Continuous–Time Probabilistic Ultimate Bounds and Invariant Sets: Computation and Assignment

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Abstract

The concepts of ultimate bounds and invariant sets play a key role in several control theory problems, as they replace the notion of asymptotic stability in the presence of unknown disturbances. However, when the disturbances are unbounded, as in the case of Gaussian white noise, no ultimate bounds nor invariant sets can in general be found. To overcome this limitation we introduced, in previous work, the notions of probabilistic ultimate bound (PUB) and probabilistic invariant set (PIS) for discrete-time systems. This article extends the notions of PUB and PIS to continuous-time systems, studying their main properties and providing tools for their calculation. In addition, the use of these concepts in robust control design by covariance assignment is presented.

Key words: Invariant sets; Ultimate bounds; Stochastic differential equations; Linear systems; Probabilistic methods

1 Introduction

Dynamical systems under the influence of non-vanishing unknown disturbances cannot achieve asymptotic stability in general. However, under certain conditions, the ultimate boundedness of the system trajectories can be guaranteed and invariant sets can be found. Consequently, the notions of ultimate bounds (UB) and invariant sets (IS) play a key role in control systems theory and design.

A necessary condition to ensure the existence of ultimate bounds and invariant sets is that the disturbances must be bounded. However, in systems theory, disturbances are often represented by unbounded signals such as Gaussian white noise, in which case ultimate bounds and invariant sets cannot be obtained in a classical sense. To overcome this problem, the authors have introduced in [9,10] the notions of *probabilistic ultimate bound* (PUB) and *probabilistic invariant set* (PIS), as sets where the trajectories converge to and stay in with a given probability.

Classic UB and IS are an important tool in modern treatments of model predictive control (see, e.g., [12,14]), fault diagnosis and fault tolerant control (see, e.g., [13,15]) and several other applications of set invariance in control problems (see [2] and the references therein). With the usage of the PUB and PIS notions, many of these applications can be extended to consider also the presence of unbounded disturbances. In fact, some recent works on model predictive control use concepts that are related to probabilistic invariant sets (see, e.g., [3,5,6]).

Although the concepts in [9,10] are limited to the discrete-time domain, ultimate boundedness and invariance are also important concepts in continuous-time systems, and they experience the same limitations regarding unbounded disturbances.

Motivated by these facts, this work firstly extends the notions, properties and tools for PUB and PIS developed in [9,10] to the continuous-time domain. While in the case of PUB the extension is almost straightforward, the concept of probabilistic invariance in continuous time needs to be redefined because of the limitations imposed by the infinite-bandwidth nature of continuous-time white noise disturbances (see, e.g., the insightful discussions in [1]).

Finally, the problem of designing a feedback controller so that the closed-loop system under white noise disturbances has a desired PUB is addressed. Preliminary results covering only single input systems in controller canonical form were presented by the authors in the conference paper [8]. The current journal version completes the contribution by presenting new results that generalise the techniques to multiple input systems given in general form.

The paper is organised as follows: Section 2 introduces the concepts of continuous time PUB and PIS and establishes their basic properties. Then, Section 3 presents closed-form formulas for the calculation of PUB and PIS, respectively. Section 4 develops the technique for control design and Section 5 illustrates the results with a numerical example.

2 Background and Definitions

We consider a continuous-time LTI system given by the following stochastic differential equation

$$\mathrm{d}x(t) = Ax(t)\mathrm{d}t + \mathrm{d}w(t) \tag{1}$$

with $x(t), w(t) \in \mathbb{R}^n$ and $A \in \mathbb{R}^{n\times n}$ being a Hurwitz matrix.

Assumption 1 *The disturbance $w(t)$ is a stochastic process whose increments are stationary and uncorrelated with zero mean values (i.e., a Lévy process, that in the case of a normal distribution becomes a Wiener process). We assume also that $w(t)$ has incremental covariance $\Sigma_w \mathrm{d}t \triangleq \operatorname{cov}[\mathrm{d}w(t)] = \mathrm{E}[\mathrm{d}w(t)\mathrm{d}w^T(t)]$ with Σ_w being a finite covariance matrix.*

2.1 Expected Value and Covariance of $x(t)$

The characterisation of probabilistic ultimate bounds and invariant sets is based on the stochastic properties of the solution $x(t)$ of Eq.(1). Given a time t, the covariance of the solution is defined as

$$\Sigma_x(t) \triangleq \operatorname{cov}[x(t)] = \mathrm{E}[(x(t) - E[x(t)])(x(t) - E[x(t)])^T] \tag{2}$$

Both, Σ_w and $\Sigma_x(t)$ are symmetric positive semidefinite matrices. The expected value $\mu_x(t) = \mathrm{E}[x(t)]$ can be computed (see e.g. [1], Theorem 6.1, page 66) as the solution of $\dot{\mu}_x(t) = A\mu_x(t)$. We assume that the initial state $x(t_0)$ is known, then $\mu_x(t_0) = x(t_0)$ and the previous equation has the solution

$$\mu_x(t) = e^{A(t-t_0)}x(t_0) \tag{3}$$

The covariance matrix $\Sigma_x(t)$ verifies (see e.g. [1], Theorem 6.1, page 66) the following differential equation:

$$\dot{\Sigma}_x(t) = A\Sigma_x(t) + \Sigma_x(t)A^T + \Sigma_w \tag{4}$$

with $\Sigma_x(t_0) = 0$ (since $x(t_0)$ is known). Since A is a Hurwitz matrix, the latter expression converges as $t \to \infty$. Then, defining $\Sigma_x \triangleq \lim_{t\to\infty} \Sigma_x(t)$ we have from Eq.(4) that Σ_x can be obtained from the Lyapunov equation

$$A\Sigma_x + \Sigma_x A^T = -\Sigma_w \tag{5}$$

2.2 Definition of PUB and γ-PIS

We next define the two notions that concern this article.

Definition 2 (Probabilistic Ultimate Bounds) *Let $0 < p \leq 1$ and let $S \subset \mathbb{R}^n$. We say that S is a PUB with probability p for system (1) if for every initial state $x(t_0) = x_0 \in \mathbb{R}^n$ there exists $T = T(x_0) \in \mathbb{R}$ such that the probability[1] $\Pr[x(t) \in S] \geq p$ for each $t \geq t_0 + T$.*

For the definition of PIS, we first introduce the product of a scalar $\gamma \geq 0$ and a set S as $\gamma S \triangleq \{\gamma x : x \in S\}$. Notice that when $0 \leq \gamma \leq 1$, and provided that S is a *star–shaped set with respect to the origin,*[2] it follows that $\gamma S \subseteq S$.

