Quantized-State Systems.
A DEVS Approach for Continuous System Simulation

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Abstract

A new class of dynamical systems, Quantized State Systems or QSS, is introduced in this paper. QSS are continuous time systems where the input trajectories are piecewise constant functions and the state variable trajectories -being themselves piecewise linear functions- are converted into piecewise constant functions via a quantization function equipped with hysteresis. It is shown that QSS can be exactly represented and simulated by a discrete event model, within the framework of the DEVS-approach. Further, it is shown that QSS can be used to approximate continuous systems, thus allowing their discrete-event simulation in opposition to the classical discrete-time simulation. It is also shown that in an approximating QSS some stability properties of the original system are conserved and the solutions of the QSS go to the solutions of the original system when the quantization goes to zero.

Keywords: Dynamical Systems, Quantized State Systems, Discrete Event Systems, DEVS, Continuous System Modeling and Simulation

1 Introduction

Digital simulation of continuous systems requires discretization. Classical methods as Euler, Runge-Kutta, Adams, etc., and their variable step versions are based on the discretization of time [4]. This approximation procedure results in a discrete time simulation model.

Some years ago, methods allowing continuous system simulation under a discrete event paradigm were developed [1] [2] [6] [7]. Discrete event simulation has some advantages over discrete time simulation since for a given accuracy the number of calculations can be reduced, especially in stiff systems. Moreover, it is possible to implement distributed simulation, with the benefit of reduction of the total computing time of the simulation [6]. Also important to mention is the fact that a discrete event model can simulate a discrete time model. Thus discrete time methods can be seen as particular cases of discrete event methods [7].

Another important advantage of these methods arises in the simulation of modern complex technical systems, modeled as hybrid systems where the continuous time, discrete time, finite state automata, and discrete event paradigms merge together: a discrete event approach allows a unified simulation framework for all parts of the hybrid system [3].

Quantization of the state variables as a method to obtain a discrete event approximation of a continuous system is an idea developed in [6] and [7]. There, the quantization is done using a piecewise constant function. However, this method could yield a system that makes an infinite number of transitions in a finite time interval, its simulation becoming thus impossible to perform. Giambiasi gives in [1] a kind of solution-approximation method, which is based on the representation of the continuous trajectories by piecewise polynomial trajectories. However, this approach cannot be seen as a simple conversion of the original equations defining the continuous model, therefore the study of properties like stability or convergence becomes almost impossible to perform.

Through the introduction of a new class of dynamical systems, Quantized State Systems (QSS), both types of disadvantages pointed-out above are removed. QSS are continuous time systems where the the trajectory of each state variable is converted into a piecewise constant function via a quantization function equipped with hysteresis. Provided that the input trajectories are piecewise constant functions the quantization results in piecewise linear functions as state trajectories. The inclusion of hysteresis in the quantization functions removes the problem of the possible infinite number of transitions, thus allowing the discrete-event simulation of the continuous system.

It is shown that QSS can be exactly represented and simulated by a discrete event model, within the framework of the DEVS-approach. Further, a DEVS coupled model for simulating a generic QSS is proposed.

It is also shown that QSS can be thought as a model-approximation method for continuous systems. This property allows showing that in an approximating QSS some stability properties of the original system are conserved, provided that certain conditions are fulfilled. Making use of these stability properties, relationships between quantization size and desired accuracy are deduced. These relationships can be translated into practical rules to guide the choice of the quantization intervals.

Further, it is shown that the solutions of a QSS converge to the trajectories of the original system when the quantization intervals go to zero. This property shows that the method can achieve an arbitrary small error in the simulation of continuous systems.

Simulation results of a simple model are reported, showing the performance of the DEVS-approach in comparison to several well-established discrete-time schemes.

2 Quantized State Systems

Before presenting the method developed, the definition of *quantization function with hysteresis* will be introduced.

2.1 Quantization Functions

Let $D = \{d_0, d_1, ..., d_r\}$ be a set of real numbers where $d_{i-1} < d_i$ with $1 \leq i \leq r$ and let $x \in \Omega$ be a continuous trajectory, where $x : \mathbb{R} \to \mathbb{R}$. Let $b : \Omega \to \Omega$ be a mapping and let $q = b(x)$ where the trajectory q satisfies:

$$q(t) = \begin{cases} d_m & if\ t = -\infty \\ d_{i+1} & if\ x(t) = d_{i+1} \wedge q(t^-) = d_i \wedge i < r \\ d_{i-1} & if\ x(t) = d_i - \varepsilon \wedge q(t^-) = d_i \wedge i > 0 \\ q(t^-) & otherwise \end{cases} \tag{1}$$

and

$$m = \begin{cases} 0 & if\ x(-\infty) < d_0 \\ r & if\ x(-\infty) \geq d_r \\ j & if\ d_j \leq x(-\infty) < d_{j+1} \end{cases}$$

Then, the map b is a *Quantization Function with Hysteresis.* The width of the hysteresis window is ε. The values d_0 and d_r are the lower and upper saturation values. Figure 1 shows a typical quantization function with uniform quantization intervals.

