = g(x(t), u(t)) \end{cases} \tag{3}$$

Related to this system, an associated QSS is defined as follows:

$$\begin{cases} \dot{x}(t) & = f(q(t), u(t)) \\ y(t) & = g(q(t), u(t)) \end{cases} \tag{4}$$

where $q(t)$ and $x(t)$ are related (componentwise) by quantization functions with hysteresis. Notice that $x(t)$ and $y(t)$ do not refer to the same trajectories in both system. The components of the vector $q(t)$ are called *quantized variables*. Figure 2 shows a block diagram of a QSS.

2.3 Properties of QSS

The most significant properties of the QSS are related to the form of the trajectories. Provided that the inputs have piecewise constant trajectories and the function f is continuous and bounded in any bounded domain, the following properties are satisfied:

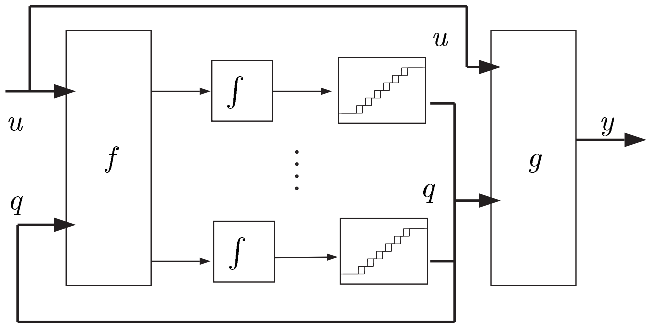

Figure 2: Block Diagram of the QSS

- The quantized variables have piecewise constant trajectories
- The state variable derivatives have also piecewise constant trajectories
- The state variables have continuous piecewise linear trajectories

As a consequence of these properties the QSS can be exactly simulated by a discrete event model within the DEVS formalism framework. Another important property is that the solutions of the QSS related to a continuous system converge to the solutions of the continuous system when the quantization size goes to zero. The DEVS model related to a QSS and the proof of the mentioned properties can be found in [10]. That DEVS model shows formaly the way of exactly simulating a QSS. In [11] a more informal explanation about the simulation of QSS is presented.

The possibility of representing a QSS by a DEVS model and the fact that DEVS models can be simulated in real time [12] by a digital device[1] allow the use of QSS as digital controllers.

3 Quantized State Control (QSC) of Continuous Systems

Consider the following plant:

$$\begin{cases} \dot{x}_p(t) &= f_p(x_p(t), u_p(t)) \\ y_p(t) &= g_p(x_p(t)) \end{cases} \tag{5}$$

Suppose that the plant is controlled by the continuous controller given by:

$$\begin{cases} \dot{x}_c(t) &= f_c(x_c(t), u_c(t)) \\ y_c(t) &= g_c(x_c(t), u_c(t)) \end{cases} \tag{6}$$

where

$$u_p(t) = y_c(t) \tag{7}$$

$$u_c(t) = y_p(t) \tag{8}$$

The corresponding Quantized State Controller (QSC controller) is defined as the Quantized State System associated to the continuous controller. This implementation implies the transformation of (6) in the new set of equations:

$$\begin{cases} \dot{x}_c(t) &= f_c(q_c(t), u_c(t)) \\ y_c(t) &= g_c(q_c(t), u_c(t)) \end{cases} \tag{9}$$

[1] The DEVS representation of QSS is exact. However, the real time simulation of DEVS has errors related to the temporal resolution and the round-off introduced by the digital device

The connections between the outputs of the plant and the inputs of the controller are made through A/D converters. However, these converters will not work in the classical way with a fixed sampling period. The conversions will be performed only in the instants of time in which the analogical input and the digital output of the converters differ in a quantity corresponding to a quantization interval. This is an asynchronous sampling technique that is used in some works to obtain fast A/D conversions with a good tradeoff between speed and resolution [13].

Thus, the digital output and the analogical input of the converters will be connected by a quantization function with hysteresis where the quantization interval and the hysteresis window have the same size. This quantization with hysteresis guarantees that the input variables of the controller will have piecewise constant trajectories. Since the QSC controller defined by (9) is a QSS with piecewise constant input variables, it can be assured that it can be exactly represented by a DEVS model.

In a similar way, the D/A converters will perform conversions only when the digital outputs change. Then, the input and output variables of the D/A converters are related by quantization functions but without hysteresis.