Definition 3 (γ-Probabilistic Invariant Sets) *Let $0 < p \leq 1$, $0 < \gamma \leq 1$ and let $S \subset \mathbb{R}^n$ be a star–shaped set with respect to the origin. We say that S is a γ–PIS with probability p for system (1) if for any $x(t_0) \in \gamma S$ the probability $\Pr[x(t) \in S] \geq p$ for each $t > t_0$.*

Remark 4 *The definitions of PUB for discrete and continuous time systems are almost identical. However, PIS for discrete-time systems were defined to ensure that any trajectory starting in the set remains in the set with a given probability. By choosing a sufficiently large set, the contractivity of the system's dynamics at the boundary of the set dominates the noise and the probability of the trajectory leaving the set at the next step can be made arbitrarily small. In continuous time, however, this is not possible. Irrespective of the contractivity, when a trajectory starts at time t_0 at the boundary of the set, taking t sufficiently close to t_0 the dynamics is always dominated by the white noise due to its infinite-bandwidth nature. Thus, for $t \to t_0^+$ the probability that $x(t)$ leaves the set S only depends on the noise and becomes independent of the size of S. In order to overcome this fundamental difficulty, the initial states of a PIS are restricted in Definition 3 to a subset γS, with γ less than one.*

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[1] In this work, the expression $\Pr[x(t) \in S \subset \mathbb{R}^n]$ denotes the probability that the solution $x(t)$, at time t, is in the set $S \subset \mathbb{R}^n$. Thus, $\Pr[\cdot]$ is the probability measure on Euclidean space induced by the stochastic process $\{w(\tau)|t_0 \leq \tau \leq t\}$ via the solution, at time t, of the stochastic differential equation (1) with known initial condition $x(t_0)$ at time t_0.

[2] A set $S \subset R^n$ is star shaped, or a star domain, with respect to the origin if $x \in S \Rightarrow \gamma x \in S$ for all $0 \leq \gamma \leq 1$

The previous remark can be simply illustrated by the solution of the scalar case of Eq. (1) with $w(t)$ a Wiener process and $A = -\lambda$, in which case $x(t) = e^{-\lambda(t-t_0)}x(t_0) + \int_{t_0}^{t} e^{-\lambda(t-\tau)}\mathrm{d}w(\tau)$. Then, it can be shown that $\lim_{t\to t_0^+} \Pr[|x(t)| > |x(t_0)|] = \lim_{t\to t_0^+} \Pr\left[\int_{t_0}^{t} \mathrm{d}w(\tau) > 0\right] = 0.5$ independently of $x(t_0)$ and λ (since $\int_{t_0}^{t} \mathrm{d}w(\tau)$ is a zero-mean Gaussian process). That is, no matter *how contractive* the term $e^{-\lambda t}$ is, nor *how big* the initial condition $|x(t_0)|$ is, the probability of confinement in $|x(t)| \le |x(t_0)|$ *is dominated* by the noise.

2.3 Some properties of PUB and γ–PIS

Here we present some basic properties of PUB and γ–PIS that are analogous to those of deterministic ultimate bounds and invariant sets. Although these properties are not used to derive the main results of the paper, they corroborate that the definitions of PUB and γ–PIS provided above are consistent with their deterministic counterparts.

The basic properties of continuous-time PUB are identical to the discrete-time ones, i.e., Lemma 3 and Corollaries 7 and 10 in [10] are also valid for continuous-time PUB. These properties establish that a PUB with probability p for (1) is also a PUB with probability $\tilde{p} \ge 0$ for any $\tilde{p} < p$ and that the union and intersection of PUB sets define PUB sets.

In the case of the unions and intersections of γ–PIS, the presence of the parameter γ introduces some changes to their discrete time counterparts. Lemma 4 in [10] is still valid (a γ–PIS with probability p is also PUB with the same probability) but the union and intersection of γ–PIS are now ruled by the following proposition.

Proposition 5 (Intersection and union of γ–PIS) *Let $\{S_i\}_{i=1}^r$ be a collection of γ_i–PIS for system (1) with probabilities p_i, $i = 1, \ldots, r$, respectively, then*

- *Provided that $\sum_{i=1}^r p_i > (r-1)$, the set $S_\cap = \cap_{i=1}^r S_i$ is a γ–PIS with probability $p = \sum_{i=1}^r p_i - (r-1)$ where $\gamma = \min\{\gamma_i : i = 1, \ldots, r\}$.*
- *The set $S_\cup = \cup_{i=1}^r S_i$ is a γ–PIS with probability $p = \min\{p_i : i = 1, \ldots, r\}$ where $\gamma = \min\{\gamma_i : i = 1, \ldots, r\}$.*

The proof of this Proposition can be derived from those of Lemma 8 in [8] and Lemma 9 in [10].

3 Characterisation of PUB and γ–PIS

We develop a method to characterise and compute Probabilistic Ultimate Bounds and Invariant Sets for (1) based on Chebyshev's inequality which can be used for arbitrary stochastic processes $w(t)$ satisfying Assumption 1. The results, summarised in Theorems 6 and 7, also provide tighter bounds for the special case of Gaussian disturbances.

Given a parameter (probability) p such that $0 < p < 1$, the method uses n arbitrary parameters $\tilde{p}_i$ chosen such that

$$0 < \tilde{p}_i < 1,\ i = 1, \ldots, n; \qquad \sum_{i=1}^{n} \tilde{p}_i = 1 - p \tag{6}$$

In the sequel, for a vector x, x_i denotes its ith component, and for a square matrix Σ, the notation $[\Sigma]_{i,i}$ indicates its ith diagonal element. The symbol $\preceq$ will denote the elementwise inequality between two vectors, i.e., for $\alpha, \beta \in \mathbb{R}^n$, $\alpha \preceq \beta$ if and only if $\alpha_i \le \beta_i$, $i = 1, \ldots, n$. For a matrix M with complex entries, M^* will denote the conjugate transpose of M.

Theorem 6 (PUB Characterisation) *Consider the system (1). Assume that $A \in \mathbb{R}^{n\times n}$ is a Hurwitz matrix and suppose that $w(t)$ is a stochastic process whose increments are uncorrelated with zero mean values and with incremental covariance matrix $\Sigma_w \mathrm{d}t$. Let $0 < p < 1$ and $\tilde{p}_i$, $i = 1, \ldots, n$, satisfy Eq.(6). Then, any set of the form $S = \{x : |x_i| \le b_i + \varepsilon;\ i = 1, \ldots, n\}$ with $\varepsilon > 0$ [3] is a PUB for the system with probability p, with*

$$b_i \triangleq \sqrt{\frac{[\Sigma_x]_{i,i}}{\tilde{p}_i}};\quad i = 1, \ldots, n \tag{7}$$

and Σ_x is the solution of the Lyapunov equation (5).

Additionally, when $w(t)$ is a Wiener process, Eq.(7) can be replaced with

$$b_i \triangleq \sqrt{2[\Sigma_x]_{i,i}}\,\mathrm{erf}^{-1}(1 - \tilde{p}_i);\quad i = 1, \ldots, n \tag{8}$$

where erf *is the* error function*:* $\mathrm{erf}(z) \triangleq \frac{2}{\sqrt{\pi}} \int_0^z e^{-\zeta^2} d\zeta$.

The proof of Theorem 6 follows those of Theorem 12 in [10] (general case) and Theorem 1 in [9] (Gaussian case) for discrete-time systems. The only difference is that Σ_x is now computed from the continuous-time Lyapunov equation (5).

Notice that the bound provided by Eq.(8) is tighter than that of Eq.(7) but it is only valid for a noise with Gaussian distribution.

[3] The only role played by $\varepsilon > 0$ is to ensure that $T(x_0)$ in Definition 2 is finite (for details, we refer to the proof of Theorem 12 in [10]).