A fundamental property of a Quantized Function with hysteresis is given by the following inequality.

$$d_0 \leq x(t) \leq d_r \Rightarrow |q(t) - x(t)| = |b(x(t)) - x(t)| \leq \max_{1 \leq i \leq r} (d_i - d_{i-1}, \varepsilon) \tag{2}$$

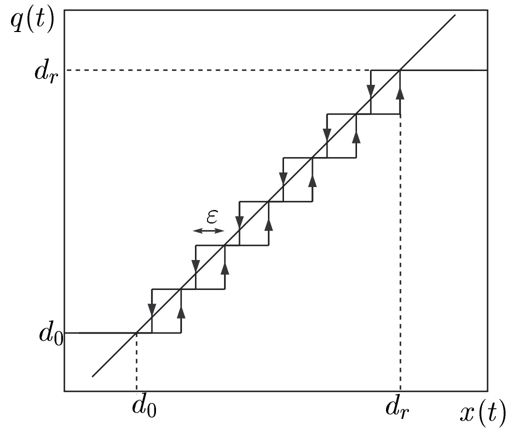

Figure 1: Quantization Function with hysteresis

2.2 QSS related to a State Equation System

Consider the following system of state and output equations:

$$\begin{cases} \dot{x}(t) &= f(x(t), u(t)) \\ y(t) &= g(x(t), u(t)) \end{cases} \tag{3}$$

Then, the following system is defined as a *Quantized State System* associated to the system given by (3):

$$\begin{cases} \dot{x}(t) &= f(q(t), u(t)) \\ y(t) &= g(q(t), u(t)) \end{cases} \tag{4}$$

where $q(t)$ and $x(t)$ are related (componentwise) by quantization functions with hysteresis. The components of the vector $q(t)$ are called *quantized variables*. Figure 2 shows a block diagram of a QSS.

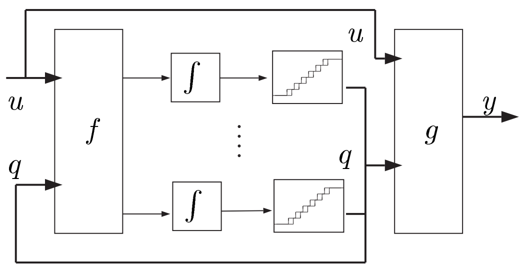

Figure 2: Block Diagram of the QSS

3 Some Properties of QSS

The most significant properties of the QSS are related with the trajectory forms. Provided that the input trajectories are piecewise constant and bounded and the function f is continuous and bounded in any bounded domain, the following properties are satisfied:

- The quantized variables have piecewise constant trajectories
- The state variable derivatives have also piecewise constant trajectories
- The state variables have continuous piecewise linear trajectories

The following theorems give the necessary conditions and prove the mentioned properties.

Theorem 1 *Given the QSS defined in* (4) *with f continuous and bounded in any bounded domain and $u(t)$ being bounded and piecewise constant, the trajectories of $q(t)$ are piecewise constant.*

Proof: Let q_j be a component of q. It follows from (1) that:

$$d_{0_j} \leq q_j(t) \leq d_{r_j} \qquad \forall j \tag{5}$$

where d_{0_j} and d_{r_j} are the lower and upper saturation values of the quantization function that relates x_j and q_j. Inequality (5) implies that $\|q(t)\|$ is bounded. From the hypothesis made about f there exists a positive number M such that

$$-M \leq \dot{x_j} \leq M \tag{6}$$

After integrating the inequality above we have

$$x_j(0) - M(t - t_0) \leq x_j(t) \leq x_j(t_0) + M(t - t_0) \tag{7}$$

Inequality (7) shows that the state variables have bounded trajectories in any finite time interval. Moreover, from (6) it follows that the state variables have also continuous trajectories.

Assume that $x_j(t) = q_j(t) = d_{i_j}, \quad (0 < i < r)$. It follows from (6) and (1) that:

$$q_j(t + \Delta t) \neq q_j(t) \quad \Rightarrow \quad \Delta t \geq \frac{\max(d_{i+1_j} - d_{i_j}, \varepsilon)}{M} = \Delta t_{min} \tag{8}$$

If we assume that $x_j(t) = d_{i+1_j} - \varepsilon = q_j(t) - \varepsilon$ the result is the same. This implies that the variable q_j needs a time interval greater than Δt_{min} to change its value twice. It means that q_j has a piecewise constant trajectory. □

Theorem 2 *In a QSS verifying the hypothesis of Theorem 1 the trajectories of the state variable derivatives are piecewise constant.*

Proof: It is straightforward from Theorem 1 since $q(t)$ and $u(t)$ are piecewise constant and f is static. □

Theorem 3 *In a QSS verifying the hypothesis of Theorem 1 the trajectories of the state variables are continuous and piecewise linear.*

Proof: It is straightforward from Theorem 2. □

Continuous systems with piecewise constant input and output trajectories can be simulated by a DEVS model [7]. However, this simulation requires the knowledge of the continuous system solution. Simulating knowing the solution is useless, but it is possible to divide the system into small subsystems, each of them composed by a single integrator and its corresponding quantizer. If theorems 1 and 2 are satisfied, each subsystem will have piecewise constant input and output trajectories and the continuous solution of the subsystems is straightforward, then the system can be simulated by a coupled DEVS structure.

Remark: Equation (8) shows the need to use hysteresis. If hysteresis were not used, (i.e. were zero) a quantized variable could change its value an infinite number of times and the resulting discrete event model would produce infinite events in a finite time interval[1], which is impossible to be simulated (see Appendix A.).

