Inner-Outer Approximation of Robust Control Invariant Sets ⋆

Román Comelli [a,*], Sorin Olaru [b], Ernesto Kofman [a]

[a] *Centro Internacional Franco-Argentino de Ciencias de la Información y de Sistemas (CIFASIS), CONICET-UNR, Ocampo y Esmeralda, 2000 Rosario, Argentina*

[b] *Laboratory of Signals and Systems, Univ. Paris-Sud-CentraleSupelec-CNRS, Université Paris Saclay, 3 rue Joliot Curie, 91190 Gif-sur-Yvette, France*

Abstract

This work proposes an approach to replace the use of a robust control invariant set by a pair of simpler sets that provide an inner and an outer approximation of the former. In the proposed approach, the outer set plays the role of the target region and the inner set is such that the trajectories that start inside it can be kept inside the outer set and be driven back to the inner set within a finite-time horizon. We show that the existence of these two sets implies the existence of a robust control invariant set between both regions. We also provide results that allow finding an inner set from a given target outer set and we show a way of using both sets in Model Predictive Control schemes such that the target region is never abandoned in spite of the fact that nor that region neither the inner set are invariant. We also illustrate the ideas with an example in which the inner and outer sets are very simple notwithstanding that any robust invariant set is not convex.

Key words: Control Invariant Sets; Set-Based Methods; Model Predictive Control; Practical Stability.

1 Introduction

In presence of disturbances and different types of non ideal assumptions, control systems are not able to drive the system state to a desired equilibrium point. In those cases, the best that can be achieved is to steer the state to some region where control goals are satisfied and to keep the trajectories bounded to it. The achievable properties are generally known under the generic term of *practical stability* [20].

A region that has the property that the state can remain in through the use of appropriate control actions defines a *positive control invariant set* (PCIS), that in presence of unknown disturbances are generally referred to as *robust positive control invariant sets* (RPCIS). In the context of this work, we shall refer to these sets simply as control invariant sets (CIS) or robust CIS (RCIS).

Set invariance plays a fundamental role in control theory [6] and there are several control design approaches that use CIS or RCIS in order to guarantee that the state remains bounded according to the control goals [4,27]. For instance, these sets are used typically as a terminal restriction in Model Predictive Control (MPC) schemes as a way of ensuring convergence to certain target region [28,1,17]. In fact, the non existence of a (R)CIS inside a target region for a control system implies that it is impossible to ensure that the state remains forever inside that region.

There are several procedures that allow computing CIS and RCIS [31,26,10,9,23] under different assumptions. However, in most situations, the characterization of these sets is not simple. In nonlinear or switched systems, or under the presence of finite input sets, their computation is not only complicated, but it also leads to very complex sets [30]. These sets may have hundreds of faces, they may be not even convex, and, in consequence, their usage in real control problems can become impractical or even impossible. Alternatives have been investigated in terms of sequences of sets with periodic, cyclic or relaxed invariance properties [24,22,27,2,21].

In this work, we propose a way of avoiding the use of control invariant sets by using two simpler sets, an outer set consisting of the target region and an inner set from which an input sequence exists such that the state does not leave the outer set and eventually (within a finite

⋆ This paper was not presented at any IFAC meeting.

* Corresponding author.

Email addresses: comelli@cifasis-conicet.gov.ar (Román Comelli), sorin.olaru@centralesupelec.fr (Sorin Olaru), kofman@cifasis-conicet.gov.ar (Ernesto Kofman).

horizon of time) the state trajectory passes back by that inner set.

We shall show that these conditions on the inner and the outer set imply the existence of a RCIS between both sets, but it is not necessary to compute it. The sole knowledge of the outer set (that we assume that is given, as it would play the role of the target region) and the computation of a simple inner set allow designing controllers that, under certain restrictions on the initial state, ensure that the outer set is never abandoned in spite of the fact that nor the outer set, neither the inner set need to be positive control invariant.

Notation

The following notation will be used along this work. An interval of the integer numbers $[E, F] \subset \mathbb{Z}$ is denoted as $\mathbb{Z}_{E:F}$. The expression $s(k+i|k)$ denotes the value of a signal s at time $k+i$ computed at time k.

Unless otherwise stated, we shall refer indistinctly to the input as $u(k)$ or $u(x(k))$ to emphasize its dependence on the state.

2 Background

This section introduces some previous definitions that are then used to present the main results of the work.

2.1 Control Invariant Sets

Consider a discrete-time perturbed system described by the following nonlinear time-invariant model:

$$x(k+1) = f(x(k), u(k), w(k)) \tag{1}$$

with $x(k) \in X \subseteq \mathbb{R}^n$, $u(k) \in U \subseteq \mathbb{R}^m$ and $w(k) \in W \subset \mathbb{R}^p$ being the state, control input and disturbance at time k, respectively. We shall assume along this work that W is bounded thus fitting in the *persistent bounded disturbances framework*.

Definition 1 (Robust Positive Control Invariant Set [12]). Given the system of Eq. (1) with the corresponding sets X, U and W, a set $\Omega \subseteq X$ is robust positive control invariant provided that for any state $x \in \Omega$, a control input $u(x) \in U$ exists such that $f(x, u(x), w) \in \Omega$ for all $w \in W$.

A way of verifying that a set Ω is a RCIS is by checking that it lies inside its robust one-step set $\mathcal{C}_1(\Omega)$ defined as follows:

Definition 2 (Robust One-step Set [16]). Given the system of Eq. (1) with the corresponding sets X, U and W, and a set $\Omega \subseteq X$, the robust one-step set to Ω is defined as

$$\mathcal{C}_1(\Omega) \triangleq \{x \in X : \exists u(x) \in U \mid f(x,u,w) \in \Omega, \\ \forall w \in W\}. \tag{2}$$

Notice that Ω is a RCIS if and only if $\Omega \subseteq \mathcal{C}_1(\Omega)$ [16].

2.2 Computation of Control Invariant Sets

While control invariant sets can take arbitrary shapes, most works in the literature are devoted to characterize ellipsoidal and polyhedral sets [8]. Ellipsoidal sets are usually obtained with procedures based on Lyapunov functions or linear matrix inequalities [19]. Polyhedral sets, in turn, are generally computed using iterative methods [14,33]. There are also constructions based on the approximations within the class of semi-algebraic sets that resort to lift and project approaches in order to benefit from the convex optimization tools [18] but remain difficult to integrate in MPC-like routines.