Theorem 7 **(γ–PIS Characterisation)** *Consider the system (1), where matrix A is assumed to be Hurwitz and diagonalisable*[4]. *Suppose that $w(t)$ is a stochastic process whose increments are uncorrelated with zero mean values and incremental covariance matrix $\Sigma_w \mathrm{d}t$. Let $0 < p < 1$ and $\tilde{p}_i$, $i = 1, \ldots, n$, satisfy Eq.(6). Then, the set $S = \{x : |V^{-1}x| \preceq b\}$ is a γ–PIS for the system with probability p, where V is a similarity transformation such that $\Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_n) = V^{-1}AV$ is the Jordan diagonal decomposition of matrix A, and the components of $b = [b_1 \ \ldots \ b_n]^T$ are computed according to*

$$b_i \triangleq \sqrt{\frac{[\Sigma_v]_{i,i}}{2|\mathbb{R}\mathrm{e}(\lambda_i)|(1-\gamma^2)\tilde{p}_i}}; \quad i = 1, \ldots, n \tag{9}$$

with $\Sigma_v = V^{-1}\Sigma_w(V^{-1})^*$.

Additionally, when $w(t)$ is a Wiener process and the parameters $\tilde{p}_i$ satisfying (6) are chosen with the restriction that for each pair of complex conjugate eigenvalues λ_i, $\lambda_j = \bar{\lambda}_i$ of matrix A we take $\tilde{p}_i = \tilde{p}_j$, Eq.(9) can be replaced with

$$b_i \triangleq \sqrt{\frac{[\Sigma_v]_{i,i}}{|\mathbb{R}\mathrm{e}(\lambda_i)|(1-\gamma^2)}} \mathrm{erf}^{-1}(1-\tilde{p}_i); \quad i = 1, \ldots, n \tag{10}$$

PROOF. We follow the idea of the proofs of Theorems 15 and 16 in [10]: With the linear transformation $x(t) = Vz(t)$, system (1) becomes

$$\mathrm{d}z(t) = \Lambda z(t)\mathrm{d}t + V^{-1}\mathrm{d}w(t) \tag{11}$$

with $z \in \mathbb{C}^n$, $w(t) \in \mathbb{R}^n$, $V^{-1} \in \mathbb{C}^{n\times n}$, and $\Lambda \in \mathbb{C}^{n\times n}$ being a diagonal matrix. Defining $v(t) \triangleq V^{-1}w(t)$, the incremental covariance of $v(t)$ satisfies $\Sigma_v \mathrm{d}t = V^{-1}\Sigma_w(V^{-1})^*\mathrm{d}t$, and the ith component of (11) is

$$\mathrm{d}z_i(t) = \lambda_i z_i(t)\mathrm{d}t + \mathrm{d}v_i(t) \tag{12}$$

The expected value of the random variable[5] $z_i(t)$ then verifies $\mathrm{E}[z_i(t)] = e^{\lambda_i(t-t_0)}z_i(t_0)$, since we assume that $z_i(t_0)$ is known. The variance of $z_i(t)$ can be computed from (12) as

$$\mathrm{var}[z_i(t)] = \frac{1-e^{2\mathbb{R}\mathrm{e}(\lambda_i)(t-t_0)}}{2|\mathbb{R}\mathrm{e}(\lambda_i)|}[\Sigma_v]_{i,i}$$

Suppose that $x(t_0) \in \gamma S$, i.e., $|V^{-1}x(t_0)| \preceq \gamma b$ with b defined by Eq.(9). Thus, $|z(t_0)| = |V^{-1}x(t_0)| \preceq \gamma b$ and $|z_i(t_0)| \leq \gamma b_i$. Then, for all $t > t_0$ it results (see, e.g., Equation (6.6) from [1], Page 66) that

$$|\mathrm{E}[z_i(t)]| = |e^{\lambda_i(t-t_0)}z_i(t_0)| \leq e^{\mathbb{R}\mathrm{e}(\lambda_i)(t-t_0)}\gamma b_i \tag{13}$$

From Inequality (13), it follows that

$$\begin{aligned}
&\Pr[|z_i(t)| \geq b_i] \\
&= \Pr[|z_i(t)| - e^{\mathbb{R}\mathrm{e}(\lambda_i)(t-t_0)}\gamma b_i \geq b_i(1-\gamma e^{\mathbb{R}\mathrm{e}(\lambda_i)(t-t_0)})] \\
&\leq \Pr[|z_i(t)| - |\mathrm{E}[z_i(t)]| \geq b_i(1-\gamma e^{\mathbb{R}\mathrm{e}(\lambda_i)(t-t_0)})] \\
&\leq \Pr[|z_i(t) - \mathrm{E}[z_i(t)]| \geq b_i(1-\gamma e^{\mathbb{R}\mathrm{e}(\lambda_i)(t-t_0)})]
\end{aligned}$$

Chebyshev's inequality establishes that

$$\begin{aligned}
\Pr\left[|z_i(t) - \mathrm{E}[z_i(t)]| \geq b_i(1-\gamma e^{\mathbb{R}\mathrm{e}(\lambda_i)(t-t_0)})\right] \\
\leq \frac{\mathrm{var}[z_i(t)]}{b_i^2(1-\gamma e^{\mathbb{R}\mathrm{e}(\lambda_i)(t-t_0)})^2}
\end{aligned} \tag{14}$$

and then it results that

$$\begin{aligned}
\Pr[|z_i(t)| \geq b_i] &\leq \frac{\mathrm{var}[z_i(t)]}{b_i^2(1-\gamma e^{\mathbb{R}\mathrm{e}(\lambda_i)(t-t_0)})^2} \\
&= \frac{1-e^{2\mathbb{R}\mathrm{e}(\lambda_i)(t-t_0)}}{2|\mathbb{R}\mathrm{e}(\lambda_i)|b_i^2(1-\gamma e^{\mathbb{R}\mathrm{e}(\lambda_i)(t-t_0)})^2}[\Sigma_v]_{i,i}
\end{aligned}$$

The expression

$$\frac{1-e^{2\mathbb{R}\mathrm{e}(\lambda_i)(t-t_0)}}{(1-\gamma e^{\mathbb{R}\mathrm{e}(\lambda_i)(t-t_0)})^2} \tag{15}$$

is maximised when $e^{\mathbb{R}\mathrm{e}(\lambda_i)(t-t_0)} = \gamma$. Then, it results that $\Pr[|z_i(t)| > b_i] \leq \Pr[|z_i(t)| \geq b_i] \leq \frac{[\Sigma_v]_{i,i}}{2|\mathbb{R}\mathrm{e}(\lambda_i)|b_i^2(1-\gamma^2)} = \tilde{p}_i$ for all $t > t_0$. Thus, the probability

$$\Pr[|z(t)| \not\preceq b] \leq \sum_{i=1}^{n}\Pr[|z_i(t)| > b_i] \leq \sum_{i=1}^{n}\tilde{p}_i = 1-p$$

for all $t > t_0$, and then,

$$\Pr[|z(t)| \preceq b] = \Pr[|V^{-1}x(t)| \preceq b] = \Pr[x(t) \in S] \geq p$$

which proves that S is a γ–PIS with probability p.

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[4] In an important number of applications, the matrix A in (1) is given by some *closed-loop matrix*, e.g., $\tilde{A} - \tilde{B}K$ or $\tilde{A} - L\tilde{C}$ [where $(\tilde{A}, \tilde{B}, \tilde{C})$ is the open-loop system and K is a feedback gain, L is an observer gain, etc.]. Under standard controllability and observability conditions on $(\tilde{A}, \tilde{B}, \tilde{C})$ the design of K, L, etc., can be readily done by pole placement techniques so that the assumptions on A made here are, without loss of generality, satisfied.

