Simplified Design of Practically Stable MPC Schemes

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Abstract

This work introduces a novel and simple way of designing MPC schemes that ensure practical stability under less restrictive assumptions than those of existing approaches. The terminal control invariant set required by most stabilizing MPC formulations is replaced by a pair of simpler inner and outer sets, which are not invariant but satisfy a weaker condition. The advantage of using this pair of sets instead of a classical invariant set is the flexibility in their design and ultimately the simplicity. Two key modifications, one in the stage cost function and other related to a constraint, are introduced in the optimal control problem that MPC solves at each time. It is shown that convergence to the outer set, which is the target region, is ensured within finite-time and that the proposed MPC scheme can keep controlling the system in the target set without requiring a different local controller. A numerical example with a nonlinear model of an inverted pendulum is used to illustrate these results.

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1. Introduction

Model Predictive Control (MPC) is one of the most employed advanced control techniques in industry. MPC computes the control actions as the result of an underlying optimization problem, which relies on model-based predictions of the future behavior of the plant and considers hard constraints on control and states [28]. The concept of receding horizon policy (RHP) turns MPC into a realizable optimization-based controller. RHP consists of solving a tractable optimization problem over a finite horizon at each time, yielding a series of control actions, of which only the first one is applied. The horizon is then "shifted" at the next time step and the process is repeated for the new system state, effectively closing the loop and enhancing robustness. The historical challenge (from a theoretical perspective) has been to propose MPC formulations that stabilize the closed-loop system, whilst guaranteeing recursive feasibility of the sequence of optimization problems solved on-line, with the largest possible domain of attraction [27]. Since the seminal work of [24], a plethora of approaches have been proposed, which not only address closed-loop stability but also focus on overcoming the limitations of the original ideas, including enlarging the domain of attraction [23], relaxing the terminal conditions [12], re-parametrizations of the control degrees of freedom [1], and avoiding the explicit (and many times prohibitive) computation of sets and functions.

A standard technique to establish closed-loop stability of MPC schemes is by using the optimal value function of the underlying optimal control problem as a Lyapunov function candidate [16, 24]. To address recursive feasibility, a typical method is to incorporate a group of terminal conditions, more specifically, a terminal cost which is a control Lyapunov function in a terminal region where the final predicted state is forced to belong to. This terminal region has to be a control invariant set for the system, which means that if the state enters this set, then there exists a feasible control input that can keep it inside that set.

The Multi-Parametric Toolbox 3.0 [14] is an interesting tool to obtain these terminal costs and sets for certain types of systems, namely linear time invariant, piecewise affine and mixed logical dynamical. Nevertheless, when considering general nonlinear dynamics, procedures are not so well established. The conventional methodology involves obtaining first these terminal elements for the linearized system and then compensating for the corresponding linearization error. Methods for continuous [25, 7] and discrete-time [17, 32] cases were developed using this idea. Other works considered a different approach to approximate the nonlinear dynamics using linear difference inclusions [15, 9]. More recently and based on finite-step control Lyapunov functions [19], the work of [20] presented a framework to obtain cyclically time-varying terminal costs and sets for discrete-time nonlinear MPC (NMPC). Building upon this, the authors of [11] developed the stabilizing NMPC toolbox that, among other things, is intended to compute ellipsoidal terminal sets and quadratic terminal costs.

The aforementioned terminal constraint for the last predicted state to belong to an invariant set [24] has been used in [29] to prove that feasibility (rather than optimality) is enough for ensuring stability and finite-time convergence. The authors of that work propose, among other controllers, optimal and suboptimal versions of a "dual-mode" MPC strategy with a stage cost function $L(x,u) \geq \ell(\|(x,u)\|)$ for all $x \notin \Omega$ and for all $u \in U$, where $\ell(s)$ is a $\mathcal{K}$-function, Ω is the target set, and U is a bounded input set. They also assume that $L(x, h_L(x)) = 0$ for all $x \in \Omega$, where $h_L(x)$ is a control law local to Ω, and that for the system in closed-loop with $u = h_L(x)$, Ω is positively invariant. Notice that in this controller, MPC is applied only in the first stage of the dual-mode operation, achieving finite-time convergence to an invariant set. Then, in the second mode, a local controller is directly assumed to attain asymptotic stability to the origin. A similar idea is presented in [4], where a positively invariant set (that is also contractive) is used as target. In this strategy, the stage cost, which includes a specific term related to the input, becomes null in this set and a terminal constraint imposes that the final predicted state must belong to it. Then, these conditions are employed to prove finite-time convergence (but not ultimate boundedness) to the target set.

Beyond implicit control limitations related to the use of invariant sets, such as the reduction of the domain of attraction of the controller, these sets are in general difficult to obtain and hard to handle, even for simple linear systems, when they are polyhedra with, typically, a high number of faces. These limitations have motivated the development of stabilizing MPC schemes that do not impose terminal constraints in their formulations. For example, in [13] a receding horizon control strategy with a stage cost reflecting economic criteria instead of distance to a reference is proposed and, by means of certain verifiable conditions, convergence to a neighborhood of the optimal steady state is proven. Another approach, presented in [22, 23], characterizes a region for which the terminal constraint of the optimization problem can be removed without losing asymptotic stability. It is then proven that, by appropriately weighting the terminal cost, this region can be enlarged so that the domain of attraction of the MPC with the terminal constraint can be practically reached. There exists also the contractive MPC idea [10] which uses a constraint on the state to be "contracted", in terms of some norm, at the end of the prediction horizon in relation to the state at the beginning. [2, 3] extend this scheme avoiding stability-related terminal constraints and using a time-varying cost function with a performance related term weighted by a controller internal state and another term associated with stability that is scaled and assumed to satisfy a contraction constraint. Convergence of the closed-loop system is proven with this strategy.