Among the different approaches, and closely related to the present work, there is a classic iterative procedure to obtain the maximal RCIS contained in certain set $R \subseteq X$ first presented in [3] and then summarized in [6,8]. This procedure starts by selecting $R_0 = R$ and then iterates to compute the sequence

$$R_{k+1} \triangleq \mathcal{C}_1(R_k) \cap R. \tag{3}$$

The sets R_k are nested, since $R_{k+1} \subseteq R_k$, and the maximal RCIS contained in R is shown to be $R_\infty \triangleq \bigcap_{k=0}^{\infty} R_k$.

The reason why this procedure is associated to polyhedral invariant sets is that, although it can be formulated for general systems, its use is mainly restricted to linear systems, and when R, U and W are polyhedral, the same occurs with R_k (but not necessarily with R_∞). Some works with remarkable results for linear, polytopic and piece-wise affine systems are respectively [13,5,16].

One of the problems with more general setups is that of calculating the robust one-step set. For nonlinear systems with constraints, computing this in a useful manner that allows the implementation of the aforementioned iterative procedure is not simple. Examples of publications that address the construction of invariant sets for the nonlinear case are [9,12,10]. Furthermore, even when computing the robust one-step set may be achievable, the invariant sets obtained could be complex to such an extent that their usage in real control problems may not be feasible.

Given the complications that arise in the computation of invariant sets, it has been studied how to obtain inner and/or outer approximations to them with different properties and under various assumptions [32,10,29,25]. In this work, even though we consider inner and outer sets that could be close to some invariant, our goal is not to approximate the invariant set but to give conditions under which this set can be replaced by a pair of inner and outer sets in order to guarantee practical stability.

2.3 Related Work

As it was mentioned in the introduction, this work proposes replacing a possibly complex RCIS by two simpler sets that provide an inner and an outer approximation of the former but do not need to be control invariant (they only need to satisfy a return condition to the inner set). While to the best of our knowledge the idea is new, some related concepts have been previously developed.

Pairs of sets related to invariance properties have been used (see for instance [7,16]), but not as simpler approximations of an invariant set.

The return condition relaxing the invariance constraint on the inner set is similar to the *cyclic* or *periodic* invariance concepts [27,35] and other related ideas such as (k, λ)-contractive sets [2,21] and dwell time invariant sets for switching systems [11]. However, on all these works, when the state abandons the target set, it is only restricted to remain on the feasible region. In our work, however, the state is constrained to remain in the outer set (which plays the role of the target region).

3 Main Results

This section presents the main results of the work, introducing first the notion of N-step Control Inner Sets and then studying their properties, providing some tools for its computation and usage in the context of control design.

3.1 N-step Control Inner Sets

The following definition provides the notion of N-step control inner set.

Definition 3 (N-step Control Inner Set)**.** Given the system of Eq. (1) and an outer set $\Omega_O \subseteq X$, we say that a set $\Omega_I \subseteq \Omega_O$ is a N-step control inner set of Ω_O if for any $x(0) \in \Omega_I$, an input sequence $u(x(0)), \dots, u(x(N-1)) \in U$ exists such that $x(k) \in \Omega_O \quad \forall k \in \mathbb{Z}_{1:N}$ and $x(j) \in \Omega_I$ for some $j \in \mathbb{Z}_{1:N}$ for any disturbance sequence $w(0), \dots, w(N-1) \in W$.

Figure 1 shows an outer set Ω_O, a N-step control inner set Ω_I, a RCIS Ω, and a trajectory starting from $x(0)$ that suggest that $N \geq 5$.

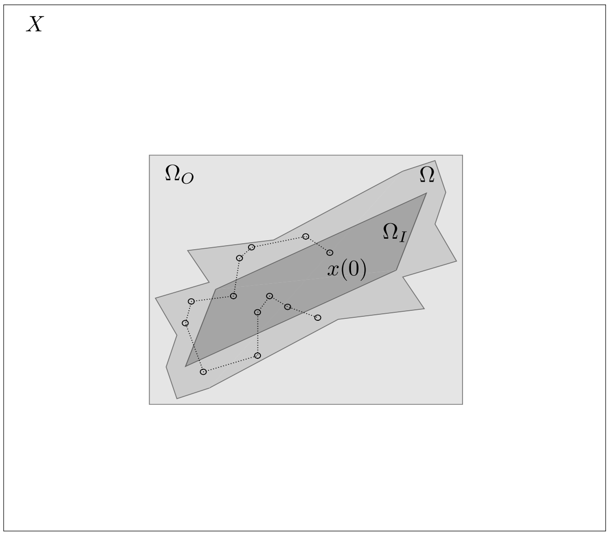

Fig. 1. A non convex RCIS Ω with a pair of simpler outer and inner sets (Ω_O and Ω_I).

We shall assume that the outer set Ω_O is given and it will play the role of a target set to which the state trajectories must converge and remain. While this target set does not need to be control invariant, we shall assume that it contains a RCIS since otherwise it will not be possible to ensure that the trajectories can be kept inside Ω_O. However, instead of computing a RCIS, the goal is to find a simpler N-step Control Inner Set Ω_I verifying Definition 1 since the trajectories that pass through it can be also kept bounded to Ω_O.

The inner set Ω_I is always associated to an outer set Ω_O. Anyway, for reasons of space and readability, and taking into account that the outer set will be always Ω_O, we shall not always explicitly mention Ω_O.

Before providing a way to characterize Ω_I, we shall first analyze the existence of a maximal N-step Control Inner Set inside Ω_O and relate that with the classic concepts of RCIS.

3.2 Maximal N-step Control Inner Set

We shall show that given an outer set, a maximal N-step control inner set exists, and, in case it is not empty, this maximal N-step control inner set turns out to be a RCIS. Moreover, we shall show that this set can be characterized as the fixed point of a simple set-iteration process.

For arriving to those results, we first provide some auxiliary propositions, lemmas and corollaries.