[5] $\mathrm{E}[z_i(t)]$ refers to the expected value of the random variable $z_i \in \mathbb{C}$ at time t with respect to the probability measure defined in footnote 1.

When $w(t)$ is a Wiener process and λ_i is real, we replace Chebyshev's inequality of Eq.(14) by the following expression valid for Gaussian distributions

$$\begin{aligned}\Pr\left[|z_i(t)-\mathrm{E}[z_i(t)]| \geq b_i(1-\gamma e^{\mathbb{R}\mathrm{e}(\lambda_i)(t-t_0)})\right] \\ = 1-\mathrm{erf}\left(\frac{b_i(1-\gamma e^{\mathbb{R}\mathrm{e}(\lambda_i)(t-t_0)})}{\sqrt{2\mathrm{var}[z_i(t)]}}\right)\end{aligned}$$

and then we obtain

$$\begin{aligned}&\Pr[|z_i(t)|>b_i] \\ &\quad\leq 1-\mathrm{erf}\left(b_i\sqrt{\frac{(1-\gamma e^{\mathbb{R}\mathrm{e}(\lambda_i)(t-t_0)})^2|\mathbb{R}\mathrm{e}(\lambda_i)|}{(1-e^{2\mathbb{R}\mathrm{e}(\lambda_i)(t-t_0)})[\Sigma_v]_{i,i}}}\right) \\ &\quad\leq 1-\mathrm{erf}\left(b_i\sqrt{\frac{(1-\gamma^2)|\mathbb{R}\mathrm{e}(\lambda_i)|}{[\Sigma_v]_{i,i}}}\right)=\tilde{p}_i \qquad (16)\end{aligned}$$

In the last step we used the fact that $\mathrm{erf}(\cdot)$ is a monotonically increasing function and we maximised the expression of Eq.(15).

In the case of complex eigenvalues, Eq.(12) can be split into real and imaginary parts $z_i(t)=\mathbb{R}\mathrm{e}[z_i(t)]+j\mathbb{I}\mathrm{m}[z_i(t)]$, where both components are Gaussian processes and the variance can be written as $\mathrm{var}[z_i(t)]=\mathrm{var}[\mathbb{R}\mathrm{e}[z_i(t)]]+\mathrm{var}[\mathbb{I}\mathrm{m}[z_i(t)]]$. Then, the proof follows that of Theorem 2 in [9] for discrete-time systems, replacing t_0+N by t and $b_i(1-|\lambda_i|^N)$ by $b_i(1-\gamma e^{\mathbb{R}\mathrm{e}(\lambda_i)(t-t_0)})$. □

Remark 8 *Notice that b_i in Eq.(9) and Eq.(10) goes to infinity as γ goes to one. This is consistent with the observation made in Remark 4 above, that a PIS cannot be defined without using a factor γ less than one to restrict the initial states due to the infinite-bandwidth nature of the continuous-time white noise disturbance (see, e.g., the discussions in [1] on continuous-time white noise).*

4 Control Design

We consider here the problem of, given a positive vector b and a probability p, find a controller gain K such that any set of the form $S=\{x:|x|\preceq b+\varepsilon\}$ with $\varepsilon>0$ is a PUB with probability p of the closed loop system

$$\mathrm{d}x(t)=(A+BK)x(t)\mathrm{d}t+BG\mathrm{d}v(t). \qquad (17)$$

We assume that the pair (A,B) is controllable where $A\in\mathbb{R}^{n\times n}$ and $B\in\mathbb{R}^{n\times m}$, the latter matrix corresponding to m control inputs. Notice also that the disturbance $v(t)\in\mathbb{R}^q$ is *matched* [6] with the control input through a matrix $G\in\mathbb{R}^{m\times q}$. We assume also that $v(t)$ has incremental covariance $\Sigma_v\mathrm{d}t$ with Σ_v a finite covariance matrix.

Theorem 6 shows that the PUB depends on the diagonal entries of the state covariance matrix Σ_x. Thus, this is a problem of *state covariance assignment*, consisting in finding a feedback gain which assigns a specified closed-loop state covariance [4], similar to the one treated in [16].

Thus, before presenting the control design procedures to obtain a desired PUB, we first derive some auxiliary results regarding covariance assignment.

4.1 Covariance Assignment in Controller Canonical Form

When matrix A is in its controller canonical form, and the system has a single input, the covariance matrix that solves Eq.(5) has a *Xiao structure* [17].

Definition 9 (Xiao matrix) *Given a vector $0\preceq z\in\mathbb{R}^k$, we define the Xiao matrix $\mathcal{X}(z)$ as*

$$\mathcal{X}(z)=\begin{bmatrix} z_1 & 0 & -z_2 & 0 & z_3 & \cdots & \cdot \\ 0 & z_2 & 0 & -z_3 & 0 & \cdots & \cdot \\ -z_2 & 0 & z_3 & 0 & -z_4 & \cdots & \cdot \\ 0 & -z_3 & 0 & z_4 & 0 & \cdots & \cdot \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & z_k \end{bmatrix} \qquad (18)$$

For the multiple input case, we derive the following Lemma:

Lemma 10 *Consider the system of Eq.(17) and assume that the pair (A,B) is in its controller canonical form, with $A\in\mathbb{R}^{n\times n}$ and $B\in\mathbb{R}^{n\times m}$; and let d_j, $j=1,\ldots,m$, be the controllability indices of (A,B). Assume also that the disturbance $v(t)$ has incremental covariance $\Sigma_v\mathrm{d}t$. Further, define*

$$\Sigma\triangleq G\Sigma_v G^T \qquad (19)$$

and assume that the pair $(A,B\Sigma^{1/2})$ is controllable. Then, the block diagonal Xiao matrix

$$\Sigma_x=\mathrm{diag}(\Sigma_x^1,\ldots,\Sigma_x^m) \qquad (20)$$

[6] The matched disturbance assumption, or 'matching condition' (see, e.g., Chapter 14 in [7]) is a usual hypothesis in robust control applications, modelling –in particular, but not limited to– all kinds of input perturbations.

where $\Sigma_x^j \in \mathbb{R}^{d_j \times d_j}$ for $j = 1, \dots, m$ are positive definite Xiao matrices, is an assignable covariance matrix for the system under the feedback law $u = Kx$ with

$$K = -B^\dagger(A\Sigma_x + \Sigma_x A^T + B\Sigma B^T)(I - BB^\dagger/2)\Sigma_x^{-1} \quad (21)$$

where $B^\dagger$ is the Moore-Penrose inverse of matrix B. Moreover, the closed-loop matrix $A + BK$ is Hurwitz.