As a consequence of the way they are implemented, the presence of the A/D and D/A converters transforms the equations (7) and (8) into:

$$
\begin{aligned}
u_p(t) &= y_{c_q}(t) \qquad (10)\\
u_c(t) &= y_{p_q}(t) \qquad (11)
\end{aligned}
$$

where the variables $y_{c_q}(t)$ and $y_{p_q}(t)$ are the quantized versions of the plant and the controller output variables.

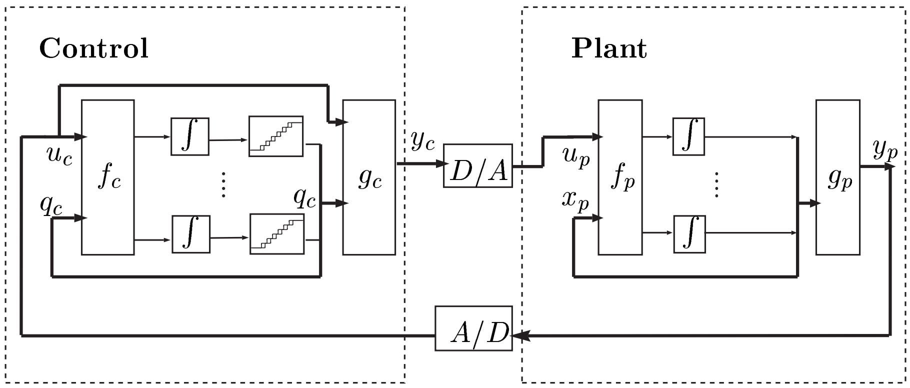

Figure 3: Block Diagram of the plant with the QSC

Figure 3 shows the scheme of a plant with a QSC controller. Here, according to (5), there is a strictly proper plant.

It is also possible to use QSC with plants having relative degree equal to 0. However, in this case, the relative degree of the continuous controller must be at least one. The algebraical loops are forbidden because they may produce an infinite number of A/D and D/A conversions in a finite interval of time. The results obtained in the following sections will be deduced for plants with relative degree greater than zero, but these results can be easily extended to plants with relative degree equal to 0 and strictly proper controllers.

4 Stability of Systems Controlled with QSC

One of the most important properties of QSC is the conservation of the stability properties given by the continuous controller even in presence of A/D and D/A quantization effects. The following theorem shows that when the system with the continuous controller has a regional stable equilibrium point it is possible to find quantization sizes so that the QSC controller also assures regional convergence of the trajectories to an arbitrary region around that equilibrium point.

Theorem 1 *Consider a plant defined as in* (5) *with a continuous control given by* (6) *through the equations* (7) *and* (8) *so that the closed loop system has an equilibrium point in the origin. Suppose that the functions f_p, g_p, f_c and g_c are continuously differentiable. Assume that it is possible to find a positive definite Lyapunov function V for the closed loop system with negative definite derivative in an open region D that contains the origin (i.e. the origin has regional stability in the closed loop system).*

Then, it is possible to find a QSC implementation of the continous control that assures the convergence (in finite time) of the system trajectories starting in an arbitrary interior region of D limited by a level surface of V to an arbitrary interior region of the former region (limited by another level surface).

Proof: From equations (5), (7) and (8) the following closed loop equations of the continuous system can be obtained:

$$\begin{cases} \dot{x}_p &= f_p(x_p, g_c(x_c, g_p(x_p))) \\ \dot{x}_c &= f_c(x_c, g_p(x_p)) \end{cases} \tag{12}$$

The implementation of the control using a quantized state controller (equations (9), (10) and (11)) transforms the equation (12) into:

$$\begin{cases} \dot{x}_p &= f_p(x_p, g_c(x_c + \Delta x_c, g_p(x_p) + \Delta y_p) + \Delta y_c) \\ \dot{x}_c &= f_c(x_c + \Delta x_c, g_p(x_p) + \Delta y_p) \end{cases} \tag{13}$$

where

$$\begin{aligned} \Delta x_c &= q_c - x_c \\ \Delta y_p &= y_{p_q} - y_p = u_c - y_p \\ \Delta y_c &= y_{c_q} - y_c = u_p - y_c \end{aligned}$$