4 DEVS Model Associated to a QSS

As it was mentioned in the previous section, a DEVS model can simulate continuous system with piecewise linear input and output trajectories. In order to do this, piecewise constant trajectories are represented by event trajectories so that each change in the value of the first trajectory is associated to an event in the second one.

The DEVS model [5] [7] will be defined as a coupling of quantized integrators (integrators with a quantizer at the output) and two systems that calculate the evolution function (f) and the output function (g) (see Figure 3).

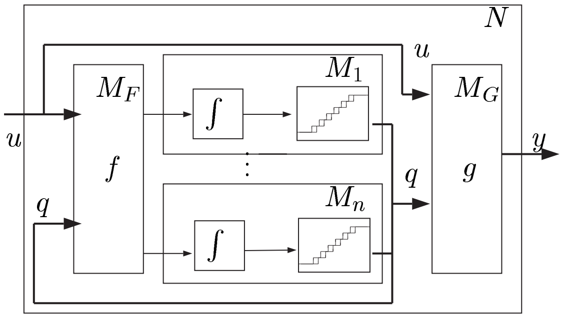

Figure 3: Partition into sub models of the QSS

The DEVS structure associated to a single quantized integrator with piecewise constant input is the following:

$$
\begin{aligned}
&M_j = <X, S, Y, \delta_{\text{int}}, \delta_{\text{ext}}, \lambda, ta>, \text{ where:}\\
&X = Y = \mathbb{R}\\
&S = \mathbb{R} \times \mathbb{R} \times \mathbb{Z} \times \mathbb{R}_0^+ \infty\\
&\delta_{\text{int}}(x, u, i, \sigma) = (x + \sigma u, u, i + sgn(u), \sigma')\\
&\delta_{\text{ext}}(x, u, i, \sigma, e, v) = (x + eu, v, i, \sigma'')\\
&\lambda(x, u, i, \sigma) = d_{i+sgn(u)}\\
&ta(x, u, i, \sigma) = \sigma
\end{aligned}
$$

with

$$
\sigma' = \begin{cases} \dfrac{d_{i+2} - (x + \sigma u)}{u} & if\ u > 0 \\ \dfrac{(x + \sigma u) - (d_{i-1} - \varepsilon)}{|u|} & if\ u < 0 \end{cases}
$$

$$
\sigma'' = \begin{cases} \dfrac{d_{i+1} - (x + eu)}{v} & if\ v > 0 \\ \dfrac{(x + eu) - (d_i - \varepsilon)}{|v|} & if\ v < 0 \\ \infty & if\ v = 0 \end{cases}
$$

[1]In that case we have an illegitimate DEVS model [7].

It can be easily verified that the structure defined simulates the behavior of a quantized integrator with piecewise constant input trajectory.

The DEVS structure for the coupled model is the following:

$$
\begin{aligned}
&N =< X, Y, D, \{M_d\}, \{I_d\}, \{Z_{i,d}\}, Select > \text{ where:}\\
&X = \{\mathbb{R} \cup \phi\}^m\\
&Y = \{\mathbb{R} \cup \phi\}^p\\
&D = \{M_F, M_G, M_1, \cdots, M_n\}
\end{aligned}
$$

D is the set of component references, where the sub models M_j are related to the n quantized integrators of the system, sub model M_F calculates the static function f and sub model M_G calculates the function g.

$$
\begin{aligned}
&M_F =< X, S, Y, \delta_{\text{int}}, \delta_{\text{ext}}, \lambda, ta >, \text{ where:}\\
&X = \{\mathbb{R} \cup \phi\}^m,\ Y = \{\mathbb{R} \cup \phi\}^n\\
&S = \mathbb{R}^{2n+m} \times \{0, \infty\}\\
&\delta_{\text{int}}(qc_1, \cdots, qc_n, uc_1, \cdots, uc_m, fc_1, \cdots, fc_n, \sigma) =\\
&\qquad (qc_1, \cdots, qc_n, uc_1, \cdots, uc_m, fc'_1, \cdots, fc'_n, \infty)\\
&\delta_{\text{ext}}(qc_1, \cdots, qc_n, uc_1, \cdots, uc_m, fc_1, \cdots, fc_n, \sigma, e, u_1, \cdots, u_m, q_1, \cdots, q_n) =\\
&\qquad (qc'_1, \cdots, qc'_n, uc'_1, \cdots, uc'_m, fc_1, \cdots, fc_n, 0)\\
&\lambda(qc_1, \cdots, qc_n, uc_1, \cdots, uc_m, fc_1, \cdots, fc_n, \sigma) = (y_1, ..., y_p)\\
&ta((qc_1, \cdots, qc_n, uc_1, \cdots, uc_m, fc_1, \cdots, fc_n, \sigma)) = \sigma
\end{aligned}
$$

where $fc'_j = f_j(qc_1, \cdots, qc_n, uc_1, \cdots, uc_m)$,

$$
qc'_i = \begin{cases} q_i & if\ q_i \neq \phi \\ qc_i & otherwise \end{cases},\quad uc'_i = \begin{cases} u_i & if\ u_i \neq \phi \\ uc_i & otherwise \end{cases}
$$

$$
\text{and } y_i = \begin{cases} fc'_i & if\ fc'_i \neq fc_i \\ \phi & otherwise \end{cases}
$$

M_G will be very similar to M_F but it will have the g_k functions instead of the f_j functions.

