Finite-time convergence results in robust Model Predictive Control

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Abstract

Robust asymptotic stability (asymptotic attractivity and ϵ-δ stability) of equilibrium regions under robust model predictive control (RMPC) strategies was extensively studied in the last decades making use of Lyapunov theory in most cases. However, in spite of its potential application benefits, the problem of finite-time convergence under fixed prediction horizon has not received - with some few exceptions - much attention in the literature.

Considering the importance in several applications of having finite-time convergence results in the context of fixed horizon MPC controllers and the lack of studies on this matter, this work presents a new set-based robust MPC for which, in addition to traditional stability guarantees, finite-time convergence to a target set is proved and moreover, an upper bound on the time necessary to reach that set is provided.

It is remarkable that the results apply to general nonlinear systems and only require some weak assumptions on the model, cost function and target set.

Keywords: Predictive Control; Robust Control; Discrete-Time Systems; Optimal Control Application.

1. Introduction

The main features of Model Predictive Control (MPC) - which make it the most employed advanced control technique in process industries - are the explicit consideration of the model, the optimal computation of the control moves, and its ability to handle, easily and effectively, hard constraints on control and states. MPC theoretical background has been widely investigated in the last two decades, showing how this technique is capable to provide stability, robustness, constraint satisfaction and tractable computation for linear and nonlinear systems [1]. Researchers achieved a consensus on that Lyapunov theory [2] is the most suitable framework to prove asymptotic stability (asymptotic attractivity and ϵ-δ stability) of certain equilibrium points [3]. To this aim, different stabilizing formulations can be found in literature: *MPC with terminal equality constraint*; *MPC with terminal cost*; or *MPC with terminal inequality constraint*. All these formulations - despite the possible use of terminal invariant sets for stability proposes - are devoted to steer the system (not in finite time) to a single equilibrium point of interest. In [4, 5], the desired equilibrium point is generalized to an entire equilibrium set, but this set does not include transient states (i.e., it is not possible in general to go from one point to any other without leaving the set).

Set invariance theory ([6, 7, 8, 9, 10]) is a very useful tool for analyzing dynamical systems subject to constraints allowing the extension of stability concepts from equilibrium points to equilibrium state space regions (i.e. invariant sets). In [1], for instance, a Lyapunov function for asymptotic stability of general invariant sets is defined, although no controller formulations are presented to explicitly account for it. In this context, set-based MPC strategies [11] represent a different concept in the field of MPC strategies, since a general invariant set is considered as control objective; i.e., once the system reaches this set, no further control actions are injected to it, meaning that the system is kept in open-loop inside the invariant set. This strategy is particularly suitable for those applications characterized by the execution of a certain task inside the target region: switching to another simpler control strategy; leaving the system in open-loop, provided that an equilibrium in the region is strongly stable; persistently exciting the system with an appropriated input signal to collect input-output data for identification ([11, 12, 13]); etc. All these potential applications, however, need finite-time convergence guarantees to ensure that the second task (which is performed inside the target region) will certainly begin.

Despite the importance of finite-time convergence, both in practice and theory, only some (few) works regarding this result - in the context of fixed control horizon formulation - can be found in the MPC literature: in [14] it is shown that, if the MPC stage cost function is bounded from below by a $\mathcal{K}$-function, then finite-time convergence to a certain terminal set can be ensured. In [15], in the context of min-max MPC, the authors propose an unconventional cost function, based on the idea of distance to a set, to show that finite-time convergence can be ensured by means of an assumption similar to the one of [14]. Other works show similar results through hard assumptions. For instance, the results in [16] are obtained making use of strong assumptions on the cost function and model system. Similarly, finite time convergence is achieved in [17] with practical certainty.

In contrast, this work presents a general set-based robust MPC formulation applicable to nonlinear disturbed systems that guarantees finite-time convergence to an invariant target set under mild assumptions. Moreover, the work provides an upper bound for the time of convergence to the target set that only depends on the initial state.

1.1. Notation

We denote, by $\mathbb{N}$ the set of natural numbers, by $\mathbb{I}$ the set of integer numbers, by $\mathbb{I}_{\geq 0}$ the set of non-negative integers, and by $\mathbb{I}_{N:M}$ the set of integers in the interval $[N, M]$. Given any real number $x \in \mathbb{R}$, the floor of x is defined by $\lfloor x \rfloor \doteq \max\{n \in \mathbb{I} : n \leq x\}$. The Euclidean distance between two points x, y on $\mathbb{R}^n$ is represented by $d(x, y)$. An open ball with center in $x \in \mathbb{R}^n$ and ratio $\varepsilon > 0$ is denoted as $\mathcal{B}_\varepsilon(x) \doteq \{y \in \mathbb{R}^n : d(x, y) < \varepsilon\}$. Let $\gamma \in \mathbb{R}$ be a constant and let $\Omega \subset \mathbb{R}^n$ be a set; then, $\gamma\Omega \doteq \{\gamma x : x \in \Omega\}$ is a scaled set of Ω. A point $x \in \Omega$ is an interior point of Ω if the there exists $\varepsilon > 0$ such that the open ball $\mathcal{B}_\varepsilon(x) \subset \Omega$. The interior of Ω is the set of all its interior points and it is denoted by Ω°. If Ω is a closed set, its boundary is denoted by $\partial\Omega$, and it is defined as $\partial\Omega \doteq \Omega \setminus \Omega^\circ$. The distance from a point $x \in \mathbb{R}^n$ to a set Ω is defined as $d_\Omega(x) \doteq \inf\{d(x, y) : \ y \in \Omega\}$. Notice that $d_\Omega(\cdot)$ is a convex and continuous function, and $d_\Omega(x) \geq 0$ for all $x \in \mathbb{R}^n$, while $d_\Omega(x) = 0$ if and only if $x \in \Omega$. A function $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is a $\mathcal{K}$-function if it is continuous, strictly increasing and $\alpha(0) = 0$.