(Notice that $x_c(t)$ and $x_p(t)$ are not the same in (12) and (13)). Define:

$$\begin{aligned} &\alpha(x, \Delta x_c, \Delta y_p, \Delta y_c) = \\ &\quad \frac{\partial V}{\partial x_p}(x) \cdot f_p(x_p, g_c(x_c + \Delta x_c, g_p(x_p) + \Delta y_p) + \Delta y_c) + \\ &\quad \frac{\partial V}{\partial x_c}(x) \cdot f_c(x_c + \Delta x_c, g_p(x_p) + \Delta y_p) \end{aligned} \tag{14}$$

where:

$$\begin{aligned} \frac{\partial V}{\partial x_p}(x) &= \left[\frac{\partial V}{\partial x_{p_1}} \cdots \frac{\partial V}{\partial x_{p_n}} \right](x) \\ \frac{\partial V}{\partial x_c}(x) &= \left[\frac{\partial V}{\partial x_{c_1}} \cdots \frac{\partial V}{\partial x_{c_k}} \right](x) \end{aligned}$$

with n and k being the order of the plant and the controller respectively and $V(x) = V(x_p, x_c)$ is the Lyapunov function of the closed loop system defined in (12). From equation (14) it can be easily seen that:

$$\alpha(x, 0, 0, 0) = \dot{V}(x)\Big|_{(12)} \tag{15}$$

Let D_1 be an interior region of D ($D \subset \mathbb{R}^{n+k}$) limited by some level surface of V. Let D_2 be an interior region of D_1 also limited by a level surface of V. Let D_3 be the region defined by $D_3 = D_1 - D_2$.

Since $\dot{V}(x)$ is negative definite, it is possible to find a positive number s so that:

$$\dot{V}(x) < -s, \forall x \in D_3 \tag{16}$$

Let α_M be the function defined by:

$$\alpha_M(\Delta x_c, \Delta y_p, \Delta y_c) = \sup_{x \in D_3} (\alpha(x, \Delta x_c, \Delta y_p, \Delta y_c)) \tag{17}$$

From (15) and (16), it follows that:

$$\alpha_M(0,0,0) < -s \tag{18}$$

Since the function α is continuous (it was assumed that the functions f_c and f_p are continuously differentiable), the function α_M is also continuous. From this property and (18), given an arbitrary number s_1 ($s > s_1 > 0$), it is possible to find a positive number r so that the condition:

$$\|(\Delta x_c, \Delta y_p, \Delta y_c)\| < r \tag{19}$$

implies that:

$$\alpha_M(\Delta x_c, \Delta y_p, \Delta y_c) < -s_1 \tag{20}$$

The condition given in (19) can be satisfied with the choice of an adequate quantization taking into account (2). Observe that it is also neccessary leaving the saturation region outside the region D_1.

For instance, considering the same uniform quantization for all the quantized variables the mentioned condition can be achieved by taking:

$$\max(\Delta q, \varepsilon) < \frac{r}{\sqrt{k+m+p}}$$

where Δq and ε are the quantization interval and the hysteresis window size respectively, k is the controller order (i.e. is the size of Δx_c), p is the number of output variables of the plant (size of Δy_p) and m is the number of input variables of the plant (size of Δy_c).

Let $x(t)$ be a solution of equation (13) for the initial condition $x(t=0) = x_0 \in D_3$. Consider that the quantization was done in order to satisfy the condition given by (19). From (13) and (14) it follows that:

$$\begin{aligned}
\alpha(x, \Delta x_c, \Delta y_p, \Delta y_c) &= \frac{\partial V}{\partial x_p}(x) \cdot \dot{x}_p + \frac{\partial V}{\partial x_c}(x) \cdot \dot{x}_c = \\
&= \frac{\partial V}{\partial x}(x) \cdot \dot{x}
\end{aligned}$$

Using (17) and (20) in the equation above, we have:

$$\frac{\partial V}{\partial x}(x) \cdot \dot{x} < -s_1 \tag{21}$$

This condition will be satisfied at least during certain time while $x(t)$ remains inside D_3 (this is guaranteed by the continuity of $x(t)$). After integrating both sides of the inequality (21), we have:

$$\begin{aligned}
\int_0^t \frac{\partial V}{\partial x}(x) \cdot \dot{x} \cdot dt &< \int_0^t -s_1 \cdot dt \\
V(x(t)) - V(x(0)) &< -s_1 \cdot t \\
V(x(t)) &< V(x_0) - s_1 \cdot t
\end{aligned}$$

This implies that V evaluated along the solution is bounded by a strictly decreasing function while that solution remains inside D_3. Since the value $V(x_0)$ is smaller than the value that V takes in the bound of D_1, it is clear that the trajectory will never leave D_1.