PROOF. Let $\Sigma_v \mathrm{d}t$ be the incremental covariance matrix of v. Defining $w(t) \triangleq BGv(t)$, the covariance of $w(t)$ is given by

$$\Sigma_w = BG\Sigma_v G^T B^T = B\Sigma B^T$$

where $\Sigma = G\Sigma_v G^T$ is the covariance of $Gv(t)$. Substituting $A + BK$ for A in (5) we have that the closed-loop state covariance matrix Σ_x satisfies the Lyapunov equation

$$(A + BK)\Sigma_x + \Sigma_x(A + BK)^T = -B\Sigma B^T \quad (22)$$

From Corollary 4.6 of [4], equation (22) has a solution K (one such solution is given by (21)) if and only if

$$(I - BB^\dagger)(A\Sigma_x + \Sigma_x A^T + B\Sigma B^T)(I - BB^\dagger) = 0 \quad (23)$$

We next analyse the form of condition (23) when (A, B) are in the controller canonical form, that is,

$$A = \begin{bmatrix} A_{11} & A_{12} & \dots & A_{1m} \\ \vdots & \vdots & \ddots & \vdots \\ A_{m1} & A_{m2} & \dots & A_{mm} \end{bmatrix} \qquad B = \begin{bmatrix} B_1 \\ \vdots \\ B_m \end{bmatrix} \quad (24)$$

where the submatrices $A_{ii} \in \mathbb{R}^{d_i \times d_i}$, $B_i \in \mathbb{R}^{d_i \times m}$, for $i = 1, \dots, m$, have the form

$$A_{ii} = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ * & * & * & \dots & * \end{bmatrix}, \quad B_i = \begin{array}{c} \begin{array}{ccccccc} & & & i & & & \\ & & & \downarrow & & & \end{array} \\ \begin{bmatrix} 0 & \dots & 0 & 0 & \dots & 0 \\ 0 & \dots & 0 & 0 & \dots & 0 \\ & \ddots & \vdots & & \ddots & \\ 0 & \dots & 1 & * & \dots & * \end{bmatrix} \end{array} \quad (25)$$

with the $*$'s representing arbitrary entries, and $A_{ij} \in \mathbb{R}^{d_i \times d_j}$, for $i, j = 1, \dots, m$, $i \neq j$, have zero elements everywhere except for possibly nonzero elements in the last row. We first find the form of $(I - BB^\dagger)$. Note that the matrix B in (24), (25) has only m nonzero rows of the form $\begin{bmatrix} 0 & \dots & 0 & 1 & * & \dots & * \end{bmatrix}$, where the 1 is in the ith position. Thus, eliminating the redundant rows containing only zeros, we can express $B^T B = \tilde{B}^T \tilde{B}$, where $\tilde{B} \in \mathbb{R}^{m \times m}$ is an upper triangular matrix with 1's in the main diagonal. Simple calculations then show that $BB^\dagger = B(B^T B)^{-1} B^T = B\tilde{B}^{-1}\tilde{B}^{-T} B^T$ has zero entries everywhere except for 1's in the (σ_i, σ_i) positions, where $\sigma_i \triangleq \sum_{j=1}^{i} d_j$, for $i = 1, \dots, m$. It follows that $(I - BB^\dagger)$ has the form

$$(I - BB^\dagger) = \mathrm{diag}(1, \dots, 1, \underset{\overset{\downarrow}{\sigma_1}}{0}, 1, \dots, 1, \underset{\overset{\downarrow}{\sigma_2}}{0}, 1 \dots, 1, \underset{\overset{\downarrow}{\sigma_m}}{0}) \quad (26)$$

Hence, $(I - BB^\dagger)B = 0$, $B^T(I - BB^\dagger) = 0$ and in the multiplication $(I - BB^\dagger)A$ we have that the rows of A with arbitrary entries $*$'s are multiplied by zero whereas the remaining rows are multiplied by one. We thus conclude that $\bar{A} \triangleq (I - BB^\dagger)A$ has the form

$$\bar{A} = \mathrm{diag}(\Gamma_1, \dots, \Gamma_m), \quad \Gamma_i \triangleq \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix} \in \mathbb{R}^{d_i \times d_i} \quad (27)$$

Based on the above calculations we can rewrite (23) as

$$\bar{A}\bar{\Sigma} + \bar{\Sigma}^T \bar{A}^T = 0, \quad \text{where} \quad \bar{\Sigma} \triangleq \Sigma_x(I - BB^\dagger). \quad (28)$$

Due to the structure of Σ_x in (20) and using (26), we can write $\bar{\Sigma} = \mathrm{diag}(\bar{\Sigma}_1, \dots, \bar{\Sigma}_m)$, where $\bar{\Sigma}_i \in \mathbb{R}^{d_i \times d_i}$, $i = 1, \dots, m$, is equal to the Xiao matrix Σ_x^i with zeros in the last column. Hence, (28) leads to the m equations

$$\Gamma_i \bar{\Sigma}_i + \bar{\Sigma}_i{}^T \Gamma_i^T = 0, \quad i = 1, \dots, m. \quad (29)$$

From the form of Γ_i in (27) and that of $\bar{\Sigma}_i$ just discussed, it is easy to see that (29) (equivalently, (23)) are satisfied and hence Σ_x in (20) is a closed-loop covariance matrix for the controller canonical form system (A, B), assignable by state feedback with gain (21). From the assumption that the pair $(A, B\Sigma^{1/2})$ is controllable and the fact that Σ_x is positive definite, it follows (see [7], Chapter 4) that $A + BK$ satisfying the Lyapunov equation (22) is Hurwitz. □

Next, the following Lemma relates the construction of a positive definite Xiao matrix with the choice of the probabilities $\tilde{p}_i$ in Theorem 6. This result will be used later to build an assignable covariance matrix having the diagonal entries required by Theorem 6 to ensure that the closed-loop system has the desired PUB.

Lemma 11 *Let $g : (0, 1] \to \mathbb{R}^+$ be a strictly monotonically decreasing function with $\mathrm{Im}(g) = [a, \infty)$ for some constant $a \geq 0$. Let $b \succ 0$ be a vector in $\mathbb{R}^n$ and let $0 < p < 1$. Then, there exist n constants $0 < \tilde{p}_i < 1$ for $i = 1, \dots, n$ such that $\sum_{i=1}^{n} \tilde{p}_i = 1 - p$ and the Xiao matrix*

$$\Sigma_x = \mathcal{X}\left(\begin{bmatrix} \frac{(b_1)^2}{g(\tilde{p}_1)^2} & \frac{(b_2)^2}{g(\tilde{p}_2)^2} & \dots & \frac{(b_n)^2}{g(\tilde{p}_n)^2} \end{bmatrix}^T\right)$$

is positive definite.

PROOF. Let us suppose that there exist $0 < \tilde{p}_i^{(k)} < 1$ for $i = 1, \dots, n$ with $\sum_{i=1}^n \tilde{p}_i^{(k)} = 1 - p$ such that the matrix

$$\Sigma_k \triangleq \mathcal{X}\left(\left[\frac{(b_1)^2}{g(\tilde{p}_1^{(k)})^2} \ \frac{(b_2)^2}{g(\tilde{p}_2^{(k)})^2} \ \cdots \ \frac{(b_k)^2}{g(\tilde{p}_k^{(k)})^2}\right]^T\right) \tag{30}$$

is positive definite. In order to use induction, we shall prove that there exist $0 < \tilde{p}_i^{(k+1)} < 1$ for $i = 1, \dots, n$ with $\sum_{i=1}^n \tilde{p}_i^{(k+1)} = 1 - p$ such that the matrix

$$\Sigma_{k+1} \triangleq \mathcal{X}\left(\left[\frac{(b_1)^2}{g(\tilde{p}_1^{(k+1)})^2} \ \frac{(b_2)^2}{g(\tilde{p}_2^{(k+1)})^2} \ \cdots \ \frac{(b_{k+1})^2}{g(\tilde{p}_{k+1}^{(k+1)})^2}\right]^T\right) \tag{31}$$

is also positive definite. As a first attempt to find Σ_{k+1}, we define the matrix

$$\begin{aligned} \tilde{\Sigma}_{k+1} &\triangleq \mathcal{X}\left(\left[\frac{(b_1)^2}{g(\tilde{p}_1^{(k)})^2} \ \frac{(b_2)^2}{g(\tilde{p}_2^{(k)})^2} \ \cdots \ \frac{(b_{k+1})^2}{g(\tilde{p}_{k+1}^{(k)})^2}\right]^T\right) \\ &= \begin{bmatrix} \Sigma_k & c_k \\ c_k^T & \frac{(b_{k+1})^2}{g(\tilde{p}_{k+1}^{(k)})^2} \end{bmatrix} = \begin{bmatrix} \Sigma_k & c_k \\ c_k^T & \tilde{d}_{k+1} \end{bmatrix} \end{aligned} \tag{32}$$