The influencer sets are:

$$
\begin{aligned}
&I_F = I_G = N \cup \{M_1, \cdots, M_n\}\\
&I_{M_j} = F\\
&I_N = G
\end{aligned}
$$

The output translation functions are:

$$
Z_{M_j, M_F}(q) = Z_{M_j, M_G}(q) = (u_1, \cdots, u_m, q_1, \cdots, q_n)
$$

where

$$
u_k = \phi,\ q_i = \begin{cases} q & \text{if} \quad i = j \\ \phi & \qquad \text{otherwise} \end{cases}
$$

$$
Z_{N, M_F}(w_1, \cdots, w_m) = Z_{N, M_G}(w_1, \cdots, w_m) = (w_1, \cdots, w_m, q_1, \cdots, q_n)
$$

with $q_i = \phi$, $i = 1 \dots n$

$$Z_{M_F,M_j}(dx_1, \cdots, dx_n) = dx_j$$
$$Z_{M_G,N}(y_1, \cdots, y_p) = (y_1, \cdots, y_p)$$

Finally, the *Select* function can be any anything because there is no need to take priorities between components.

This DEVS structure is not the only one that is capable to simulate the quantized system defined by (4). An alternative is to define a single atomic model that represents all the system, which will improve the velocity of simulation and the memory used. There are also many options to construct coupled models.

5 Simulation of Continuous Systems by QSS

The addition of quantization functions with hysteresis at the output of all the integrators of a continuous system transforms it into a QSS that can be simulated. However, before claiming that this approximation really constitutes a simulation method it is necessary to prove that the resulting QSS and the original system have *similar* trajectories. Then, it is important to guarantee that the resulting simulation model conserves some properties of the original system like equilibrium points and stability. It is also important to show that the solutions of the simulation model converge to the solutions of the original model when the discretization goes to zero. This property implies that the method can be implemented achieving an arbitrary small error.

The following theorems give sufficient conditions to assure that such properties are verified in the method developed.

5.1 Stability and equilibrium points of QSS

Theorem 4 *Consider a continuous system without inputs* (9) *and its associated Quantized State System* (10)*:*

$$\begin{cases} \dot{x}(t) &= f(x(t)) \\ y(t) &= g(x(t)) \end{cases} \tag{9}$$

$$\begin{cases} \dot{x}(t) &= f(q(t)) \\ y(t) &= g(q(t)) \end{cases} \tag{10}$$

The point $x = \bar{x}$ is an equilibrium point of (9) *if and only if the point $q = \bar{x}$ is an equilibrium point of* (10).

The proof of this theorem is straightforward and it can be extended to system with constant inputs.

Theorem 4 implies that the quantized versions of the state variables in the quantized system have the same possible values in the equilibrium as the state variables of the original system. However, it does not imply that state variables in the quantized system will have such values or that the quantized variables can reach them.

Theorem 5 (Stability of QSS). *Consider a system as the one defined in* (9) *that has an equilibrium point in the origin with the function f being continuously differentiable. Assume that it is also possible to find a Lyapunov function $V(x)$, which is continuous in an open region D including the origin, and has a negative definite time derivative. Let $D_1 \subset D$ be a region limited by a level surface of function V. Then, given an arbitrary open region $D_2 \subset D_1$ also limited by a level surface of V it is always possible to find a quantization so that any trajectory of the resulting associated quantized state system starting into D_1 converges to the interior of D_2.*

Proof: Defining $\Delta x(t) = q(t) - x(t)$ equation (10) can be rewritten

$$\begin{cases} \dot{x}(t) & = f(x(t) + \Delta x(t)) \\ y(t) & = g(x(t) + \Delta x(t)) \end{cases} \tag{11}$$

Let D_3 be the region defined by $D_3 = D_1 - D_2$. Since $\dot{V}(x)$ is negative definite, it exists a positive number s such that:

$$\dot{V}(x) < -s, \forall x \in D_3 \tag{12}$$

Define:

$$\alpha(x, \Delta x) = \nabla V(x)^T \cdot f(x + \Delta x) \tag{13}$$

This is a continuous function in Δx since it is the scalar product of a constant vector and a continuous function. It is also verified:

$$\alpha(x, 0) = \dot{V}(x) \tag{14}$$

Then, it is defined the following function

$$\alpha_M(\Delta x) = \sup_{x \in D_3} (\alpha(x, \Delta x)) \tag{15}$$

It can be easily seen that this function is continuous and verifies

$$\alpha_M(0) < -s \tag{16}$$

Thus, a positive number r can be found satisfying

$$\alpha_M(\Delta x) < -s_1 < 0 \tag{17}$$

if

$$\|\Delta x\| < r \tag{18}$$

for any positive number s_1 $(0 < s_1 < s)$. The condition given by (18) can be satisfyied with the choice of an adequate quantization taking into account (2).