2. Plant model and control scheme

This section introduces the model of the plant and the robust MPC scheme.

2.1. Plant model

Consider a discrete time disturbed system described by the following time-invariant model

$$x(k+1) = F(x(k), u(k), w(k)), \quad x(0) = x_0, \tag{1}$$

where $x(k) \in X \subset \mathbb{R}^n$ is the system state at the k–th sample time, x_0 is the initial state, $u(k) \in U \subset \mathbb{R}^m$ is the current control input and $w(k) \in W \subset \mathbb{R}^p$ is the current disturbance. We assume that set X is closed, set U is compact, and both sets contain the origin in their interior. Furthermore, we assume that W is compact and convex, and it contains the origin in its interior. We also assume that function $F : X \times U \times W \to X$ is continuous on $Z \doteq X \times U \times W$ and $F(0, 0, 0) = 0$.

2.2. Robust MPC scheme with fixed control horizon

For a given (fixed) horizon $N \in \mathbb{N}$, and a given compact and convex set $\Omega \subseteq X$, that contains the origin in its interior, the following (set-dependent) cost function is proposed:

$$V_N(x, \Omega; \mathbf{u}) \doteq \max_{\mathbf{w} \in W^N} \sum_{j=0}^{N-1} L(x(j), u(j); \Omega), \tag{2}$$

where $x = x(0)$ is the current state; $x(j+1) = F(x(j), u(j), w(j))$, for $j \in \mathbb{I}_{0:N-1}$; $\mathbf{u} \doteq \{u(0), \cdots, u(N-1)\}$ is an input sequence; and $\mathbf{w} \doteq \{w(0), \cdots, w(N-1)\}$

is a disturbance realization. Furthermore, it is assumed that $L(\cdot) \geq 0$, and $L(x, \cdot, \cdot) = 0$ when $x \in \Omega$. Ω is the *target set* where we want the closed-loop system to converge to in finite-time.

Let

$$\mathcal{X}_N = \{x \in X \mid x(j) \in X, u(j) \in U,\ j = 0, 1, ..N-1 \text{ and } x(N) \in \Omega,\ \forall \mathbf{w} \in W^N\},$$

be the initially feasible region. Then, for all $x \in \mathcal{X}_N$ we can denote by $\mathcal{U}_N(x)$ the set of control sequences, $\mathbf{u}$, satisfying the state and control constraints ($x(j) \in X, u(j) \in U$, for $j \in \mathbb{I}_{0:N-1}$), together with a terminal constraint of the form $x(N) \in \Omega$, for every admissible disturbance sequence $\mathbf{w} \in W^N$, when the initial state is x. By Definition 1, for all $i \geq 0$, the set $\mathcal{C}^i(\Omega, U)$ denotes the set of states x such that $\mathcal{U}_i(x) \neq \emptyset$. At each time instant k, the Robust MPC control law is derived from the solution of the following optimization problem:

$$\mathcal{P}_N(x, \Omega):\ V^0(x, \Omega) \doteq \min\{V_N(x, \Omega; \mathbf{u}) \mid \mathbf{u} \in \mathcal{U}_N(x)\} \tag{3}$$

where Ω and the initial sate $x \in X$ are the optimization parameters and the sequence $\mathbf{u}$ is the optimization variable.

Associated to the optimal control sequence $\mathbf{u}^0(x) \doteq \{u^0(0; x), u^0(1; x), \ldots, u^0(N-1; x)\}$ there is a bundle of optimal state trajectories $\{\mathbf{x}^0(x, \mathbf{w})\}$, where each trajectory corresponds to an admissible disturbance realization $\mathbf{w}$:

$$x^0(x, \mathbf{w}) \doteq \{x^0(0; x, \mathbf{w}), x^0(1; x, \mathbf{w}), \ldots, x^0(N; x, \mathbf{w})\} \in \{\mathbf{x}^0(x, \mathbf{w})\}. \tag{4}$$

By definition of $\mathcal{P}_N(x, \Omega)$, $x^0(N; x, \mathbf{w}) \in \Omega$ for each admisible $\mathbf{w}$. The control law, derived from the application of a receding horizon control policy (RHC) is given by $\kappa_{MPC}(x) = u^0(0; x)$, where $u^0(0; x)$ is the first element of the solution sequence $\mathbf{u}^0(x)$. This way, the closed-loop system under the Robust MPC law is described as:

$$x(k+1) = F(x(k), \kappa_{MPC}(x(k)), w(k)). \tag{5}$$

and the optimal cost function is given by:

$$V_N^0(x, \Omega) = V_N(x, \Omega, \mathbf{u}^0(x)). \tag{6}$$

3. Convergence analysis

3.1. Some previous results

In [14] and [18] the finite-time convergence to Ω, for the nominal case, $W = \{0\}$, is ensured by imposing the following conditions to the stage cost: (i) there exits a $\mathcal{K}$-function $\ell(\cdot)$ such that $L(x, u; \Omega) \geq \ell(\|(x, u)\|)$ for all $x \notin \Omega$ and for all $u \in U$; and (ii) $L(x, h_L(x); \Omega) = 0$ for all $x \in \Omega$, where $h_L(\cdot)$ is a local control law imposed once the state enters the target set Ω. Furthermore, Ω is assumed to be an invariant set for $x(k+1) = F(x(k), h_L(x(k)), 0)$, which establishes an undesired dependence of Ω on the arbitrary control law $h_L(x)$.