If the product $c_k^T(\Sigma_k)^{-1}c_k < \tilde{d}_{k+1}$ then $\tilde{\Sigma}_{k+1} > 0$ and we can choose $\tilde{p}_i^{(k+1)} = \tilde{p}_i^{(k)}$ and the matrix Σ_{k+1} defined as in Eq.(31) is positive definite.

Otherwise, if $c_k^T(\Sigma_k)^{-1}c_k \geq \tilde{d}_{k+1}$, we first compute

$$r_{k+1} = \frac{c_k^T(\Sigma_k)^{-1}c_k}{\tilde{d}_{k+1}} \tag{33}$$

and choose a constant $\alpha > 1$ to calculate

$$[\Sigma_{k+1}]_{i,i} = \frac{[\Sigma_k]_{i,i}}{\alpha r_{k+1}} \quad \text{for } 1 \leq i \leq k \tag{34}$$

Then, noticing that

$$\frac{b_i}{\sqrt{[\Sigma_{k+1}]_{i,i}}} = \frac{b_i\sqrt{\alpha r_{k+1}}}{\sqrt{[\Sigma_k]_{i,i}}} > \frac{b_i}{\sqrt{[\Sigma_k]_{i,i}}} = g(\tilde{p}_i^{(k)})$$

it results that

$$\frac{b_i}{\sqrt{[\Sigma_{k+1}]_{i,i}}} \in \mathrm{Im}(g)$$

and we can take

$$\tilde{p}_i^{(k+1)} = \begin{cases} g^{-1}\left(\dfrac{b_i}{\sqrt{[\Sigma_{k+1}]_{i,i}}}\right) & \text{for } 1 \leq i \leq k \\ \tilde{p}_i^{(k)} \dfrac{1 - p - \sum_{j=1}^k \tilde{p}_j^{(k+1)}}{1 - p - \sum_{j=1}^k \tilde{p}_j^{(k)}} & \text{for } i > k \end{cases} \tag{35}$$

resulting $\tilde{p}_i^{(k+1)} < \tilde{p}_i^{(k)}$ for $1 \leq i \leq k$, which implies that $\tilde{p}_i^{(k+1)} > \tilde{p}_i^{(k)} > 0$ for $k+1 \leq i \leq n$. Thus, the parameters defined in Eq.(35) satisfy $\tilde{p}_i^{(k+1)} > 0$ and $\sum_{i=1}^n \tilde{p}_i^{(k+1)} = 1 - p$. Then, we have

$$\begin{aligned} \Sigma_{k+1} &= \mathcal{X}\left(\left[\frac{(b_1)^2}{g(\tilde{p}_1^{(k+1)})^2} \ \frac{(b_2)^2}{g(\tilde{p}_2^{(k+1)})^2} \ \cdots \ \frac{(b_{k+1})^2}{g(\tilde{p}_{k+1}^{(k+1)})^2}\right]^T\right) \\ &= \begin{bmatrix} \dfrac{\Sigma_k}{\alpha r_{k+1}} & \dfrac{c_k}{\alpha r_{k+1}} \\ \dfrac{c_k^T}{\alpha r_{k+1}} & \frac{(b_{k+1})^2}{g(\tilde{p}_{k+1}^{(k+1)})^2} \end{bmatrix} = \begin{bmatrix} \dfrac{\Sigma_k}{\alpha r_{k+1}} & \dfrac{c_k}{\alpha r_{k+1}} \\ \dfrac{c_k^T}{\alpha r_{k+1}} & d_{k+1} \end{bmatrix} \end{aligned}$$

with $d_{k+1} > \tilde{d}_{k+1}$. Thus, it results that

$$\begin{aligned} \frac{c_k^T}{\alpha r_{k+1}}\left(\frac{\Sigma_k}{\alpha r_{k+1}}\right)^{-1}\frac{c_k}{\alpha r_{k+1}} &= \frac{c_k^T(\Sigma_k)^{-1}c_k}{\alpha r_{k+1}} \\ &= \frac{\tilde{d}_{k+1}}{\alpha} < d_{k+1} \end{aligned}$$

and then Σ_{k+1} is positive definite.

The proof by induction then concludes by observing that Σ_1 is positive definite for any choice of the parameters $\tilde{p}_i^{(1)} > 0$ such that $\sum_{i=1}^n \tilde{p}_i^{(1)} = 1 - p$. We can initially take, in particular, $\tilde{p}_i^{(1)} = \frac{1-p}{n}$. □

4.2 Control Design with PUB guarantee

Making use of Lemmas 10 and 11, the following theorem establishes that a feedback gain K can be computed so that the closed–loop system of Eq.(17) has an arbitrarily prescribed PUB.

Theorem 12 (PUB Assignment) *Given a system*

$$\mathrm{d}x(t) = Ax(t)\mathrm{d}t + Bu(t)\mathrm{d}t + BG\mathrm{d}v(t) \tag{36}$$

where the pair (A, B) is controllable, the disturbance vector $v(t)$ is a zero mean stationary stochastic process with uncorrelated increments and incremental covariance $\Sigma_v\mathrm{d}t$, and the pair $(A, B\Sigma^{1/2})$ with Σ defined in (19) being controllable, and given a vector $b \succ 0$ and a probability $0 < p < 1$, there exists a control law $u(t) = Kx(t)$ such that any set of the form $S = \{x : |x_i| \leq b_i + \varepsilon;\ i = 1, \dots, n\}$ with $\varepsilon > 0$, is a PUB with probability p of the closed-loop system (17).

PROOF. Given a control gain K such that $A + BK$ is Hurwitz, if we find certain constants $\tilde{p}_i > 0$ subject to $\sum_{i=1}^n \tilde{p}_i = 1 - p$, then, according to Theorem 6, the set $S = \{x : |x_i| \leq b_i + \varepsilon;\ i = 1, \dots, n\}$ would be a PUB with probability p for the closed loop system of Eq.(17) provided that

$$b_i = \sqrt{[\Sigma_x]_{i,i}} g(\tilde{p}_i)$$

with

$$g(\tilde{p}_i) = \begin{cases} 1/\sqrt{\tilde{p}_i} & \text{general distribution} \\ \sqrt{2}\mathrm{erf}^{-1}(1-\tilde{p}_i) & \text{Gaussian distrib.} \end{cases} \tag{37}$$

and where Σ_x is the solution of the Lyapunov equation (22).