Let $x(t)$ be a solution of equation (11) for the initial condition $x(t = 0) = x_0 \in D_3$. Consider that the quantization was done in order to satisfy the condition given by (18). From (11) and (13) it follows that:

$$\alpha(x, \Delta x) = \nabla V(x)^T \cdot \dot{x}$$

Using (15) and (17) in the equation above, it can be seen that:

$$\frac{\partial V}{\partial x}(x) \cdot \dot{x} < -s_1 \tag{19}$$

This condition will be satisfied at least during certain time while $x(t)$ remains inside D_3 (this is guaranteed by the continuity of $x(t)$). After integrating both sides of the inequality (19), we have:

$$\begin{aligned} \int_0^t \frac{\partial V}{\partial x}(x) \cdot \dot{x} \cdot dt &< \int_0^t -s_1 \cdot dt \\ V(x(t)) - V(x(0)) &< -s_1 \cdot t \\ V(x(t)) &< V(x_0) - s_1 \cdot t \end{aligned}$$

This implies that V evaluated along the solution is bounded by a strictly decreasing function while that solution remains inside D_3. Since the value $V(x_0)$ is smaller than the value that V takes in the bound of D_1, it is clear that the trajectory will never leave D_1.

Let V_1 be the value that V takes in the bound of region D_2. Then, it can be easily seen that the trajectory will reach the region D_2 in a finite time t_1 with:

$$t_1 < \frac{V(x_0) - V_1}{s_1}$$

□

Theorem 5 requires the choice of the quantization intervals satisfying (18). For instance, considering the same uniform quantization Δq and hysteresis window ε for all the quantized variables, each component of Δx is less than $\max(\Delta q, \varepsilon)$. Then, $\|\Delta x\|$ is less than $\sqrt{n}$ times that value, being n the dimension of the state space. Thus, the condition given by (18) can be achieved by taking:

$$\max(\Delta q, \varepsilon) < \frac{r}{\sqrt{n}}$$

Theorem 5 can be easily extended to systems with constant inputs and with equilibrium points others than the origin. Theorems 4 and 5 show that the method presented can be implemented achieving a result with a given final error. They also show the way of doing the quantization in order to obtain a final error bounded to some arbitrary value (given by the choice of the region D_2).

5.2 Simulation with Arbitrary Small Error

The stability properties showed give rules to implement the method with arbitrary small steady state error. However, they do not say anything about the error during the transient of the continuous system.

The following theorem give sufficient conditions to assure that the trajectories of a QSS converge to the trajectories of the continuous system when the quantization goes to 0. Then, the mentioned error can be reduced to arbitrary small values.

Theorem 6 (Convergence of QSS). *Consider the system* (3) *and its associated QSS* (4). *Let D be the* non-saturation region *defined by*

$$D = \{x = (x_1, ..., x_n)/d_{0_i} < x_i < d_{r_i}\} \tag{20}$$

Assume that the input $u(t) \in D_u$ being D_u a bounded region and suppose that function $f(x,u)$ is Lipschitz on $D\times < D_u$. Let $\phi(t)$ be the solution of (3) *from the initial condition $x(0) = x_0$ and let $\phi_1(t)$ be a solution of the related QSS* (4) *starting in the same initial condition x_0. Assume that $\phi(t) \in D_1$ where $D_1 \subset D$ (the continuous system solution is in the non-saturation region). Then, $\phi_1(t) \to \phi(t)$ when the quantization intervals go to* 0.[2]

The proof of this theorem is in Appendix B. A QSS can be simulated by a DEVS when $u(t)$ is piecewise constant. If it does not occur, $u(t)$ can be approximate by a piecewise constant function $u_q(t)$. Provided that the norm of the difference between $u(t)$ and $u_q(t)$ is bounded by a constant, the result of Theorem 6 can be easily extended for that case.

Theorems 5 and 6 show that the approximation of a continuous system by a QSS is a well posed simulation method since the stability properties are conserved from the original system and the error can be reduced to arbitrary small values.

[2]Without modifying the saturation bounds (i.e. the number of quantization levels goes to ∞).

6 Examples and Results

In order to show the qualities of the developed method, some results obtained from the simulation of a second order stiff system (21) are presented here.

$$\begin{cases} \dot{x_1} &= \frac{1}{L}x_2 \\ \dot{x_2} &= U - \frac{1}{C}x_1 - \frac{R}{L}x_2 \\ y &= \frac{1}{L}x_2 \end{cases} \tag{21}$$

The quantization was done using uniform intervals for each state variable and with a hysteresis value (ε) equal to the difference between consecutive quantization values. The parameters were $R = 100.01, L = 0.01, C = 0.01, U = 100$, resulting a stiff systems since the eigenvalues are -1 and -10000.

The first simulation was done using quantization intervals of 10^{-2} and 10^{-4} for x_1 and x_2 respectively. The simulation result can be seen in Figures 4 and 5.

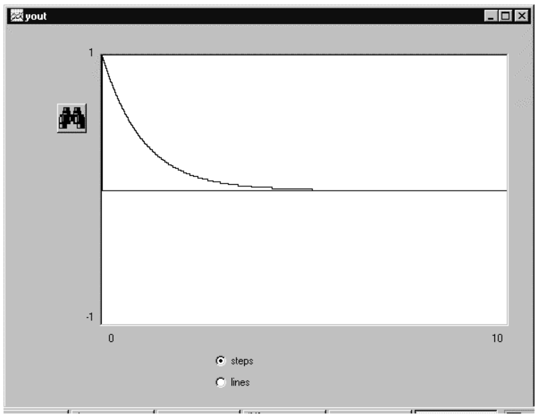

Figure 4: Output trajectory of the quantized system

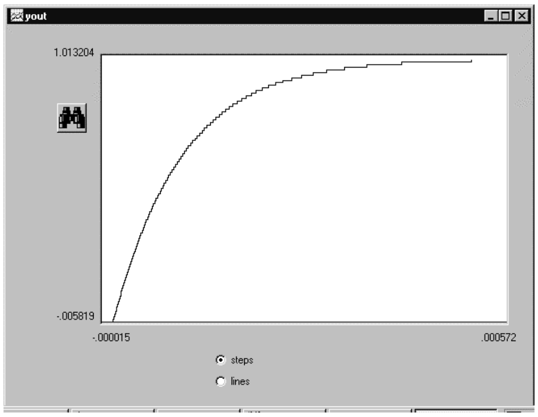

Figure 5: Start of the output trajectory

The resulting DEVS model completed the simulation doing 304 internal transitions.