Proposition 4. *Given the system of Eq.* (1) *and the outer set* $\Omega_O \subseteq X$*, a maximal N-step control inner set* $\Omega_I^{\max}$ *(that could be the empty set) exists.*

Proof. Given two N-step control inner sets, Ω_I^1, Ω_I^2, it can be straightforwardly seen that their union $\Omega_I^1 \cup \Omega_I^2$ is also a N-step control inner set. Thus, a maximal N-step control inner set exists and it consists of the union of all possible N-step control inner sets. If there is no N-step control inner sets, then $\Omega_I^{\max} = \emptyset$. □

Lemma 5. *Let Ω_I be a N-step control inner set of Ω_O. Then, the set $\Omega_I \cup (\mathcal{C}_1(\Omega_I) \cap \Omega_O)$ is also a N-step control inner set of Ω_O.*

Proof. Let $x(0) \in \mathcal{C}_1(\Omega_I) \cap \Omega_O$, then an input $u(x(0)) \in U$ exists such that $x(1) = f(x(0), u(x(0)), w(0)) \in \Omega_I$ for any disturbance $w(0) \in W$. Then, from Definition 3, a sequence of inputs $u(x(1)), \ldots, u(x(N)) \in U$ exists such that $x(k) \in \Omega_O \ \forall k = \mathbb{Z}_{2:N+1}$ and $x(j) \in \Omega_I$ for some $j \in \mathbb{Z}_{2:N+1}$ for any disturbance sequence. Thus, $x(0) \in \mathcal{C}_1(\Omega_I) \cap \Omega_O$ implies that an input sequence $u(x(0)), \ldots, u(x(N-1)) \in U$ exists such that $x(k) \in \Omega_O \ \forall k = \mathbb{Z}_{1:N}$. Moreover, this and the fact that $x(j) \in \Omega_I$ means that $x(j_1) \in \mathcal{C}_1(\Omega_I) \cap \Omega_O$ for $j_1 = j - 1 \in \mathbb{Z}_{1:N}$ for any disturbance sequence. Then, it results that $\mathcal{C}_1(\Omega_I) \cap \Omega_O$ is a N-step control inner set so it follows that $\Omega_I \cup (\mathcal{C}_1(\Omega_I) \cap \Omega_O)$ is also a N-step control inner set. □

Lemma 5 establishes that given a N-step control inner set Ω_I, a larger (or equal) N-step control inner set can be constructed by adding to it the set $\mathcal{C}_1(\Omega_I) \cap \Omega_O$. Since the maximum N-step control inner set $\Omega_I^{\max}$ cannot be enlarged, then the following corollary can be stated:

Corollary 6. *The set $\Omega_I^{\max}$ verifies $C_1(\Omega_I^{\max}) \cap \Omega_O \subseteq \Omega_I^{\max}$.*

Lemma 7. *If the set $\Omega_I^{\max}$ is not empty, then it is a RCIS.*

Proof. Given any $x(0) \in \Omega_I^{\max}$, an input sequence exists such that $x(k)$ remains in Ω_O and passes inside $\Omega_I^{\max}$ at some time $j \in \mathbb{Z}_{1:N}$ for any disturbance sequence. Then, $x(j-1) \in \Omega_O$ and $x(j-1) \in \mathcal{C}_1(\Omega_I^{\max})$. Thus, from Corollary 6, $x(j-1) \in \Omega_I^{\max}$. Repeating this argument, it results that $x(j-i) \in \Omega_I^{\max} \ \forall i \in \mathbb{Z}_{1:j-1}$, and finally $x(1) \in \Omega_I^{\max}$ proving that $\Omega_I^{\max}$ is a RCIS. □

Lemma 8. *For all $N \in \mathbb{N}$, the maximal N-step control inner set $\Omega_I^{\max}$ is the maximal RCIS contained in Ω_O.*

Proof. Let $\Omega^{\max}$ be the maximal RCIS contained in Ω_O. From Definitions 1 and 3, it can be easily seen that $\Omega^{\max}$ is a N-step control inner set for all $N \in \mathbb{N}$. Thus, for any $N \in \mathbb{N}$, it results that $\Omega^{\max} \subseteq \Omega_I^{\max}$.

According to Lemma 7, given $N \in \mathbb{N}$ and the outer set Ω_O, the maximal N-step inner set $\Omega_I^{\max}$ is a RCIS and then $\Omega_I^{\max} \subseteq \Omega^{\max}$.

Then, considering that $\Omega^{\max} \subseteq \Omega_I^{\max}$ and $\Omega_I^{\max} \subseteq \Omega^{\max}$ it results that $\Omega_I^{\max} = \Omega^{\max}$. □

This last result tells that the maximal control inner set does not depend on N and that it coincides with the maximal RCIS contained in Ω_O. A direct consequence of this result is that a trajectory that starts in any N-step control inner set can be held forever inside Ω_O, as expressed in the following corollary.

Corollary 9. *If Ω_I is a N-step control inner set, then given any $M > 0$, the condition $x(0) \in \Omega_I$ implies that there exists an input sequence $u(x(0)), \ldots, u(x(M-1)) \in U$ such that $x(k) \in \Omega_O \ \forall k \in \mathbb{Z}_{1:M}$ for any disturbance sequence.*

Once the structural properties of the maximal N-step control set are brought into light in Lemma 8 we can make a link with the classical control invariant constructions. The following result states that the maximal N-step control inner set can be seen as a limit set of a set-iterations of the standard procedure of Eq. (3) initialized in Ω_O.

Theorem 10. *Consider the following succession of sets:*

$$\Omega_0 = \Omega_O, \quad \Omega_{k+1} = \mathcal{C}_1(\Omega_k) \cap \Omega_O. \tag{4}$$

Then, $\Omega_I^{\max} = \lim_{k \to \infty} \Omega_k$.

Proof. Notice first that since $\Omega_0 = \Omega_O$, then $\Omega_1 = \mathcal{C}_1(\Omega_0) \cap \Omega_O \subseteq \Omega_0$. In order to use induction, suppose that for some k, $\Omega_k \subseteq \Omega_{k-1}$. Hence, $\mathcal{C}_1(\Omega_k) \subseteq \mathcal{C}_1(\Omega_{k-1})$. Then, $\Omega_{k+1} = \mathcal{C}_1(\Omega_k) \cap \Omega_O \subseteq \Omega_k = \mathcal{C}_1(\Omega_{k-1}) \cap \Omega_O$. This proves that for any $M > 0$, $\Omega_M \subseteq \Omega_{M-1} \ldots \subseteq \Omega_1 \subseteq \Omega_0$. This implies that the sequence Ω_k converges to some set $\Omega_\infty \triangleq \lim_{k \to \infty} \Omega_k$ (which could be the empty set).