However, given that Ω is assumed to contain the origin in its (non empty) interior, assumptions (i) and (ii) imply that function $L(\cdot)$ is discontinuous on the boundary of Ω, which is a strong assumption that can produce some problems.

The way this formulation ensures finite-time convergence is summarized as follows. By usual procedures in MPC stability theory, the optimal cost function, $V_N^0(x,\Omega) = V_N(x,\Omega;\mathbf{u}^0)$, is shown to satisfy

$$V_N^0(x(k+1),\Omega) - V_N^0(x(k),\Omega) \leq -L(x(k),\kappa_{MPC}(x(k));\Omega), \qquad \forall x(k) \notin \Omega,$$

Given that for all $x \notin \Omega$ there is $r > 0$ such that $\|x\| > r$, by assumption (i), it follows that

$$L(x,u;\Omega) \geq \ell(\|(x,u)\|) \geq \ell(\|x\|) \geq \ell(r), \ \forall x \notin \Omega,$$

and then

$$V_N^0(x(k+1),\Omega) - V_N^0(x(k),\Omega) \leq -\ell(r). \tag{7}$$

This means that at each step there is a cost drop of at least $\ell(r) > 0$, for $x \notin \Omega$, which is much stronger than the usual radial cost drop where the drop magnitude depends on the distance to Ω. That way, the finite-time convergence to Ω is achieved.

The main drawback of the proposal in [14] and [18] is clearly the strong assumption (i), which leads to a discontinuous stage cost. The use of discontinuous stage costs is a major obstacle for implementation [15] when using standard solvers for linear, quadratic, semi-definite or other smooth, convex nonlinear programming problems. To overcome this drawback, [15] proposes - in the context of robust linear MPC, i.e., with $W \neq \{0\}$ - the following stage cost:

$$L(x,u;\Omega) \doteq \min_{y\in\Omega} \|Q(x-y)\|_p + \|R(u-h_L(x))\|_p,$$

with $Q \in \mathbb{R}^{n\times n}$ and $R \in \mathbb{R}^{m\times m}$ being weighting matrices and $\|\|_p$ being a particular norm. The system is given by $x(k+1) = Ax(k) + Bu(k) + w(k)$ with $w(k) \in W$, and the local control law $h_L(x)$ is a fixed linear feedback gain $K \in \mathbb{R}^{m\times n}$. The set Ω is here a disturbance invariant set for $x(k+1) = Ax(k) + BKx(k) + w(k)$, and again, it depends on K, as it is usual in the dual MPC context.

This cost is continuous, and, in fact, it does not meet condition (i). However, this formulation ensures finite-time convergence by assuming the following control law:

$$\begin{cases} \kappa_{MPC}(x) & \text{if } x \in X \setminus \mathcal{O}_\infty, \\ h_L(x) & \text{if } x \in \mathcal{O}_\infty, \end{cases}$$

where $\mathcal{O}_\infty$ is the the maximal disturbance invariant set (MDIS). If it is assumed that $\Omega \subset \mathcal{O}_\infty^\circ$, then the minimal disturbance invariant set (mDIS) $\mathcal{F}_\infty \subset \mathcal{O}_\infty$ is asymptotically stable for the closed-loop system with the control law defined above. In addition, if $\mathcal{F}_\infty \subset \Omega^\circ$, there is a finite time convergence to the set Ω. This way, the real target set is given by $\mathcal{F}_\infty$, and so, the finite-time convergence to Ω is trivially achieved by the asymptotic convergence to $\mathcal{F}_\infty$ and the assumption $\mathcal{F}_\infty \subset \Omega^\circ$.

Remark 1. *As a particular case, if the nominal scenario is considered ($W = \{0\}$), then set mDIS is given by $\mathcal{F}_\infty = \{0\}$, and classical attractivity of the origin is achieved. So, any set including the origin in its interior - as the proposed Ω - will trivially be finite-time attractive.*

Other works devoted to achieve finite-time convergence in the context of dual-mode model predictive control are [19] and [20]. However, these approaches consider a variable-horizon MPC formulation, which constitutes a different scenario to the one considered in this work.

Before introducing our main results, we give below some previous definitions.

3.2. Controllable set definitions and properties

The next definitions and properties will be referred to system (1), and the corresponding state, input and disturbance constraints.

Definition 1. ***(One-step disturbance controllable set)*** *Given two sets $\Omega \subset X$ and $\Delta \subset U$, the one step disturbance controllable set to Ω corresponding to Δ, $\mathcal{C}(\Omega, \Delta)$, is the set of all $x \in X$ for which there exists $u \in \Delta$ such that $F(x, u, w) \in \Omega$, for all $w \in W$, i.e.,*

$$\mathcal{C}(\Omega, \Delta) \doteq \{x \in X : \exists\ u \in \Delta \textit{ such that } F(x, u, w) \in \Omega, \forall w \in W\}.$$

In other words, $\mathcal{C}(\Omega, \Delta)$ is the set of states in X for which an admissible control input in set Δ exists, that guarantees that the system will be driven to Ω in one step, independently of the (bounded) disturbance effect.