The error (Figure 6) can be evaluated comparing the simulation result with the exact solution given by (22).

$$y(t) = \frac{10000}{9999}(e^{-t} - e^{-10000t}) \tag{22}$$

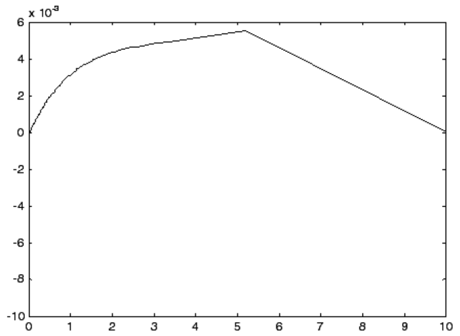

Figure 6: Error in the discrete event simulation

The greatest absolute value is lower than 10^{-2}. To obtain a similar error using Euler's algorithm is necessary to use a step size that results in more than 150000 steps, while fourth order Runge-Kutta uses more than 90000 steps.

A Variable step algorithm as Runge-Kutta 4-5 [4](Matlab's ode45) needs more than 30000 steps while Adams-Bashforth-Moulton method with variable step size (Matlab's ode113) needs more than 60000 steps to achieve this error.

Only Matlab's ode15s can achieve a similar error with 81 steps. But we must take in account that we are comparing a 5th order implicit method against an explicit and very simple first order approach. Thus, the number of calculations made by ode15s is bigger because it must calculate the inverse of a matrix at each step.

Using a quantization four times bigger, the result of Figure 7 was obtained.

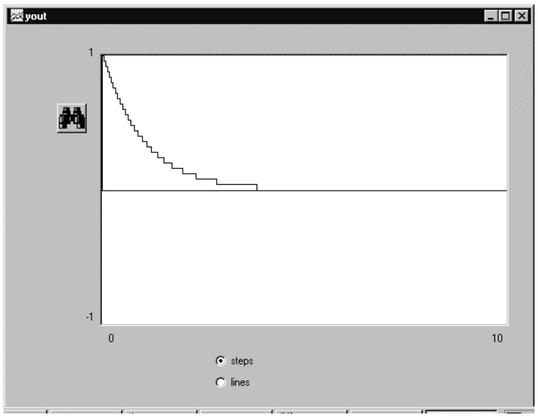

Figure 7: Output trajectory with a bigger quantization

Taking into account that we are using Quantized State Systems in order to simulate continuous systems it is useful to represent the trajectories by lines instead of steps. In this way it is possible to obtain a better approach to the continuous curve. Figure 8 shows the result of drawing the curve in that way (the simulation is the same as in Figure 7.

The greatest absolute value of the error is now lower than 4×10^{-2}. The number of internal transitions is now 78.

Matlab's ode15s needs at least 64 steps to achieve a similar error but the number of calculations with this method is much bigger.

When discrete time approaches (Euler, Runge-Kutta, variable step and implicit methods) are comparing against

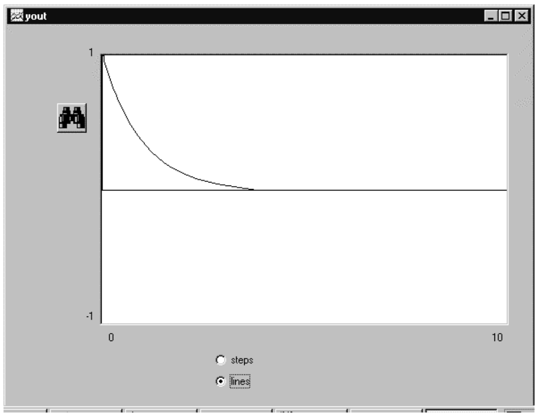

Figure 8: Output trajectory using lines instead of steps

discrete event methods, it is also important to take into account that discrete time methods in each step makes calculations over each part of the system. However, discrete event methods in each step only make calculations over the components doing the transition and the components directly related to them. In big systems, this implies a considerable advantage.

7 Conclusions

Aiming at simulating continuous time systems with a discrete event approach, and building-up on the concept of Quantized Systems introduced in [7], this paper proposed Quantized State Systems (QSS for short) as a DEVS-representable tool for approximating differential equation models.

QSS feature a quantization function with hysteresis for the conversion of state trajectories into piecewise-constant functions. Vis-à-vis other discrete event tools for continuous system simulation, hysteresis is a distinctive feature of QSS, essential in order to exclude the possibility of having infinite number of transitions in a finite time interval of simulation. In this manner, simulateability of the DEVS-approximation of the continuous system is ensured.