Let us suppose first that the pair (A, B) is in controller canonical form. Notice that in both cases (general and Gaussian distribution), the function $g(\cdot)$ verifies the hypothesis of Lemma 11 (with $a = 1$ and $a = 0$, respectively).

Then, taking m constants $q_j > 0, j = 1, \dots, m$, such that $\sum_{j=1}^m q_j = 1$, and defining σ_j for $j = 1, \dots, m$ as in the proof of Lemma 10, we can use the result of Lemma 11 and, for each $j \in \{1, \dots, m\}$, find d_j constants $\tilde{p}_i$, with $i = \sigma_j - d_j + 1, \dots, \sigma_j$ such that

$$\sum_{i=\sigma_j - d_j + 1}^{\sigma_j} \tilde{p}_i = q_j(1-p) = 1 - \hat{p}_j$$

and the Xiao matrix

$$\Sigma_x^j = \mathcal{X}\left(\left\{b_i^2/g(\tilde{p}_i)^2;\ i = \sigma_j - d_j + 1, \dots, \sigma_j\right\}\right)$$

is positive definite.

Then, according to Lemma 10, matrix $\Sigma_x = \mathrm{diag}(\Sigma_x^1, \dots, \Sigma_x^m)$ is an assignable positive definite covariance matrix under the control law $u = Kx$ with K computed from Eq.(21), and the closed-loop matrix $A + BK$ is Hurwitz.

Notice that the n diagonal entries of Σ_x are $b_i^2/g(\tilde{p}_i)^2$, and

$$\sum_{i=1}^n \tilde{p}_i = \sum_{j=1}^m \sum_{i=\sigma_j - d_j + 1}^{\sigma_j} \tilde{p}_i = \sum_{j=1}^m q_j(1-p) = 1 - p$$

Then, using Theorem 6, the set $S = \{x : |x_i| \leq b_i + \varepsilon;\ i = 1, \dots, n\}$, for any given $\varepsilon > 0$, is a PUB with probability p of the closed loop system (17).

In case the pair (A, B) is not in the controller canonical form, there exists a linear transformation U (see, e.g., [11]) that brings it into that form. Under this transformation, the system of Eq.(36) becomes

$$\mathrm{d}x_c(t) = A_c x_c(t)\mathrm{d}t + B_c u(t)\mathrm{d}t + B_c G \mathrm{d}v(t) \tag{38}$$

with $A_c = U^{-1}AU$, and $B_c = U^{-1}B$.

Let $\tilde{\Sigma}_c$ be a positive definite block diagonal Xiao matrix that according to Lemma 10 is an assignable covariance matrix for the pair (A_c, B_c). Define $\tilde{\Sigma}_x \triangleq U\tilde{\Sigma}_c U^T$ and let $\mu > 0$ be a number that also verifies

$$\mu \leq \mu_{\max} \triangleq \min_{1 \leq i \leq n} \frac{b_i^2}{(a+\delta)^2 [\tilde{\Sigma}_x]_{i,i}}$$

with $a = g(1)$, $g(\cdot)$ defined in Eq.(37), and $\delta \triangleq g(1-p) - a > 0$. Notice that

$$\frac{b_i}{\sqrt{\mu[\tilde{\Sigma}_x]_{i,i}}} \geq \frac{b_i}{\sqrt{\mu_{\max}[\tilde{\Sigma}_x]_{i,i}}} \geq a + \delta$$

which implies that the first term on the left hand side of the last inequality is in $\mathrm{Im}(g) = [a, \infty)$. Define

$$\check{p}_i(\mu) \triangleq g^{-1}\left(\frac{b_i}{\sqrt{\mu[\tilde{\Sigma}_x]_{i,i}}}\right), i = 1, \dots, n \tag{39}$$

Notice that each function $\check{p}_i(\mu)$ monotonically grows with μ, taking values in the interval $(0, \check{p}_i(\mu_{\max})]$. Moreover, there exists at least one function $\check{p}_{i^*}(\mu)$ where

$$i^* = \arg\min_{1 \leq i \leq n} \frac{b_i^2}{(a+\delta)^2[\tilde{\Sigma}_x]_{i,i}}$$

that takes values in the interval $(0, 1-p]$. Define then

$$s(\mu) \triangleq \sum_{i=1}^n \check{p}_i(\mu) \tag{40}$$

and notice that the continuous function $s(\mu)$ monotonically grows with μ and can take any value from 0 to $s_{max} \geq 1 - p$. Thus, a value $\bar{\mu}$ exists such that $s(\bar{\mu}) = 1 - p$.

Then, taking $\Sigma_c = \bar{\mu}\tilde{\Sigma}_c$, it results that Σ_c is an assignable covariance matrix of system (38) under the control law $u(t) = K_c x_c(t)$ with

$$K_c = -B_c^\dagger(A_c\Sigma_c + \Sigma_c A_c^T + B_c \Sigma B_c^T)(I - B_c B_c^\dagger/2)\Sigma_c^{-1} \tag{41}$$

Then, the covariance matrix of system (36) under the control law $u(t) = Kx(t)$ with $K = K_c U^{-1}$ results

$$\Sigma_x = U\Sigma_c U^T = U\bar{\mu}\tilde{\Sigma}_c U^T = \bar{\mu}\tilde{\Sigma}_x$$

Taking $\tilde{p}_i = \check{p}_i(\bar{\mu})$ for $i = 1, \dots, n$, from Eq.(39) it results that

$$b_i = g(\tilde{p}_i)\sqrt{\bar{\mu}[\tilde{\Sigma}_x]_{i,i}} = g(\tilde{p}_i)\sqrt{[\Sigma_x]_{i,i}}$$

with $\sum_{i=1}^{n} \tilde{p}_i = 1 - p$. Then, according to Theorem 6, the set $S = \{x : |x_i| \leq b_i + \varepsilon;\ i = 1, \ldots, n\}$ is a PUB with probability p for the closed-loop system (17). $\square$

From Theorem 12 and Lemmas 10 and 11 the following algorithm can be devised to find an assignable covariance matrix Σ_x and the corresponding control law $u(t) = Kx(t)$ such that system (36) has a PUB of size b with probability p. We consider that the system has m inputs and that the corresponding controllability indexes are d_j with $j = 1, \ldots, m$. We also define the cumulative indexes $\sigma_i \triangleq \sum_{j=1}^{i} d_j$, for $i = 1, \ldots, m$ as in the proof of Lemma 10.

Algorithm 1 *PUB Design – Controller Canonical Form*

(1) *Choose m constants q_j such that*[7] $\sum_{j=1}^{m} q_j = 1$.
(2) *For each j in $1, \ldots, m$,*
  (a) *Compute $\hat{p}_j = 1 - q_j(1-p)$.*
  (b) *Take $b^j = [b_{\sigma_j - d_j + 1}, \ldots, b_{\sigma_j}]$.*
  (c) *Take $k = 1$ and $\tilde{p}_i^{(1)} = (1 - \hat{p}_j)/d_j$ for $i = 1, \ldots, d_j$.*
  (d) *Using b^j instead of b, form Σ_k from Eq. (30). If $k = d_j$ go to step (2i).*
  (e) *Using b^j instead of b form $\tilde{\Sigma}_{k+1}$ from Eq.(32).*
  (f) *If $\tilde{\Sigma}_{k+1} > 0$, take $\tilde{p}_i^{(k+1)} = \tilde{p}_i^{(k)}$ and go to step (2h).*
  (g) *Otherwise, choose $\alpha > 1$ and compute $\tilde{p}_i^{(k+1)}$ from Eqs.(33)–(35).*
  (h) *Let $k := k + 1$ and go back to step (2d).*
  (i) *Take $\Sigma_x^j = \Sigma_k > 0$*
(3) *Compute the block diagonal Xiao matrix Σ_x from Eq.(20).*
(4) *Calculate $\Sigma = G\Sigma_v G^T$ and compute K from Eq.(21).*

When the pair (A, B) is not in its controller canonical form, the following algorithm can be devised to find a control law such that the closed–loop system has a PUB of size b with probability p.