Notice also that $\Omega_I^{\max} \subseteq \Omega_0$ implies that $\Omega_I^{\max} = \mathcal{C}_1(\Omega_I^{\max}) \cap \Omega_O \subseteq \mathcal{C}_1(\Omega_0) \cap \Omega_O = \Omega_1$ and then $\Omega_I^{\max} \subseteq \Omega_1$. Applying this recursively, we arrive to $\Omega_I^{\max} \subseteq \Omega_k$ for all k and then $\Omega_I^{\max} \subseteq \Omega_\infty$ On the other hand, the fact that $\Omega_\infty = \mathcal{C}_1(\Omega_\infty) \cap \Omega_O$ implies that Ω_∞ is a RCIS contained in Ω_O. Thus, it is also contained in $\Omega_I^{\max}$ that, according to Lemma 8 is the maximal RCIS contained in Ω_O. Hence, $\Omega_\infty \subseteq \Omega_I^{\max}$.

The facts that $\Omega_I^{\max} \subseteq \Omega_\infty$ and $\Omega_\infty \subseteq \Omega_I^{\max}$ mean that $\Omega_I^{\max} = \Omega_\infty$ completing the proof. □

An alternative proof for Theorem 10 can be obtained using the fact that $\Omega_I^{\max}$ is the maximal RCIS inside Ω_O and that this set can be obtained using the procedure of Eq. (3).

Corollary 11. *Given an outer set Ω_O, any $N > 0$, and the succession of Eq.* (4)*, a non empty N-step control inner set exists if and only if* $\lim_{k\to\infty} \Omega_k \neq \emptyset$.

Another version of this corollary is

Corollary 12. *Given an outer set Ω_O, for any $N > 0$, a non empty N-step control inner set exists if and only if Ω_O contains a RCIS in its interior.*

3.3 N-step Control Inner Set Characterization

The existence of the maximal N-step control inner set and its characterization as the result of an infinite iteration is of theoretical value. However, in practice, we are interested in characterizing simpler inner sets, a goal that can be achieved with the help of the following theorem.

Theorem 13. *Let $S_0 \subseteq \Omega_O$ be a non-empty set and consider the succession of sets*

$$S_{k+1} = \mathcal{C}_1(S_k) \cap \Omega_O \tag{5}$$

and the set $T_N \triangleq \bigcup_{k=1}^{N} S_k$. Then,

(1) S_0 is a N-step control inner set if and only if $S_0 \subseteq T_N$.
(2) Any set Ω such that $S_0 \subseteq \Omega \subseteq T_N$ is a N-step control inner set.

Proof. 1. $\Longrightarrow$) Being $x(0) \in \Omega_O$, the condition $x(0) \notin T_N$ implies that $x(0) \notin \bigcup_{j=1}^{N} S_j = \bigcup_{j=1}^{N} \mathcal{C}_1(S_{j-1}) \cap \Omega_O$. Then, an input $u(0) \in U$ cannot be found such that $x(1) \in \bigcup_{j=0}^{N-1} S_j$ for all possible disturbances $w(0) \in W$. Thus, selecting any input $u(0) \in U$, a disturbance $w(0) = w(x(0), u(0)) \in W$ exists such that $x(1) \notin \bigcup_{j=0}^{N-1} S_j$.

In order to proceed by induction, assuming that $x(k) \in \Omega_O$, the condition $x(k) \notin \bigcup_{j=1}^{N-k} S_j = \bigcup_{j=1}^{N-k} \mathcal{C}_1(S_{j-1}) \cap \Omega_O$ implies that selecting any input $u(k) \in U$, a disturbance $w(k) = w(x(k), u(k)) \in W$ exists such that $x(k+1) \notin \bigcup_{j=0}^{N-k-1} S_j$.

Then, for any input sequence $u(0), \ldots, u(N-1) \in U$, a disturbance sequence $w(0), \ldots, w(N-1) \in W$ exists such that $x(k) \notin S_0 \subseteq \bigcup_{j=0}^{N-k-1} S_j$ for all $k \in \mathbb{Z}_{1:N}$ assuming that $x(k) \in \Omega_O$. This implies that when we can chose $x(0)$ such that $x(0) \in S_0 \subseteq \Omega_O$ and $x(0) \notin T_N$, then S_0 is not a N-step control inner set either because $x(k) \notin S_0$ for any $k \in \mathbb{Z}_{1:N}$ or because $x(k) \notin \Omega_O$ for some $k \in \mathbb{Z}_{1:N}$.

Thus, $S_0 \subseteq T_N$ is a necessary condition for S_0 to be a N-step control inner set.

$\Longleftarrow$) The condition $S_0 \subseteq T_N$ implies that given any $x(0) \in S_0$, $x(0) \in S_j = \mathcal{C}_1(S_{j-1}) \cap \Omega_O$ for some $j \in \mathbb{Z}_{1:N}$. Thus, an input $u(0) \in U$ exists such that $x(1) \in S_{j-1}$ for any disturbance $w(0) \in W$. Then, by induction an input sequence $u(0), \ldots, u(j-1) \in U$ exists such that $x(1) \in S_{j-1}, \ldots, x(j) \in S_0$ for any disturbance sequence. This implies that S_0 is a N-step control inner set.

2. Take $R_0 = \Omega$ and consider the succession $R_{k+1} = \mathcal{C}_1(R_k) \cap \Omega_O$ for $k \in \mathbb{Z}_{1:N-1}$. Notice that $S_0 \subseteq R_0$ implies that $S_k \subseteq R_k$ for all $k \in \mathbb{Z}_{1:N}$ and then $T_N \subseteq \bigcup_{k=1}^{N} R_k$. Thus, the condition $\Omega = R_0 \subseteq T_N$ implies that $R_0 \subseteq \bigcup_{k=1}^{N} R_k$ and according to the first part of this Theorem, this means that $\Omega = R_0$ is a N-step inner set. ☐

This last result allows computing simple inner sets, by starting with some candidate inner set and iterating until it is covered by their successive controllable sets. If after N iterations the candidate set is not covered by T_N, then the chosen set was not a N-step control inner set. In that case, a possible workaround is to keep iterating until S_0 is covered, or until T_N converges without covering S_0 so we can ensure that S_0 is not a N-step control inner set for any $N > 0$.