Being a model-approximation method (as opposed to a solution-approximation one), QSS allow the analytical proof of properties that could be relevant to determine the degree of convergence of the approximated to the exact solutions of the original continuous system model. Through the proof of some stability properties, relationships between quantization size and desired accuracy were presented, which were translated into practical rules for the choice of the quantization intervals. Some simulation results confirmed the advantage of discrete-event vis-à-vis discrete-time based simulation of continuous systems, in the sense of providing a good trade-off between computational burden and accuracy.

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Appendix A. Why is hysteresis necessary?

It was mentioned that in absence of hysteresis the quantized variables of the QSS can perform an infinite number of changes in a finite interval of time and then the DEVS model will be illegitimate [7].

In fact, two types of illegitimate DEVS models can be distinguished:

- a DEVS model that performs an infinite number of transitions at the same time (but not simultaneously!)
- a DEVS model that performs an infinite number of transitions in a finite time interval greater than zero

The first case is very common in systems having cycles in the internal transition function where all the states of the cycle have their time advance function equal to zero.

The second case only can be present in systems with an infinite set of states. Here, starting from an initial state, the sum of the time advance function of all the successive states that the system reaches can converge to a finite value. Then, the system makes an infinite number of transitions in a finite interval of time (like in Zeno's paradox of *Achilles and the Tortoise*)

The following example shows a case of simulation of QSS that lies in the second type of illegitimacy. Consider the linear second order continuous system given by:

$$\left\{ \begin{array}{ll} \dot{x_1} & = -0.5x_1 + 1.5x_2 \\ \dot{x_2} & = -x_1 \end{array} \right. \tag{23}$$

Suppose that the system is aproximated by a QSS without hysteresis using quantization functions where the quantized values are the odd numbers (Figure 9).

This quantization over both variables divides the state space as Figure 10 shows:

The derivative of the state vector of the QSS in a point is given by the derivative of the continuous system (23) evaluated in the bottom left corner of the square cointaining that point. For instance, at the point $(0, 2)$ the derivative is given by the right hand of (23) at the point $(-1, 1)$, that is $(2, 1)$.

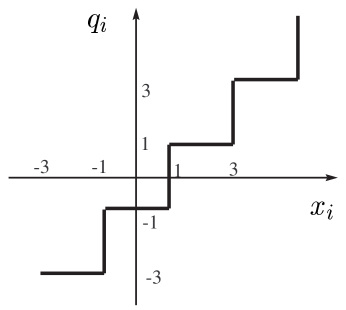

Figure 9: A quantization function without hysteresis

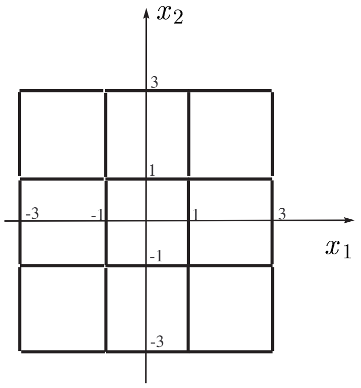

Figure 10: Partition of the state space

Then, if the initial condition is $(0, 2)$ the trajectory will go following the direction $(2, 1)$ until it reaches the next square (here we have a transition since there is a change in the quantized variable q_1 corresponding to x_1). The transition will occur at the time $t = 0.5$(the speed in x_1 is 2 and the distance to the square is 1). The point in which the trajectory reaches the new square is $(1, 2.5)$.

After this transition, the derivative is calculated at the point $(1, 1)$. The direction is now $(1, -1)$. After 1.5 units of time the system will reach the point $(2.5, 1)$ arriving to a new square. The new direction is $(-2, -1)$ (calculated at the point $(1, -1)$) and after 0.75 units of time the system will reach the point $(1, 0.25)$ in the bound of a new square. Then, the direction is $(-1, 1)$ and after 0.75 units of time the system reaches the initial square at the point $(0.25, 1)$. Then, after 0.375 units of time the system goes back to the second square, arriving at the point $(1, 1.375)$.

The elapsed time from the first time the system reaches the second square to the second arrival to that square is 3.375. Then, it can be easily seen that the system will follow again a similar cycle but starting from the new initial condition $(1, 1.375)$ and it will take $3.375/4 = 0.84375$ units of time. Each cycle will be done four times faster than the previous one. Then, the sum of all the cycle times will converge to 4.5 units of time. Since the first transition occurs at time 0.5, before 5 unit of time the system performs an infinite number of transitions.

Figure 11 shows that trajectory in the space state while Figure 12 shows the temporal evolution of the quantized variable q_1.