This concept can be generalized to the N-step disturbance controllable set $\mathcal{C}^N(\Omega, \Delta)$, for any $N \in \mathbb{N}$, by applying the above definition recursively, i.e., $\mathcal{C}^n(\Omega, \Delta) \doteq \mathcal{C}(\mathcal{C}^{n-1}(\Omega, \Delta), \Delta)$, for $n = 1, \cdots, N$, with $\mathcal{C}^0(\Omega, \Delta) \doteq \Omega$.

Definition 2. ***(Disturbance γ-control invariant set, Dγ-CIS)*** *Given $\gamma \in (0, 1]$, the set $\Omega \subseteq X$ is a disturbance γ-control invariant set if for all $x \in \Omega$ there exists an input $u \in U$ such that $F(x, u, w) \in \gamma\Omega$, for all $w \in W$. Associated to the Dγ-CIS Ω, is the corresponding input set $\Pi(\Omega) \doteq \{u \in U : \ \exists\ x \in \Omega$ such that $F(x, u, w) \in \gamma\Omega,\ \forall w \in W\}$.*

This set Ω is such that once the system enters it, there exists an admissible control input, belonging to $\Pi(\Omega)$, that is able to keep the system inside the set, independently of the (bounded) disturbance effect. When $\gamma = 1$, the latter set is simply denoted as disturbance control invariant set (DCIS).

From [21], it is known that every Dγ−CIS, Ω, is by definition such that $\Omega \subseteq \mathcal{C}(\Omega, U)$. We shall need later a slightly stronger geometric condition telling that Ω is in the interior of $\mathcal{C}(\Omega, U)$. The property below establishes it.

Property 1. *Let $\Omega \subset X^{\circ}$ be a closed and convex Dγ-CIS, with $\gamma < 1$, containing the origin as an interior point, with the corresponding input set $\Pi(\Omega) \subseteq U$. Then, $\Omega \subseteq \mathcal{C}(\Omega, \Pi(\Omega))^{\circ}$.*

The proof of Property 1 can be found in the Appendix.

The idea in the next subsection is to show that if set Ω is selected to be a D$\gamma-$CIS for the open-loop system and its associated input set $\Pi(\Omega)$ is also considered in the stage cost, then a finite-time convergence to Ω can be ensured, with no further assumption on the controller formulation.

3.3. Main result

Consider the following set-based (in both, state and input sets) stage cost for a particular realization of the disturbance:

$$L(x, u; \Omega) = d_{\Omega}(x) + d_{\Pi(\Omega)}(u), \tag{8}$$

where the function $d_{\Omega}(x)$ is the euclidean distance from the state x to the set Ω, and $d_{\Pi(\Omega)}(u)$ is the euclidean distance from the control u to the set $\Pi(\Omega)$. Ω is assumed to be a compact and convex D$\gamma-$CIS for the open-loop system (1), with $\gamma < 1$, and containing the origin as an interior point (in such a way that Property 1 holds). Furthermore, the corresponding input set is denoted as $\Pi(\Omega)$.

Remark 2. *Note that according to this formulation, the control objective is considered to be reached once the system enters Ω, and no further implicit objectives are considered. However, the interesting point is that the system will not be completely in open-loop, since the controller will act every time the state goes outside Ω, by the effect of an unknown strong disturbance.*

The next Lemma establishes the asymptotic convergence of the closed loop derived from Problem (3), to Ω, when the stage cost is given by (8)[1].

Lemma 1. *Let $x = x(0) \in \mathcal{C}^N(\Omega, U)$. Consider the Robust MPC formulation $\mathcal{P}_N(x, \Omega)$, (3), with the stage cost (8). Then, the closed-loop system $x(k+1) = F(x(k), \kappa_{MPC}(x(k)), w(k))$ satisfies*

$$V_N^0(x(k+1), \Omega) - V_N^0(x(k), \Omega) \leq -d_{\Omega}(x(k)) - d_{\Pi(\Omega)}(u(k)),$$

for all $k \geq 0$, and so $\lim_{k\to\infty} d_{\Omega}(x(k)) = 0$ and $\lim_{k\to\infty} d_{\Pi(\Omega)}(u(k)) = 0$.

Proof: Let $x \in \mathcal{C}^N(\Omega, U)$, at a given time k. Suppose that the optimal cost function is given by

$$V_N^0(x, \Omega) = \max_{\mathbf{w} \in W^N} \sum_{j=0}^{N-1} d_{\Omega}(x^0(j; x)) + d_{\Pi(\Omega)}(u^0(j; x)), \tag{9}$$

[1] As it is known, convergence is the main condition to ensure stability by means of Lyapunov classical methods.

where $u^0(j;x)$, for $j \in \mathbb{I}_{0:N-1}$, is the optimal input trajectory, and $x^0(j;x)$ is the corresponding optimal state trajectory that belongs to a bundle of optimal states $\{x^0(j;x,\mathbf{w})\}$, one for each admissible disturbance realization $w \in W$.