Let V_1 be the value that V takes in the bound of region D_2. Then, it can be easily seen that the trajectory will reach the region D_2 in a finite time t_1 with:

$$t_1 < \frac{V(x_0) - V_1}{s_1}$$

□

A corollary of this theorem is that when the Lyapunov function derivative is negative definite in all the state space, the QSC implementation can assure semiglobal stabilization (global stability cannot be assured in general cases due to the presence of saturation).

5 A Procedure for QSC Implementation

Based on Theorem 1 the following procedure can be given in order to design QSC controllers satisfying some conditions about convergence.

1. Design a continuous controller that allows finding an appropriate Lyapunov function for the closed loop system.
2. Identify the region D where the derivative of the Lyapunov function is negative
3. Define the region $D_1 \subset D$ from which the trajectories should converge.
4. Choose the region $D_2 \subset D_1$ according to the desired steady state error.(D_1 and D_2 must be limited by level surfaces of the Lapunov function).
5. Calculate the function α using equation (14).
6. Obtain the function α_M with equation (18).
7. Find the maximum s satisfying (16) and choose the lower bound of convergence speed s_1 so that $0 < s_1 < s$.
8. Estimate the maximum value of r so that (19) implies (20).
9. Choose the quantization intervals in order to satisfy (19).
10. Choose the saturation bounds of the quantization functions outside the region D_1

It can be seen that this procedure for the choice of the quantization intervals guarantees the convergence of the system trajectories to the region D_2. An example about the use of this procedure can be found in [11].

6 Convergence of QSC

It was shown that the QSC implementation can approximate the stability properties of the original controller designed in continuous time. The following theorem shows that the trajectories of the system with the QSC controller go to the trajectories of the system with the continuous controller when the quantization go to zero.

Thus, any performance measure achieved by the original continuous controller can be approximately achieved by the QSC controller by the choice of sufficiently small quantization intervals.

Theorem 2 *Consider the system corresponding to a continuous plant with a continuous controller given by* (12) *and the associated QSC implementation* (13). *Let D_{x_c}, D_{y_c} and D_{y_p} be the non-saturation regions of the QSC controller, the D/A converters and the A/D converters respectively. D_{x_c} is defined as $D_{x_c} = \{x = (x_1, ..., x_k)/d_{0_i} < x_i < d_{r_i}\}$, while D_{y_c} and D_{y_p} are defined in a similar way. Let D_{x_p} be a bounded region in $\mathbb{R}^n$ and let D be defined by*

$$D = \{(x_p, x_c) | x_p \in D_{x_p}, x_c \in D_{x_c}, g_p(x_p) \in D_{y_p}, g_c(x_c, g_p(x_p)) \in D_{y_c}\}$$

Assume that the functions f_c and g_c are Lipschitz on $D_{x_c} \times D_{y_p}$, the function f_p is Lipschitz on $D_{x_p} \times D_{y_c}$ and the function g_p is Lipschitz on D_{x_p}. Let $\phi(t)$ be the solution of (12) *from the initial condition $x(0) = x_0$ and let $\phi_1(t)$ be a solution of* (13) *starting in the same initial condition x_0. Assume that $\phi(t) \in D_1$ $\forall t$ being D_1 a closed interior region of D. Then, $\phi_1(t) \to \phi(t)$ when the quantization intervals go to* 0.

Proof: Let $S = \mathbb{R}^{n+k} - D$ and let

$$D_2 = D_{x_p} \times D_{x_c} \times D_{y_p} \times D_{y_c}$$

let F be

$$F = \sup_{(x_p, x_c, y_p, y_c) \in D_2} \left\| \begin{array}{c} f_p(x_p, y_c) \\ f_c(x_c, y_p) \end{array} \right\| \tag{22}$$

Let d be defined by

$$d = \inf_{x \in S} (\inf_{t \in [0, \infty]} \|\phi(t) - x\|) \tag{23}$$

Since d is positive and F is finite, a positive number t_1 can be taken satisfying

$$t_1 < \frac{d}{F} \tag{24}$$

From (22),(23) and (24) it can be seen that $\phi_1(t)$ does not leave the region D during the interval $[0, t_1]$.