As a result of this behavior, the simulation will be stuck after 5 units of time. However, the use of hysteresis

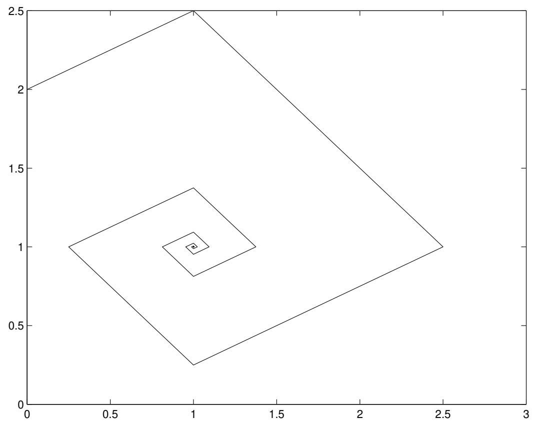

Figure 11: State Space trajectory with infinite transitions

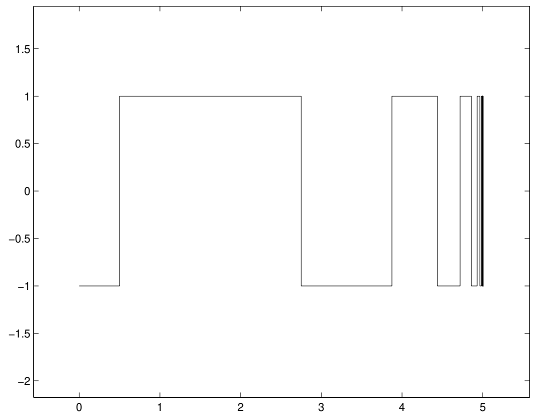

Figure 12: Quantized variable trajectory with infinite transitions

solves this problem, as Theorem 1 shows.

Appendix B. Proof of Theorem 6

(Convergence of QSS - Simulation with arbitrary small error)

Proof: Let S be $\mathbb{R}^n - D$ and let F be

$$F = \sup_{u \in D_u} (\sup_{x \in D} \|f(x,u)\|)$$

Let d be defined by

$$d = \inf_{x \in S} (\inf_{t \in [0,\infty]} \|\phi(t) - x\|) \tag{24}$$

Taking into account the assumptions on f and $\phi(t)$, a positive constant t_1 can be found satisfying

$$t_1 < \frac{d}{F} \tag{25}$$

It can be easily seen that during the interval $[0, t_1]$ the trajectory of $\phi_1(t)$ will remain inside D.

The equation of the QSS (4) can be written

$$\begin{cases} \dot{x}(t) & = f(x(t) + \Delta x(t), u(t)) \\ y(t) & = g(x(t) + \Delta x(t), u(t)) \end{cases} \tag{26}$$

where $\Delta x(t)$ satisfies

$$\|\Delta x(t)\| \leq \Delta_x \quad (0 \leq t \leq t_1) \tag{27}$$

being Δ_x a constant defined by the quantization intervals. Let $t \in [0, t_1]$. It follows from (26), (3) and the fact that $\phi_1(0) = \phi(0)$ that:

$$\phi_1(t) - \phi(t) = \int_0^t (f(\phi_1(\tau) + \Delta x(t), u(\tau)) - f(\phi(\tau), u(\tau)))d\tau$$

Thus, appliyng the euclidean norm we obtain

$$\|\phi_1(t) - \phi(t)\| = \|\int_0^t (f(\phi_1(\tau) + \Delta x(\tau), u(\tau)) - f(\phi(\tau), u(\tau)))d\tau\|$$

and then,

$$\|\phi_1(t) - \phi(t)\| \leq \int_0^t \|f(\phi_1(\tau) + \Delta x(\tau), u(\tau)) - f(\phi(\tau), u(\tau))\|d\tau \tag{28}$$

Let M be the Lipschitz constant of function f on $D \times D_u$. Since the argument of the function f in (28) is inside that region we have

$$\begin{aligned}
\|\phi_1(t) - \phi(t)\| &\leq \int_0^t M\|\phi_1(\tau) + \Delta x(\tau) - \phi(\tau)\|d\tau \\
\Rightarrow \|\phi_1(t) - \phi(t)\| &\leq \int_0^t M(\|\phi_1(\tau) - \phi(\tau)\| + \|\Delta x(\tau)\|)d\tau \\
\Rightarrow \|\phi_1(t) - \phi(t)\| &\leq \int_0^t M(\|\phi_1(\tau) - \phi(\tau)\| + \Delta_x)d\tau \\
\Rightarrow \|\phi_1(t) - \phi(t)\| &\leq \int_0^t M\|\phi_1(\tau) - \phi(\tau)\|d\tau + Mt\Delta_x
\end{aligned}$$

The functions ϕ and ϕ_1 are continuous, as well as the term $Mt\Delta_x$. Since M is positive, it is possible to apply the *Gronwall-Bellman Inequality* [8], resulting in

$$\begin{aligned}
\|\phi_1(t) - \phi(t)\| &\leq Mt\Delta_x + \int_0^t M^2 s\Delta_x e^{\int_s^t M d\tau} ds \\
\Rightarrow \|\phi_1(t) - \phi(t)\| &\leq (e^{Mt} - 1)\Delta_x
\end{aligned}$$

Then, since M and t_1 do not depend on Δ_x, for $0 \leq t \leq t_1$ we have

$$\lim_{\Delta_x \to 0} \|\phi_1(t) - \phi(t)\| = 0 \tag{29}$$

From (24) we have

$$d \leq \inf_{x \in S}(\|\phi(t_1) - x\|) \tag{30}$$

Taking into account (25), (29) and (30), it is possible to find a sufficiently small quantization such that

$$t_1 < \frac{\inf_{x \in S}(\|\phi_1(t_1) - x\|)}{F}$$

This inequality implies that the solution $\phi_1(t)$ does not leave the region D during the interval $[t_1, 2t_1]$. Then, the validity of equations (27) to (29) hold for the interval $[0, 2t_1]$. Repeating that argument we can assure that (29) holds for all t. $\square$