Let $x^+ = x^0(1;x)$ be the successor state of x under the closed-loop system, so that $x^+ \in \{x^0(1;x,\mathbf{w})\}$ and $\{x^0(N;x,\mathbf{w})\} \subset \Omega$. One feasible solution to Problem $\mathcal{P}_N(x^+,\Omega)$ at time $k+1$ is given by the sequence $\hat{\mathbf{u}} \doteq \left\{u^0(1;x),\cdots,u^0(N-1;x),\hat{u}\right\}$, where $\hat{u}$ is a control action belonging to $\Pi(\Omega)$, such that $\hat{x} \doteq F(x^0(N;x),\hat{u},w) \in \Omega$ for all $w \in W$ (at least one input $\hat{u} \in \Pi(\Omega)$ exists since $x^0(N;x) \in \{x^0(N;x,\mathbf{w})\} \subset \Omega$, and Ω is a D$\gamma-$CIS). Since the cost function takes into account the worst case scenario at any time, the corresponding feasible cost function of Problem $\mathcal{P}_N(x^+,\Omega)$ can be written as

$$V_N(x^+,\Omega;\hat{\mathbf{u}}) = V_N^0(x,\Omega) + d_\Omega(\hat{x}) + d_{\Pi(\Omega)}(\hat{u}) - d_\Omega(x) - d_{\Pi(\Omega)}(u^0(0;x)),$$

where $d_\Omega(\hat{x}) = 0$ and $d_{\Pi(\Omega)}(\hat{u}) = 0$. Furthermore, by optimality is $V_N^0(x^+,\Omega) \leq V_N(x^+,\Omega;\hat{\mathbf{u}})$, which implies that

$$V_N^0(x^+,\Omega) - V_N^0(x,\Omega) \quad \leq \quad -d_\Omega(x) - d_{\Pi(\Omega)}(u^0(0;x)).$$

In other words, for any time $k \in \mathbb{I}_{\geq 0}$, the optimal cost function satisfies $V_N^0(x(k+1),\Omega) - V_N^0(x(k),\Omega) \leq -d_\Omega(x(k)) - d_{\Pi(\Omega)}(u(k))$ and, given that $V_N^0(\cdot)$ is positive, then it results that $\lim_{k\to\infty} d_\Omega(x(k)) = 0$ and $\lim_{k\to\infty} d_{\Pi(\Omega)}(u(k)) = 0$. ■

Before presenting the main result of this work, the following lemma - which is a particular property of the proposed set-based stage cost (8) - is introduced.

Lemma 2. *Consider the MPC formulation $\mathcal{P}_N(x,\Omega)$, (3), with the stage cost (8), and the control law, $\kappa_{MPC}(\cdot)$. If $x \in \mathcal{C}(\Omega,\Pi(\Omega))$ then $F(x,\kappa_{MPC}(x),w) \in \Omega$, for all $w \in W$.*

Proof: Let $x(0) = x \in \mathcal{C}(\Omega,\Pi(\Omega))$, so there is a control action $u(0) \in \Pi(\Omega)$ such that

$$x(1) = F(x(0),u(0),w(0)) \in \Omega, \quad \forall\ w(0) \in W$$

From the disturbance $\gamma-$invariance of the target set Ω, there exist control actions $u(k) \in \Pi(\Omega)$, $k = 1,\ldots,N-1$, for which $x(k) \in \gamma\Omega \subseteq \Omega$, for $k = 2,\ldots,N-1$, and for all $w(k) \in W$, $k = 1,2,\ldots,N-1$. The use of such a control sequence produces the cost

$$\begin{aligned} V_N(x,\Omega,\mathbf{u}) &= d_\Omega(x(0)) + \underbrace{d_{\Pi(\Omega)}(u(0))}_{=0} \\ &+ \underbrace{\max_{\mathbf{w}\in W^{N-1}} \sum_{j=1}^{N-1} (d_\Omega(x(j)) + d_{\Pi(\Omega)}(u(j)))}_{=0} \\ &= d_\Omega(x(0)) \end{aligned}$$

while any control action that leaves $x(1)$ outside Ω produces a cost greater than $d_\Omega(x(0))$. Thus, the MPC will drive the state to the target set in one step. ■

The result of the above Lemma is basically due to the fact that a control within the set $\Pi(\Omega)$ does not add positive cost to the stage cost (8) and it can drive the state to the set Ω from $\mathcal{C}(\Omega, \Pi(\Omega))$, and $\mathcal{C}(\Omega, \Pi(\Omega)) \setminus \Omega \neq \emptyset$ by Property 1. Furthermore, this is possible due to the continuity of System (1) and the contractivity of the target set Ω.

To illustrate the role of set $\Pi(\Omega)$ in the MPC formulation, Figures 1 and 2 show the closed-loop state evolution corresponding to two different starting point. The controller is given by the Robust MPC (Problem $\mathcal{P}(\Omega, x)$) with the stage cost (8), and the simulations were made in a nominal scenario, $W = \{0\}$, for clarity. The controller is stopped when the state enters the target set Ω. Note that when the initial state $x(0) \in \mathcal{C}(\Omega, \Pi(\Omega))$ (Fig. 1) the controller steers the state inside Ω in one time step, as Lemma 2 claims. On the other hand, if the initial state is in $x(0) \in \mathcal{C}(\Omega, U) \setminus \mathcal{C}(\Omega, \Pi(\Omega))$ - even when it is possible to reach it in only one time step, by the one step set definition - the MPC controller reaches Ω in two time steps (Fig. 2). The crucial point that this counterexample highlights is that the use of the set $\mathcal{C}(\Omega, U)$ in the stage cost (8) may lead to the loss of the finite-time convergence property. The model and MPC parameters of this scenario simulations are described in the appendix.