In this time interval, in the system (13) the following conditions are fulfilled

$$\begin{aligned} \|\Delta x_c\| &\leq \Delta_{x_c} \\ \|\Delta y_p\| &\leq \Delta_{y_p} \\ \|\Delta y_c\| &\leq \Delta_{y_c} \end{aligned} \tag{25}$$

being $\Delta_{x_c}, \Delta_{y_p}$ and Δ_{y_c} constants defined by the quantization intervals in the QSC controller, A/D converters and D/A converters respectively.

It follows from (12), (13) and the fact that $\phi_1(0) = \phi(0)$ that:

$$\phi_1(t) - \phi(t) = \left[\begin{array}{c} \phi_{1_p}(t) - \phi_p(t) \\ \phi_{1_c}(t) - \phi_c(t) \end{array} \right] = \int_0^t \left[\begin{array}{c} \psi_{1_p}(\tau) - \psi_p(\tau) \\ \psi_{1_c}(\tau) - \psi_c(\tau) \end{array} \right] d\tau \tag{26}$$

where

$$\begin{aligned} \psi_{1_p}(\tau) &= f_p(\phi_{1_p}, g_c(\phi_{1_c} + \Delta x_c, g_p(\phi_{1_p}) + \Delta y_p) + \Delta y_c) & (27) \\ \psi_p(\tau) &= f_p(\phi_p, g_c(\phi_c, g_p(\phi_p))) & (28) \\ \psi_{1_c}(\tau) &= f_c(\phi_{1_c} + \Delta x_c, g_p(\phi_{1_p}) + \Delta y_p) & (29) \\ \psi_c(\tau) &= f_c(\phi_c, g_p(\phi_p)) \end{aligned}$$

Then, applying the Euclidean norm to the first row in (26),

$$\begin{aligned} \|\phi_{1_p}(t) - \phi_p(t)\| &= \| \int_0^t \psi_{1_p}(\tau) - \psi_p(\tau) d\tau \| \\ &\leq \int_0^t \|\psi_{1_p}(\tau) - \psi_p(\tau)\| d\tau \end{aligned} \tag{30}$$

In a similar way, for the second row, we have

$$\|\phi_{1_c}(t) - \phi_c(t)\| \leq \int_0^t \|\psi_{1_c}(\tau) - \psi_c(\tau)\| d\tau \tag{31}$$

From (30) and (31) it follows that

$$\|\phi_{1_p} - \phi_p\| + \|\phi_{1_c} - \phi_c\| \leq \int_0^t (\|\psi_{1_p}(\tau) - \psi_p(\tau)\| + \|\psi_{1_c}(\tau) - \psi_c(\tau)\|) d\tau \tag{32}$$

Taking into account that $\phi(t)$ and $\phi_1(t)$ are inside region D, and taking Δx_c, Δy_c and Δy_p enough small we can assure that the arguments of the functions f_p, f_c, g_c and g_p in (27) and (29) are inside the regions where these functions satisfy the Lipschitz conditions. Using these conditions and the definitions (27) and (28), we have

$$\begin{aligned}
\|\psi_{1_p}(\tau) - \psi_p(\tau)\| &\leq M_{f_p}\|(\phi_{1_p}, g_c(\phi_{1_c} + \Delta x_c, g_p(\phi_{1_p}) + \Delta y_p) + \Delta y_c) \\
&\quad -(\phi_p, g_c(\phi_c, g_p(\phi_p)))\| \\
&\leq M_{f_p}(\|\phi_{1_p} - \phi_p\| + \|\Delta y_c\| + M_{g_c} \\
&\quad \|(\phi_{1_c} + \Delta x_c, g_p(\phi_{1_p}) + \Delta y_p) - (\phi_c, g_p(\phi_p))\|) \\
&\leq M_{f_p}(\|\phi_{1_p} - \phi_p\| + \|\Delta y_c\| + M_{g_c} \\
&\quad (\|\phi_{1_c} - \phi_c\| + \|\Delta x_c\| + \|\Delta y_p\| + M_{g_p}\|\phi_{1_p} - \phi_p\|)) \\
&\leq M_1(\|\phi_{1_p} - \phi_p\| + \|\phi_{1_c} - \phi_c\| + \Delta)
\end{aligned} \tag{33}$$

where M_{f_p}, M_{g_c} and M_{g_p} are the Lipschitz constants of the corresponding functions. $M_1 = \max(M_{f_p}(1 + M_{g_c}M_{g_p}), M_{f_p}M_{g_c})$, and $\Delta = \Delta_{x_c} + \Delta_{y_c} + \Delta_{y_p}$.