If we find that S_0 is a N-step control inner set but N is too large, then the following result can be used for finding an inner set with less steps.

Theorem 14. *Let S_0 be a N-step control inner set. Then the set $V_M = \bigcup_{k=0}^{M} S_k$ for $M \in \mathbb{Z}_{0:N-1}$ with S_k defined in Eq.* (5) *is a $(N-M)$-step control inner set.*

Proof. Let $Z_0 = V_M$ and consider the succession $Z_{k+1} = \mathcal{C}_1(Z_k) \cap \Omega_O$. We shall prove first that

$$Z_k = \bigcup_{j=k}^{M+k} S_j, \ \forall k \in \mathbb{Z}_{0:N-M}. \tag{6}$$

Notice that $Z_0 = V_M = \bigcup_{j=0}^{M} S_j$. Then, in order to proceed by induction, assuming that $Z_k = \bigcup_{j=k}^{M+k} S_j$ for some $k \in \mathbb{Z}_{0:N-M}$, it results

$$\begin{aligned} Z_{k+1} &= \mathcal{C}_1(Z_k) \cap \Omega_O = \mathcal{C}_1(\bigcup_{j=k}^{M+k} S_j) \cap \Omega_O \\ &= \bigcup_{j=k}^{M+k} \mathcal{C}_1(S_j) \cap \Omega_O = \bigcup_{j=k}^{M+k} S_{j+1} = \bigcup_{j=k+1}^{M+k+1} S_j, \end{aligned}$$

showing that Eq. (6) holds. Then, it results

$$\hat{T}_{N-M} \triangleq \bigcup_{k=1}^{N-M} Z_k = \bigcup_{j=1}^{N} S_j = T_N.$$

Since S_0 is a N-step control inner set, according to Theorem 13, it results that $S_0 \subseteq T_N$. Then,

$$\hat{T}_{N-M} = T_N = \bigcup_{j=1}^{N} S_j = \bigcup_{j=0}^{N} S_j = \bigcup_{j=0}^{N-M} Z_j$$

meaning that $Z_0 \subseteq \hat{T}_{N-M}$. This, according to Theorem 13, implies that $Z_0 = V_M$ is a $(N-M)$-step control inner set completing the proof. □

While the results of Theorems 13-14 are useful for checking that a candidate set S_0 is a N-step control inner set and for finding a set with a smaller value of N, they do not tell how to choose the candidate set S_0.

We shall not provide a general answer to this problem since the solution may depend on the different assumptions made: the system may be linear or nonlinear, it can include one or multiple equilibria or limit sets and these can be open loop stable or unstable, the input set may be convex or finite, etc. Anyway, there are some general considerations that can be taken into account which are valid for all cases:

- Since every inner set verifies $\Omega_I \subseteq \Omega_I^{\max}$, then the candidate set S_0 must be inside any set Ω_k in the succession of Eq. (4). Thus, a possible procedure may consist in performing some iterations of that succession and then taking a simple set contained in Ω_k as candidate.
- If the procedure fails and S_0 is not covered by $T_N = \bigcup_{k=1}^{N} S_k$ the iterations can be continued for larger values of N until some set S_k is covered by T_N. If that occurs, we can then adopt $\Omega_I = S_k$. In case S_k has a complex shape, we can also exploit the second part of Theorem 13 and take Ω_I such that $S_k \subseteq \Omega_I \subseteq T_N$.
- In several occasions, choosing a small candidate set S_0 works. However, as we shall see in Section 3.4, we may want Ω_I to be as large as possible. Thus, as before, we can exploit the second part of Theorem 13 to construct a larger inner set by placing some simple set Ω between the candidate set S_0 and the set T_N. Moreover, we can use this idea recursively by starting back the procedure with $S_0 = \Omega$ and enlarge further the inner set.
- Taking $S_0 = \Omega$ where Ω is a non-maximal RCIS contained in Ω_O will always work since it results that $S_0 \subseteq S_1 \subseteq T_N$ in Theorem 13. Then, after computing T_N and provided that $T_N \neq S_0$, a simpler set Ω_I can be adopted satisfying $S_0 \subseteq \Omega_I \subseteq T_N$ and proceeding recursively like in the previous item. A way of finding a non-maximal RCIS Ω is using the procedure of Theorem 10 replacing Ω_O by some smaller set $\tilde{\Omega}_O \subset \Omega_O$ verifying $\Omega_I^{\max} \not\subseteq \tilde{\Omega}_O$. Provided that $\tilde{\Omega}_O$ contains a RCIS, that procedure will compute the maximal RCIS contained on it.
- In any case, using $\Omega_I = \Omega_I^{\max}$ will work and Theorem 10 provides a procedure to compute that maximal inner set. However, the goal of this work is to use simpler sets than those that are the result of an iteration that may lead to very complex shapes. We are assuming in this work that $\Omega_I^{\max}$ results too complex to be used in practice, what justifies the need for using inner and outer approximations.

3.4 Application to Control Design

The existence of a RCIS like $\Omega_I^{\max}$ inside a target region Ω_O is a necessary condition for the existence of a control law that can keep the state inside this region. Moreover, in several MPC schemes the RCIS need to be explicitly computed since they are used as terminal constraints for the predicted states in order to ensure recursive feasibility and practical stability.

However, as it was already mentioned, in many situations the computation of the control invariant set can be very difficult or can lead to a complex set that cannot be used in practice. In such case, this complex set can be replaced by the target region itself Ω_O and an inner approximation Ω_I that verifies Definition 3. In order to show how to proceed with this replacement, we shall consider a MPC scheme where the predicted states for an input sequence $\hat{u}(k|k), \ldots, \hat{u}(k+N-1|k)$ and a disturbance sequence $\hat{w}(k|k), \ldots, \hat{w}(k+N-1|k)$ are given by

$$\hat{x}(k+j+1|k) = f(\hat{x}(k+j|k), \hat{u}(k+j|k), \hat{w}(k+j|k)) \quad (7)$$

with $\hat{x}(k|k) = x(k)$.

The following lemma then shows that using a pass-through constraint on the inner set Ω_I ensures recursive feasibility of the MPC scheme. More precisely, we consider that the admissible input sequences are those that ensure that the predicted state sequence pass through Ω_I within the prediction horizon and remain in Ω_O after passing through Ω_I. We then show that if an admissible input trajectory is found at time k, then an admissible input trajectory can be found at time $k+1$.