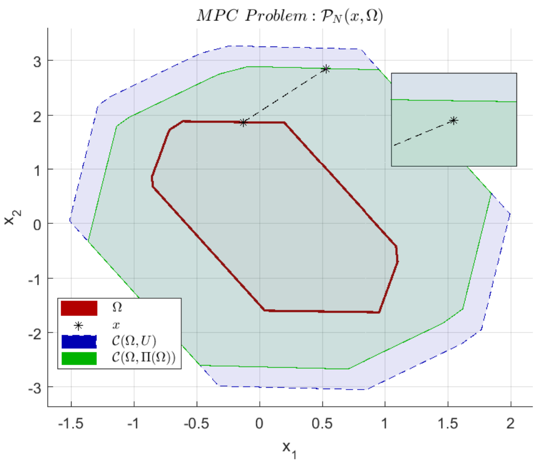

Figure 1: Closed-loop state evolution, starting at $\mathcal{C}(\Omega, \Pi(\Omega))$.

In the following theorem, the main result of this note is presented, where both, the finite-time convergence and an upper bound for this time is established.

Theorem 1. *Consider the MPC formulation $\mathcal{P}_N(x, \Omega)$, (3), with the stage cost (8). Then, for any $x = x(0) \in \mathcal{C}^N(\Omega, U)$, Ω is locally reached in finite-time by the system $x(k+1) = F(x(k), \kappa_{MPC}(x(k)), w(k))$. Moreover, the system reaches*

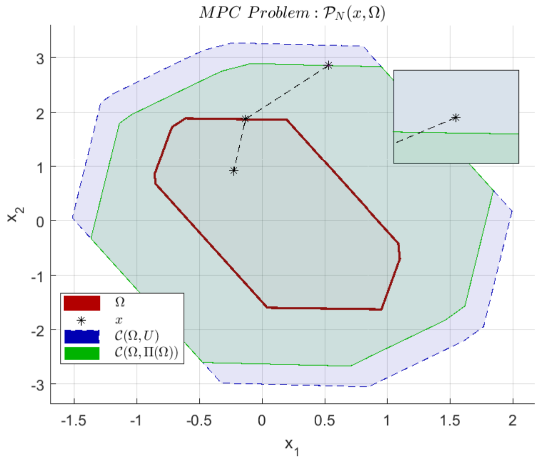

Figure 2: Closed-loop state evolution, starting at $\mathcal{C}(\Omega, U) \setminus \mathcal{C}(\Omega, \Pi(\Omega))$. The state does not reach the target set Ω in one step.

Ω in at most $\lfloor K \rfloor$ steps, with [2]

$$K = \frac{V_N^0(x,\Omega)}{\min\limits_{x \in \partial\mathcal{C}(\Omega,\Pi(\Omega))} d_\Omega(x)} + 1. \tag{10}$$

Proof: The proof proceeds by contradiction. Let consider an initial state $x(0) \in \mathcal{C}^N(\Omega, U) \setminus \mathcal{C}(\Omega, \Pi(\Omega))$ and a scalar m such that $m > \frac{V_N^0(x(0),\Omega)}{\min\limits_{x \in \partial\mathcal{C}(\Omega,\Pi(\Omega))} d_\Omega(x)} = K - 1$, and assume that $x(k) \notin \mathcal{C}(\Omega, \Pi(\Omega))$, for any $k = 1, 2, ..., m$. Then

$$-d_\Omega(x(k)) \leq - \min_{x \in \partial\mathcal{C}(\Omega,\Pi(\Omega))} d_\Omega(x) < 0, \quad k = 0, 1, ..., m. \tag{11}$$

Moreover, from Lemma 1, it follows that $V_N^0(x(k+1), \Omega) - V_N^0(x(k), \Omega) \leq -d_\Omega(x(k))$, which implies that

$$V_N^0(x(k+1), \Omega) - V_N^0(x(k), \Omega) \leq - \min_{x \in \partial\mathcal{C}(\Omega,\Pi(\Omega))} d_\Omega(x) < 0, \tag{12}$$

for $k = 0, 1, ..., m$. Summing up the terms of latter inequality from $k = 0$ to m, it follows that

$$V_N^0(x(m), \Omega) - V_N^0(x(0), \Omega) \leq -m \min_{x \in \partial\mathcal{C}(\Omega,\Pi(\Omega))} d_\Omega(x), \tag{13}$$

[2]Note that K is well defined since, by Property 1, $\Omega \subset \mathcal{C}(\Omega, \Pi(\Omega))^\circ$, and so $\min\limits_{x \in \partial\mathcal{C}(\Omega,\Pi(\Omega))} d_\Omega(x) \neq 0$.

which means that

$$V_N^0(x(m),\Omega) \le -m \min_{x\in\partial\mathcal{C}(\Omega,\Pi(\Omega))} d_\Omega(x) + V_N^0(x(0),\Omega) < 0 \tag{14}$$

This latter inequality represents a contradiction and, so, $x(k)$ must be inside $\mathcal{C}(\Omega,\Pi(\Omega))$ for some $k \le K-1$. Furthermore, Lemma 2 ensures that $x(k) \in \Omega$ for some $k \le K$, which concludes the proof. ∎

Remark 3. *Note that the only requirement for the target set in the latter result is that Ω needs to be a γ-invariant set, with $0 < \gamma < 1$. So, if one has a desired equilibrium set-point x_{sp}, it is possible to select Ω as $\hat{\Omega} \oplus x_{sp}$, where $\hat{\Omega}$ is an arbitrary small γ-invariant set (large enough to ensure that Property 1 still holds), with an arbitrary small value of γ. In this case, Ω approximates the equilibrium set-point, and the set-point tracking problem is (approximately) recovered.*