In a similar way, we can obtain that

$$\|\psi_{1_c}(\tau) - \psi_c(\tau)\| \leq M_2(\|\phi_{1_p} - \phi_p\| + \|\phi_{1_c} - \phi_c\| + \Delta) \tag{34}$$

with $M_2 = \max(M_{f_c}, M_{f_c}M_{g_p})$. Let $M = M_1 + M_2$. From (33) and (34) it follows that

$$\|\psi_{1_p}(\tau) - \psi_p(\tau)\| + \|\psi_{1_c}(\tau) - \psi_c(\tau)\| \leq M(\|\phi_{1_p} - \phi_p\| + \|\phi_{1_c} - \phi_c\| + \Delta) \tag{35}$$

Defining $\Delta_\phi(t) = \|\phi_{1_p} - \phi_p\| + \|\phi_{1_c} - \phi_c\|$ and using (35) into (32) we have

$$\Delta_\phi(t) \leq \int_0^t (M\Delta_\phi(\tau) + M\Delta) d\tau$$

and finally

$$\Delta_\phi(t) \leq \int_0^t M\Delta_\phi(\tau) d\tau + Mt\Delta$$

It can be easily seen that the function Δ_ϕ is continuous. Being also M positive, it is possible to apply the *Gronwall-Bellman Inequality*, resulting in

$$\begin{aligned}
\Delta_\phi(t) &\leq Mt\Delta + \int_0^t M^2 s\Delta e^{\int_s^t M d\tau} ds \\
\Rightarrow \Delta_\phi(t) &\leq (e^{Mt} - 1)\Delta
\end{aligned}$$

Then, for any $t \in [0, t_1]$ it results that

$$\lim_{\Delta \to 0} \|\phi_{1_p}(t) - \phi_p(t)\| + |\phi_{1_c}(t) - \phi_c(t)\| = 0$$

and then

$$\lim_{\Delta \to 0} \|\phi_1(t) - \phi(t)\| = 0 \tag{36}$$

From (23) we have

$$d \leq \inf_{x \in S} (\|\phi(t_1) - x\|) \tag{37}$$

Taking into account (24), (36) and (37) an enough small quantization interval can be found such that

$$t_1 < \frac{\inf_{x \in S} \|\phi_1(t_1) - x\|}{F}$$

This inequality implies that ϕ_1 does not leave the region D during the interval $[t_1, 2t_1]$. Thus, all the analysis done from (25) is also valid for the interval $[0, 2t_1]$. Repeating this argument the result of (36) holds for all t. □

7 Conclusions and Future Research

QSC constitutes a new class of digital controllers that can be designed taking into account the effects of D/A and A/D converters.

One of the most important advantages of QSC is that following the procedure developed for the implementation, the regional convergence of the trajectories to small regions around the equilibrium points can be assured, even in presence of the quantization effects of A/D and D/A converters.

Another advantage is related to the sampling scheme. The A/D conversions are performed in a way that allows the controller having as much information as possible but without taking useless data. Similarly, the D/A conversions are performed only when the corresponding variables change. It can be easily seen that the system is always in closed loop (even in the intervals between successive conversions). In applications in which the optimization of conversions could be important (for instance in distributed control systems in which each conversion implies some transmission over a channel) the scheme of A/D and D/A conversion of QSC might be convenient. (See the second example in [11])

In large complex hybrid systems, the supervisory controllers for the higher hierarchical layers can be implemented (and even designed) using DEVS models [15]. The implementation of the different controllers for the continuous parts in the lower layers via QSC allows the programming of a unique DEVS model to control the entire system.

The future research in the line opened by this paper is pointed to the particularization of the methodology of QSC implementation for linear systems and for some classical control methods. Dealing with particular cases we expect to obtain closed formulas to calculate the quantization intervals. It is also important to obtain results related to other performance measures.

As it was said, the QSC scheme can guarantee semiglobal stabilization. The presence of saturation does not allow assuring global stabilization for general cases. However, under some conditions, global stabilization could be guaranteed even in presence of saturation. Some results about global stabilization in presence of saturation ([16] for instance) might be useful at least for the cases of linear plants.

Finally we should mention that the QSS represent just a small class of the systems that can be represented within the DEVS framework. In fact, any discrete time system (as classical digital controllers) can be also represented by a DEVS model. The possibility of finding new classes of DEVS models having nice properties for the control of continuous system is more than promising.

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