Lemma 15. *Consider the system of Eq.* (1)*, an outer set Ω_O and an associated N-step control inner set Ω_I. Suppose that in an MPC scheme the admissible input sequences $\hat{u}(k|k), \ldots, \hat{u}(k+N-1|k) \in U$ are those that ensure that, for any disturbance sequence, the predicted states verify $\hat{x}(k+j|k) \in \Omega_I$ for some $j \in \mathbb{Z}_{1:N}$ and $\hat{x}(k+i|k) \in \Omega_O$ $\forall i \in \mathbb{Z}_{j:N}$. If the state $x(k)$ is such that* *an admissible input sequence exists, then an admissible input sequence exists also for* $x(k+1)$.

Proof. Let $\check{u}(k|k), \ldots, \check{u}(k+N-1|k) \in U$ be the admissible input sequence chosen by the MPC scheme such that $u(k) = \check{u}(k|k)$. Let $w(k)$ be the actual disturbance and let $x(k+1) = f(x(k), u(k), w(k))$ be the state at time $k+1$.

Let $\check{x}(k+1|k), \ldots, \check{x}(k+1|k)$ be the predicted state sequence for the chosen input sequence and take $j \in \mathbb{Z}_{1:N}$ as the minimum value such that $\check{x}(k+j|k) \in \Omega_I$ for any disturbance sequence. Then, if $j = 1$ it results that $\check{x}(k+1|k) \in \Omega_I$ for any disturbance $\hat{w}(k|k)$. Thus, $x(k+1) \in \Omega_I$ and, according to Definition 3, an admissible input sequence exists for $x(k+1)$.

In case $j > 1$, consider the input sequence $\hat{u}(k+i|k+1) = \check{u}(k+i|k)$ for $i \in \mathbb{Z}_{1:j-1}$. Then, according to the definition of j, it results that $\hat{x}(k+j|k+1) \in \Omega_I$. Thus, from Definition 3, an input sequence $\hat{u}(k+j|k+1), \ldots, \hat{u}(k+N|k+1) \in U$ exists such that $\hat{x}(k+i|k+1) \in \Omega_O$ for all $i \in \mathbb{Z}_{j+1:N+1}$ for any disturbance sequence.

This implies that starting from $x(k+1)$ an admissible input sequence exists $\hat{u}(k+1|k+1), \ldots, \hat{u}(k+N|k+1) \in U$ such that the predicted states pass through Ω_I and after that never abandon Ω_O within the prediction horizon. □

The next lemma shows that, provided that the initial state is already in Ω_O, admitting only input sequences that keep the state inside Ω_O and pass through Ω_I allows keeping forever the state inside the target set Ω_O ensuring practical stability.

Lemma 16. *Consider the system of Eq.* (1)*, an outer set* Ω_O *and an associated* N*-step control inner set* Ω_I*. Suppose that in a MPC scheme the admissible input sequences* $\hat{u}(k|k), \ldots, \hat{u}(k+N-1|k) \in U$ *are those ensuring that, for any disturbance sequence, the predicted states verify* $\hat{x}(k+j|k) \in \Omega_I$ *for some* $j \in \mathbb{Z}_{1:N}$ *and* $\hat{x}(k+i|k) \in \Omega_O \ \forall i \in \mathbb{Z}_{1:N}$*. Then, if the state* $x(k) \in \Omega_O$ *is such that an admissible input sequence exists, then* $x(k+1) \in \Omega_O$ *and an admissible input sequence exists also for* $x(k+1)$.

This lemma, whose proof is very similar to that of the previous one, tells that given a target region Ω_O, a predictive control scheme can be formulated such that the state remains bounded to that region even when it is not invariant. Moreover, it can be easily seen that the state is in fact bounded to the maximal control invariant set $\Omega_I^{\max}$ without actually knowing the latter set.

In order to exploit these features in the design of a controller, given the target region Ω_O we need to compute an inner set Ω_I for which Theorem 13 can be used. In practice, the N-step control inner set Ω_I should be *large*. Otherwise, the horizon length N could result too long or the feasibility region may result too small. The reason is that Ω_I must be reached in up to N-steps from the initial state, so using a small inner set may limit the region from which it is reached within the prediction horizon.

4 Application Example

This section presents a numerical example that illustrates the advantages of replacing a RCIS by an outer target region and a N-step control inner set. We consider a nonlinear discrete-time perturbed system with a finite input set, given by:

$$
\begin{aligned}
x_1(k+1) &= x_1(k) + a_1(1-x_3(k))x_2(k) + a_2 + w_1(k),\\
x_2(k+1) &= a_3(1-x_3(k))x_1(k) + a_4 x_2(k) + w_2(k),\\
x_3(k+1) &= x_3(k) + a_5 u(k),
\end{aligned} \tag{8}
$$

where the state is $x(k) = [x_1(k), x_2(k), x_3(k)]^T \in X \subseteq \mathbb{R}^3$, the input is $u(k) \in U = \{-1, 1\}$ and $w(k) = [w_1(k), w_2(k)]^T \in W = \{w \in \mathbb{R}^2 : |w_1| \leq 1 \cdot 10^{-4},\ |w_2| \leq 2 \cdot 10^{-3}\}$ is a perturbation. Parameters are $a_1 = -0.01$, $a_2 = 0.1$, $a_3 = 10$, $a_4 = 0.5$, and $a_5 = 0.02$. This system represents the model of a boost converter, being x_1 the inductance current, x_2 the capacitor voltage and x_3 the duty cycle. According to the last state equation, each control action modifies the duty cycle in discrete steps of $\pm a_5 = \pm 0.02$.

We shall suppose that the target region is the outer set $\Omega_O = \{x \in X : |x_1 - \bar{x}_1| \leq 0.06,\ |x_2 - \bar{x}_2| \leq 0.5,\ |x_3 - \bar{x}_3| \leq 0.015\}$, where $\bar{x} = [\bar{x}_1, \bar{x}_2, \bar{x}_3]^T = [2, 20, 0.5]^T$. Given the system non-linearity, the computation of a RCIS contained in Ω_O is not simple. Moreover, it can be seen that any RCIS inside Ω_O is not connected (and therefore non-convex). The reason is that if $|x_3 - \bar{x}_3| < 0.005$, then there is no control action in U that keeps the state inside Ω_O. In addition, when $(x_3 - \bar{x}_3) \in [0.005, 0.015]$ the only control action that keeps the state in Ω_O is $U = -1$, and after that action we have $(x_3 - \bar{x}_3) \in [-0.015, -0.005]$. The inverse situation occurs when $(x_3 - \bar{x}_3) \in [-0.015, -0.005]$. Thus, the only possible way of keeping the state inside Ω_O is alternating the control action which drives the state between two disconnected regions.