4. Simulation results

Although the main contribution of the work is of theoretical nature, this section introduces simple simulation results to illustrate the way finite–time convergence is achieved. The selected system to test the controller is the double integrator $d^2y(t)/dt^2 = u(t)$, which comes from a simple mass in one-dimensional space y, under the effect of a time-varying force input u. By means of a discretization (with a sampling time of $T = 1sec$) the following discrete-time system is obtained:

$$x(k+1) \quad = \quad Ax(k) + Bu(k),$$

$$\text{where} \qquad A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \qquad B = \begin{bmatrix} 0.5 \\ 1 \end{bmatrix},$$

$x = [x_1,\ x_2]^T$, with x_1, x_2 being the position and velocity of the mass, respectively. The inputs an state constraints are given by $U = \{u \in \mathbb{R} : \|u\|_\infty \le 4\}$, $X = \{(x_1,x_2) \in \mathbb{R}^2 : \|x_1\|_\infty \le 15,\ \|x_2\|_\infty \le 12\}$. For the sake of simplicity, only the nominal case will be considered as the goal is to show the finite-time convergence, a property that is independent of the controller robustness.

The MPC controller is derived from Problem (3), with the stage cost given by (8), where Ω is an arbitrary γ-invariant set (see Figure 3), with $\gamma < 1$, and $\Pi(\Omega)$ is its corresponding input set according to Definition 2. The simulations, running along 10 time steps, start at four different initial states outside the target set Ω: $x_{01} = [-7,\ -7]^T$, $x_{02} = [7,\ 7]^T$, $x_{03} = [-15,\ 11]^T$ and $x_{04} = [15,\ -11]^T$.

The following remarks can be made after the trajectories shown in Figures 3 and 4:

- The controlled system reaches the target set in a number of steps significantly smaller than the upper bound computed by (10), which is greater than 50 in all the cases.
- Given that Ω is a γ-invariant set, with $\gamma < 1$, it is contained in the interior of $\mathcal{C}(\Omega, \Pi(\Omega))$.
- The state of the system does not converges to the origin, but to an arbitrary point in the steady state set X_{ss} (in black, in Figure 3), given that no penalization is made once it enters Ω.
- Every time the controlled system enters $\mathcal{C}(\Omega, \Pi(\Omega))$, the set Ω is reached after one step, as claimed in Lemma 2.
- For the initial state $x_{01} = [-7,\ -7]^T$ (or $x_{03} = [7,\ 7]^T$), the system enters $\mathcal{C}(\Omega, U)$, at time step 3. Then, although it can reach Ω in the next time step according to the open-loop dynamics, it does not do it because it is not the optimal path. This is another case reinforcing the counterexample of subsection 3.3.
- The costs are null once the system enters Ω, as it can be seen in Figure 4.

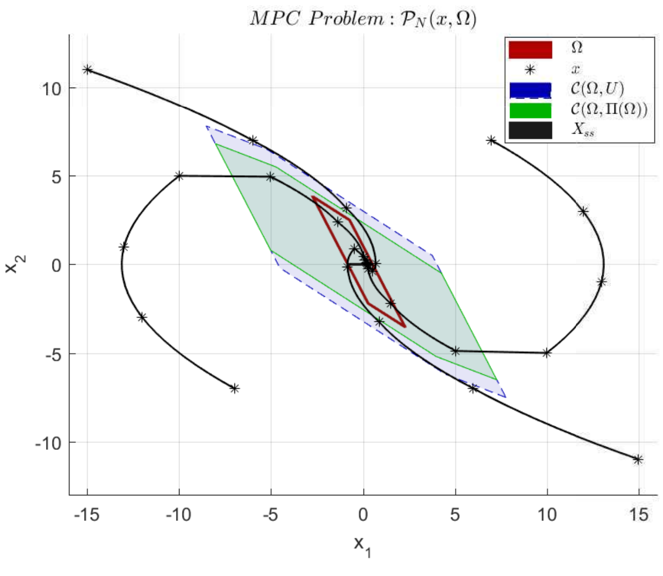

Figure 3: Closed-loop state evolution toward the set Ω.

Remark 4. *The sets used in the algorithm (γ-invariant set Ω and its corresponding input set $\Phi(\Omega)$) are obtained offline with a computational complexity depending on the form of the dynamic function involved in the model ([8, 9, 10]).*

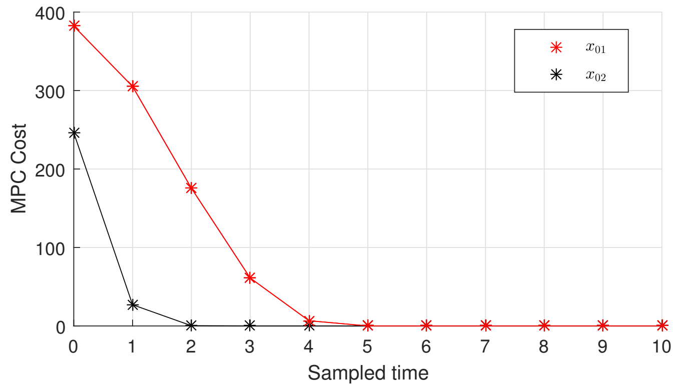

Figure 4: MPC costs for the different starting states.

Regarding online calculations, the cost function measures distances between sets and points. Except for the calculation of these distances (whose complexity depends on the shape of the sets), the proposed algorithm is not more computationally expensive than any other traditional MPC strategy.

5. Conclusions

In this work, a new and simple robust MPC formulation based on a nonlinear model of the system has been presented. This formulation ensures robust finite-time convergence to a given state space region under a fixed prediction horizon. The necessary conditions to ensure this property have been studied in detail and an upper bound on the convergence time is provided. In spite of its simplicity, the result increases the applicability of two-stage MPC controllers, in which a second stage controls the system once it reaches - under the first stage controller - some particular objective region.