Algorithm 2 *PUB Design – General Form*

(1) *Compute the matrix U that brings the system (36) into its controller canonical form (see, e.g., [11]).*
(2) *Choose a positive vector $\tilde{b}_c$ and using Algorithm 1 up to step (3), compute an assignable covariance matrix $\tilde{\Sigma}_c$ for the system in controller canonical form.*
(3) *Find $\bar{\mu}$ such that $s(\bar{\mu}) = 1 - p$, with $s(\cdot)$ defined in Eqs.(39)–(40).*
(4) *Compute $\Sigma_c = \bar{\mu}\tilde{\Sigma}_c$ and K_c from Eq.(41).*
(5) *Calculate the controller gain $K = K_c U^{-1}$*

[7] A reasonable choice would be $q_j = d_j/n$, which assigns an exit probability proportional to the dimension of each controllability subspace.

5 Example

We consider a system described by Eq.(36) with

$$A = \begin{bmatrix} 2 & 2 & -1 & -1 & -3 \\ 1 & 0 & -1 & 0 & -1 \\ 0 & 1 & 0 & -1 & -1 \\ 1 & 0 & -1 & 0 & 0 \\ 2 & 1 & -1 & -1 & -3 \end{bmatrix}; \quad B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ -1 & -2 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}; \tag{42}$$

$G = [1\ 1]^T$, and $v(t)$ is a Wiener process with incremental covariance $\Sigma_v \mathrm{d}t = 0.01\mathrm{d}t$. We want this system to have a PUB $S = \{x : |x_i| \leq b_i + \varepsilon;\ i = 1, \ldots, n\}$, with $b = \begin{bmatrix} 0.1 & 0.1 & 0.1 & 0.1 & 0.1 \end{bmatrix}^T$, for all $\varepsilon > 0$ with probability $p = 0.9$. We next follow Algorithm 2:

(1) The matrix that brings the system into its controller canonical form is

$$U = \begin{bmatrix} 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix};$$

(2) The positive vector $\tilde{b}_c$ is chosen here as $\tilde{b}_c = b$. Then, following Algorithm 1 with $q_1 = 3/5$, $q_2 = 2/5$, we obtain the following assignable covariance matrix for the system in controller canonical form:

$$\tilde{\Sigma}_c = 10^{-3} \begin{bmatrix} 0.9239 & 0 & -0.9239 & 0 & 0 \\ 0 & 0.9239 & 0 & 0 & 0 \\ -0.9239 & 0 & 2.783 & 0 & 0 \\ 0 & 0 & 0 & 1.848 & 0 \\ 0 & 0 & 0 & 0 & 1.848 \end{bmatrix}$$

(3) Computing $\tilde{\Sigma}_x = U\tilde{\Sigma}_c U^T$ and $\check{p}_i(\mu)$ from Eq.(39), the value $\bar{\mu}$ such that $s(\bar{\mu}) = 1 - p$ in Eq.(40) is $\bar{\mu} = 0.4911$. Then, the exit probabilities $\tilde{p}_i = \check{p}_i(\bar{\mu})$ result $\tilde{p}_1 = \tilde{p}_4 = 0.0189$, $\tilde{p}_2 = 0.000003$, $\tilde{p}_3 = 0.055597$, and $\tilde{p}_5 = 0.0067$.
(4) We compute $\Sigma_c = \bar{\mu}\tilde{\Sigma}_c$ and

$$K_c = \begin{bmatrix} -16.4396 & -7.0118 & -15.4396 & -1 & -8.0102 \\ -16.9308 & 1 & -16.9308 & 0 & -4.5102 \end{bmatrix}$$

(5) We calculate the controller gain for the system in the original coordinates:

$$K = \begin{bmatrix} -30.8792 & 1.9984 & 15.4396 & -1 & 22.8689 \\ -33.8616 & 5.5102 & 16.9308 & 0 & 29.3513 \end{bmatrix}$$

This gain ensures that system (17) has the desired PUB with probability $p = 0.9$. This fact can be checked by applying Theorem 6 to the closed loop system of Eq.(17) with the values of $\tilde{p}_i$ computed in the step 3 above.

In order to verify the results, we performed 5,000 simulations of the system from the initial state $x(t_0) = 10 \cdot b$ (outside S) and for each instant of time $t_k = 0.01k$, with $k = 0, \ldots, 120{,}000$, we evaluated the *exit ratio* e as the number of times $x(t_k)$ lies outside the PUB divided by 5,000. We found that for any $t_k \geq 1{,}050$, between 7.14% and 10% of the simulations lie outside the calculated PUB, which agrees with the maximum theoretical probability $(1 - p)$ of 10%. The computed exit ratio as a function of the time is depicted in Figure 1.

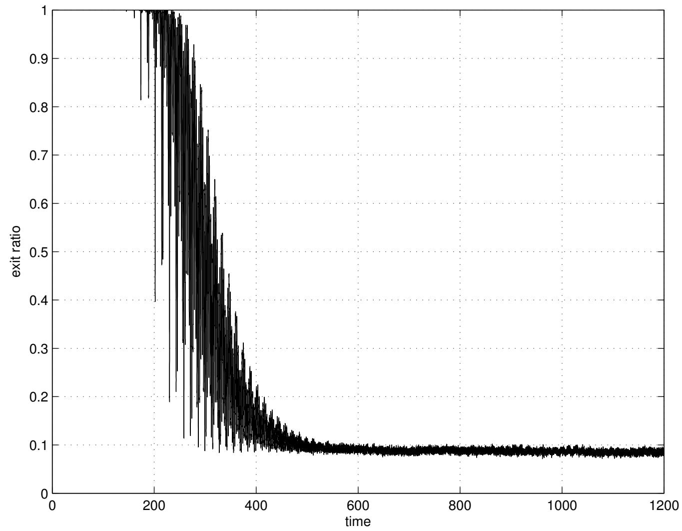

Fig. 1. Exit ratio vs. t for the PUB

6 Conclusions

We have extended the notions of PUB and PIS to the continuous-time domain, deriving their main properties and providing formulas for their calculation. In the case of PIS, a redefinition was required to take into account the fundamental limitations imposed by the infinite-bandwidth nature of continuous-time white noise. Then, a controller design technique was presented to assign a predetermined PUB having a given probability p. The results were illustrated with a numerical example.

Future work will include the use of the results in control applications such as fault tolerant control and model predictive control, where the notions of invariance and ultimate boundedness play a fundamental role and the concepts and tools developed here can allow to deal with unbounded disturbances in a probabilistic framework.

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