In spite of the fact that any RCIS is non-connected, a simple control inner set could be found with just six faces. Taking $\Omega_I = \{x \in X : Fx \leq g\}$, where

$$F = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 2.53 \cdot 10^{-2} & -9.9967 \cdot 10^{-1} \\ 1.24 \cdot 10^{-1} & -3.41 \cdot 10^{-4} & -9.92 \cdot 10^{-1} \\ -1.24 \cdot 10^{-1} & 3.41 \cdot 10^{-4} & 9.92 \cdot 10^{-1} \\ 0 & -2.53 \cdot 10^{-2} & 9.9967 \cdot 10^{-1} \\ 0 & 0 & -1 \end{bmatrix},$$

$g = \begin{bmatrix} 49.5 & 1.5716 & -24.318 & 24.576 & -1.0652 & -48.5 \end{bmatrix}^T \cdot 10^{-2}$, it can be verified that this set is indeed a 10-step control inner set. This and the outer set are shown in Figure 2 with some state evolutions.

In order to show the usage of these sets in the context of MPC, we designed a very simple controller, where the input trajectories within the prediction horizon (of length $H = N = 10$) were restricted to satisfy the constraints of Lemma 15 before reaching Ω_I. This way, recursive feasibility was ensured. After Ω_I was reached, the constraints were those of Lemma 16 ensuring that the trajectories never leave the outer region. Besides verifying the constraints, the input trajectories were chosen such that the cost function

$$J(\hat{\mathbf{u}}(k)) = \sum_{i=1}^{H} \|\hat{x}(k+i|k) - \bar{x}\|_2 \qquad (9)$$

is minimized. Here, the feasible input sequences are written as $\hat{\mathbf{u}}(k) = [\hat{u}(k|k), \hat{u}(k+1|k), \ldots, \hat{u}(k+N-1|k)]$, and the predicted states are computed considering the nominal case as

$$\hat{x}(k+i+1|k) = f(\hat{x}(k+i|k), \hat{u}(k+i|k), 0)$$

for $i \in \mathbb{Z}_{0:N-1}$ with $\hat{x}(k|k) = x(k)$ and being $f(\cdot)$ given by the right hand side of Eq. (8). Then, the input at time k is selected as $u(k) = \hat{u}^*(k|k)$ where the sequence $\hat{\mathbf{u}}^*(k)$ is the one that minimizes Eq. (9).

Figure 2 shows the state trajectories that were obtained in the simulation of the resulting scheme from different initial states that verify the aforementioned restrictions. These state trajectories verify that, as stated in Lemmas 15 and 16, recursive feasibility and practical stability are achieved using only the simple inner and outer sets rather than a complicated non-connected RCIS.

Notice that while Ω_I is a 10-step control inner set, the trajectories of Figure 2 enter back that set after only two steps. However, for initial states very close to the vertices of Ω_I, there are possible disturbance trajectories that do not allow the state to be back in that set before 10 steps.

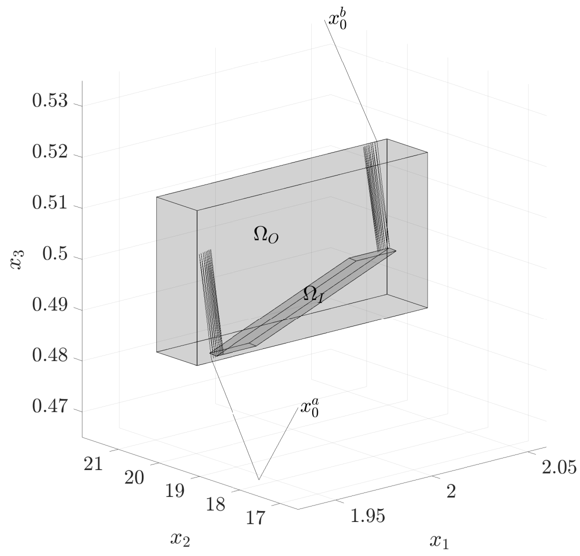

Fig. 2. MPC state trajectories.

5 Conclusions

We introduced a methodology to replace the use of complicated robust positive control invariant sets by simpler outer and inner approximations for control design purposes. We showed that given a target outer set Ω_O, instead of computing and using a complicated RCIS contained in its interior, we may use a simpler N-step control inner set Ω_I and still ensure that the state trajectories will not leave the target region. We also provided theoretical results that relate N-step control inner sets with the existence of robust positive control invariant sets and some more practical results that can help to construct and characterize these inner sets. In addition, we showed how these inner and outer approximations can be used in the context of model predictive controllers as a way of replacing the usual terminal restriction to a RCIS that ensures practical stability, illustrating the ideas with a numerical example.

Regarding future work, we are currently working on specific MPC schemes that exploit the presence of inner and outer sets to ensure practical stabilization and finite time convergence to the target region. In addition, we are working in the design of systematic procedures to compute inner set approximations in some particular cases (linear systems with finite input sets, for instance).

References

[1] Alejandro Anderson, Alejandro H. González, Antonio Ferramosca, and Ernesto Kofman. Finite-time convergence results in robust model predictive control. *Optimal Control Applications and Methods*, 39(5):1627–1637, 2018.

[2] Nikolaos Athanasopoulos, Alina I Doban, and Mircea Lazar. On constrained stabilization of discrete-time linear systems. In *21st Mediterranean Conference on Control and Automation*, pages 830–839. IEEE, 2013.

[3] Dimitri P. Bertsekas. Infinite time reachability of state-space regions by using feedback control. *IEEE Transactions on Automatic Control*, 17(5):604–613, 1972.

[4] George Bitsoris and Eliana Gravalou. Design techniques for the control of discrete-time systems subject to state and control constraints. *IEEE transactions on automatic control*, 44(5):1057–1061, 1999.

[5] Franco Blanchini. Ultimate boundedness control for uncertain discrete-time systems via set-induced lyapunov functions. *IEEE Transactions on Automatic Control*, 39(2):428–433, 1994.