6. Appendix

6.1. Proof of property 1:

Before presenting the proof of Property 1, it is necessary to introduce the next property, which states the closed nature of the controllable set $\mathcal{C}^N(\Omega, U)$ when Ω is closed.

Property 2. *([22]) Let $\Omega \subset \mathbb{R}^n$ be a closed set and let $\mathcal{C}^N(\Omega, U)$ be the N step disturbance controllable set to Ω for system* (1)*. Then, $\mathcal{C}^N(\Omega, U)$ is a closed set.*

Proof: Consider a sequence $\{x_k\}_{k\in\mathbb{I}_{\geq 0}} \subset \mathcal{C}(\Omega, U)$ converging to $\bar{x}$. Proving that $\mathcal{C}(\Omega, U)$ is closed is equivalent to show that $\bar{x} \in \mathcal{C}(\Omega, U)$. Indeed, by definition of $\mathcal{C}(\Omega, U)$, there exists a corresponding sequence $\{u_k\} \subset U$ such that

$$F(x_k, u_k, w_k) \in \Omega, \quad \forall w_k \in W, \quad k \in \mathbb{I}_{\geq 0}. \tag{15}$$

By the compactness of U, the sequence $\{u_k\}$ admits a subsequence $\{u'_k\} \subset U$ which converges to $\bar{u} \in U$. Let us consider the subsequence $\{x'_k\} \subset \mathcal{C}(\Omega, U)$ corresponding to the subsequence $\{u'_k\}$. Clearly, $x'_k \to \bar{x}$.

Therefore, since $x'_k \to \bar{x}$, and $u'_k \to \bar{u}$, by continuity of the function F we have that

$$F(x'_k, u'_k, w_k) \to F(\bar{x}, \bar{u}, w_k), \quad \forall w_k \in W$$

Since the sequence $\{F(x'_k, u'_k, w_k)\} \subset \Omega$, and Ω is closed, then $F(\bar{x}, \bar{u}, w_k) \in \Omega$ for each $w_k \in W$, which means that $\bar{x} \in \mathcal{C}(\Omega, U)$, and so $\mathcal{C}(\Omega, U)$ is closed. The fact that $\mathcal{C}^N(\Omega, U)$ is closed follows by induction concluding the proof. ∎

Now, the proof of Property 1 is presented:

Proof: By the result in [21], we know that Ω is not only contained in $\mathcal{C}(\Omega, U)$, but also $\Omega \subseteq \mathcal{C}(\Omega, \Pi(\Omega))$. It remains to show that every point of Ω is an interior point of $\mathcal{C}(\Omega, \Pi(\Omega))$. Let a state $\bar{x} \in \Omega$. Given that by hypothesis Ω is a D$\gamma-$CIS, then there exists $\bar{u} \in \Pi(\Omega)$ such that $F(\bar{x}, \bar{u}, w) \in \gamma\Omega$, for all $w \in W$. Furthermore, since $\gamma < 1$, and Ω contains the origin in its interior, it follows that $\gamma\Omega \subset \Omega^\circ$. Then,

$$\varepsilon \doteq \inf\{d(y, z) : \ y \in \partial\Omega, \ z \in \gamma\Omega\} \tag{16}$$

is such that $\varepsilon > 0$, being $\partial\Omega$ the boundary of Ω.

Since F is continuous at $\bar{x}$, there exits $\delta > 0$ such that for all $x \in \mathcal{B}_\delta(\bar{x})$ it follows that

$$d(F(x, \bar{u}, w), F(\bar{x}, \bar{u}, w)) < \varepsilon, \quad \forall w \in W. \tag{17}$$

Since $F(\bar{x}, \bar{u}, w) \in \gamma\Omega$ and $d(F(x, \bar{u}, w), F(\bar{x}, \bar{u}, w)) < \varepsilon$, for all $w \in W$, and given that Ω is closed and convex, it follows, from (16), that $F(x, \bar{u}, w) \in \Omega$ for all $w \in W$, and then $x \in \mathcal{C}(\Omega, \Pi(\Omega))$, since $\mathcal{C}(\Omega, \Pi(\Omega))$ is closed by Property 2. So $\mathcal{B}_\delta(\bar{x}) \subset \mathcal{C}(\Omega, \Pi(\Omega))$, i.e. $\Omega \subseteq \mathcal{C}(\Omega, \Pi(\Omega))^\circ$. ∎

6.2. Simulation parameters in Subsection 3.3

The simulations shown in Figures 1 and 2 correspond to a second order stable linear system, $x(k+1) = Ax(k) + Bu(k)$, similar to the one presented in [11], with

$$A = \begin{bmatrix} 0.7476 & -0.4984 \\ 0.0356 & 1.0680 \end{bmatrix}, \quad B = \begin{bmatrix} 0.3 \\ -0.4 \end{bmatrix}. \tag{18}$$

The system constraints are given by $X = \left\{x \in \mathbb{R}^2 : \|x\|_\infty \leq 10\right\}$ and $U = \{u \in \mathbb{R} : \|u\|_\infty \leq 4\}$. The horizon of the MPC controller $\mathcal{P}(\Omega, x)$, (3), with the stage cost (8), is given by $N = 7$, being Ω a D$\gamma-$CIS ($W = \{0\}$) with $\gamma = 0.65$, and its corresponding input set $\Pi(\Omega) = \{u \in \mathbb{R} : \|u\|_\infty \leq 3\}$.

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