[6] Franco Blanchini. Set invariance in control. *Automatica*, 35(11):1747–1767, 1999.

[7] Franco Blanchini, Fouad Mesquine, and Stefano Miani. Constrained stabilization with an assigned initial condition set. *International Journal of Control*, 62(3):601–617, 1995.

[8] Franco Blanchini and Stefano Miani. *Set-theoretic methods in control*, volume 78. Birkhäuser Cham, 2008.

[9] José M. Bravo, Daniel Limon, Teodoro Alamo, and Eduardo F. Camacho. On the computation of invariant sets for constrained nonlinear systems: An interval arithmetic approach. *Automatica*, 41(9):1583–1589, 2005.

[10] Benjamin Decardi-Nelson and Jinfeng Liu. Computing robust control invariant sets of constrained nonlinear systems: A graph algorithm approach. *Computers & Chemical Engineering*, 145:107177, 2021.

[11] Masood Dehghan and Chong-Jin Ong. Discrete-time switching linear system with constraints: Characterization and computation of invariant sets under dwell-time consideration. *Automatica*, 48(5):964–969, 2012.

[12] Mirko Fiacchini, Teodoro Alamo, and Eduardo F. Camacho. On the computation of convex robust control invariant sets for nonlinear systems. *Automatica*, 46(8):1334–1338, 2010.

[13] Elmer G. Gilbert and Kok T. Tan. Linear systems with state and control constraints: the theory and application of maximal output admissible sets. *IEEE Transactions on Automatic Control*, 36(9):1008–1020, 1991.

[14] Per-Olof Gutman and Michael Cwikel. An algorithm to find maximal state constraint sets for discrete-time linear dynamical systems with bounded controls and states. *IEEE Transactions on Automatic Control*, 32(3):251–254, 1987.

[15] S Keerthi and E Gilbert. Computation of minimum-time feedback control laws for discrete-time systems with state-control constraints. *IEEE Transactions on Automatic Control*, 32(5):432–435, 1987.

[16] Eric C. Kerrigan. *Robust constraint satisfaction: Invariant sets and predictive control.* PhD thesis, University of Cambridge UK, 2001.

[17] Markus Kögel and Rolf Findeisen. Stability of nmpc with cyclic horizons. *IFAC Proceedings Volumes*, 46(23):809–814, 2013.

[18] Milan Korda, Didier Henrion, and Colin N. Jones. Convex computation of the maximum controlled invariant set for polynomial control systems. *SIAM Journal on Control and Optimization*, 52(5):2944–2969, 2014.

[19] Alexander B. Kurzhanski and Pravin Varaiya. Ellipsoidal techniques for reachability analysis. In *International workshop on hybrid systems: Computation and control*, pages 202–214. Springer, 2000.

[20] Joseph La Salle and Solomon Lefschetz. *Stability by Liapunov's Direct Method with Applications by Joseph L Salle and Solomon Lefschetz.* Elsevier, 2012.

[21] Mircea Lazar, Alina I Doban, and Nikolaos Athanasopoulos. On stability analysis of discrete-time homogeneous dynamics. In *2013 17th International Conference on System Theory, Control and Computing (ICSTCC)*, pages 297–305. IEEE, 2013.

[22] Mircea Lazar and Martin Tetteroo. Computation of terminal costs and sets for discrete–time nonlinear mpc. *IFAC-PapersOnLine*, 51(20):141–146, 2018.

[23] Young I. Lee, Mark Cannon, and Basil Kouvaritakis. Extended invariance and its use in model predictive control. *Automatica*, 41(12):2163–2169, 2005.

[24] Young Il Lee and Basil Kouvaritakis. Constrained robust model predictive control based on periodic invariance. *Automatica*, 42(12):2175–2181, 2006.

[25] Benoît Legat, Paulo Tabuada, and Raphaël M Jungers. Computing controlled invariant sets for hybrid systems with applications to model-predictive control. *IFAC-PapersOnLine*, 51(16):193–198, 2018.

[26] Chengyuan Liu, Furqan Tahir, and Imad M. Jaimoukha. Full-complexity polytopic robust control invariant sets for uncertain linear discrete-time systems. *International Journal of Robust and Nonlinear Control*, 29(11):3587–3605, 2019.

[27] Warody Lombardi, Sorin Olaru, Georges Bitsoris, and Silviu-Iulian Niculescu. Cyclic invariance for discrete time-delay systems. *Automatica*, 48(10):2730–2733, 2012.

[28] David Q. Mayne, James B. Rawlings, Christopher V. Rao, and Pierre O. M. Scokaert. Constrained model predictive control: Stability and optimality. *Automatica*, 36(6):789–814, 2000.

[29] Pierre-Jean Meyer, Alex Devonport, and Murat Arcak. *Interval Reachability Analysis: Bounding Trajectories of Uncertain Systems with Boxes for Control and Verification.* Springer Nature, 2021.

[30] Meeko Oishi, Ian Mitchell, Claire Tomlin, and Patrick Saint-Pierre. Computing viable sets and reachable sets to design feedback linearizing control laws under saturation. In *Proceedings of the 45th IEEE Conference on Decision and Control*, pages 3801–3807. IEEE, 2006.

[31] Saša V. Rakovic and Miroslav Baric. Parameterized robust control invariant sets for linear systems: Theoretical advances and computational remarks. *IEEE Transactions on Automatic Control*, 55(7):1599–1614, 2010.

[32] Matthias Rungger and Paulo Tabuada. Computing robust controlled invariant sets of linear systems. *IEEE Transactions on Automatic Control*, 62(7):3665–3670, 2017.

[33] Mohamed A. B. Sassi, Antoine Girard, and Sriram Sankaranarayanan. Iterative computation of polyhedral invariants sets for polynomial dynamical systems. In *53rd IEEE Conference on Decision and Control*, pages 6348–6353. IEEE, 2014.

[34] Francesco Scibilia, Sorin Olaru, and Morten Hovd. On feasible sets for mpc and their approximations. *Automatica*, 47(1):133–139, 2011.

[35] Martin Soyer, Sorin Olaru, and Zhou Fang. A novel motion cueing algorithm based on real-time optimization and periodic invariant sets. In *2020 IEEE Conference on Control Technology and Applications (CCTA)*, pages 19–24. IEEE, 2020.