Despite the latter works, the use of invariant sets for the design of MPC strategies may nowadays be considered standard—its ultimate form is known as set-based MPC [4, 26]. However due to the difficulties to compute and to handle these sets, their applications are limited.

To address this problem, we propose here an alternative set-based MPC scheme in which the invariant terminal set is replaced by two simpler sets, one containing the other. It will be shown that more classical formulations involving invariant sets can be retrieved as a particular case when these inner and outer sets are equal to each other. Practical stability of the closed-loop system is established even though these sets only satisfy a weaker condition than invariance. Moreover, it will be shown that by means of:

- a constraint for the predicted state trajectory to have at least one point of intersection with the inner set (a "passing" condition at any time, not as a terminal constraint), and

- a modification of the stage cost function to account for the set membership of the state to the outer set,

recursive feasibility of the optimization problem, convergence and ultimate boundedness to the latter set can be guaranteed.

The ideas involved in the set-based MPC design presented in this work bring along two independent contributions:

(C1) the replacement of the invariant set by a pair of simpler inner-outer sets that, nevertheless, keep the practical stability property of the controller, with finite-time convergence and ultimate boundedness to the target set, and

(C2) the possibility to continue controlling inside the target region without knowing nor switching to another local controller.

Contribution (C1) is particularly important when considering challenging systems with nonlinearities, switching modes, finite input sets, etc. Generally, in these cases it is not feasible in practice to compute or to use invariant sets, which means that stability properties of MPC strategies cannot be guaranteed in this way. Contribution (C2) is a notable difference with respect to the previously mentioned dual-mode MPC [29], which requires a local controller (with certain specific properties) for the invariant target region. The example presented in this work, based on a third-order nonlinear model of an inverted pendulum, attempts to illustrate these contributions.

The remainder of this paper is organized as follows. First, in Section 2 the problem is described and some assumptions are made. Then, the proposed MPC scheme is presented in Section 3. Section 4 contains the stability analysis of the closed-loop system. Section 5 presents a procedure for designing a controller based on our formulation. The results are discussed in Section 6. Section 7 presents the aforementioned simulation example and finally, conclusions are given in Section 8.

Notation

Let $\mathbb{R}$ and $\mathbb{R}_{\geq 0}$ denote the real and non-negative real numbers. $\mathbb{N}$ denotes the set of natural numbers $\{1, 2, 3, \ldots\}$, $\mathbb{Z}$, the set of integer numbers and $\mathbb{Z}_{N:M}$, the set of integers in the interval $[N, M]$. The ceiling of $b \in \mathbb{R}$ is defined by $\lceil b \rceil \triangleq \min\{z \in \mathbb{Z} : z \geq b\}$. For a sequence $\{a_1, \ldots, a_n\}$ with $n \in \mathbb{N}$ and a set A, the notation $\{a_1, \ldots, a_n\} \in A^n$ means that every element of that sequence belongs to A. Given a system with control input u_k, possible future values of the input at time $k+i$ assumed at the kth sample are denoted by $u_{k+i|k}$. Similarly, predictions of the system state, x_k, at time $k+i$, based on knowledge up to sampling instant k and considering future values of inputs up to $k+i-1$, are indicated as $x_{k+i|k}$. An open ball with radius $\epsilon > 0$ centered at $x \in \mathbb{R}^n$ is denoted by $B(x, \epsilon) \triangleq \{y \in \mathbb{R}^n : |x - y| < \epsilon\}$. When this ball is centered at the origin, notation is simplified to B_ϵ.

2. Problem Description

Consider the discrete-time nonlinear system given by:

$$x_{k+1} = f(x_k, u_k), \tag{1}$$

where $x_k \in X \subseteq \mathbb{R}^n$ and $u_k \in U \subseteq \mathbb{R}^m$ are the state and input vectors at time k, respectively, and X, U are closed sets which define the corresponding constraints. The problem is to design an MPC controller, with a horizon length $N \in \mathbb{N}$, which ensures that state trajectories converge to and remain in a compact *target set* $\Omega_O \subset X$ whilst satisfying state and control constraints at all times.

The results presented are based on the existence of an *inner set* $\Omega_I \subseteq \Omega_O$ satisfying the following assumption:

Assumption 1 (Existence of an N-steps Control Inner Set Ω_I)**.** Given the system of Eq. (1), the horizon length N and the compact target set Ω_O, a compact N-*steps control inner set* $\Omega_I \subseteq \Omega_O$ exists with the following property: if $x_k \in \Omega_I$, then a sequence of control inputs $\{u_k, \ldots, u_{k+i-1}\} \in U^i$ for some $i \in \mathbb{Z}_{1:N}$ exists such that $x_{k+i} \in \Omega_I$ and $\{x_k, \ldots, x_{k+i-1}\} \in \Omega_O^i$.

In plain words, an N-steps control inner set has the property that, if the system state "visits" the set at some time k, then, it is feasible for the system to revisit the set, given the available control authority, in at most N steps without leaving Ω_O at any time.

Our MPC problem will be solved for initial states x_k belonging to the *up-to-N-steps controllable set* to the inner set Ω_I, defined as follows:

Definition 1 (Up-to-N-steps Controllable Set $X_N^{\Omega_I}$)**.** Given the system of Eq. (1), the horizon length N and the inner set Ω_I, we say that $X_N^{\Omega_I} \subseteq X$ is the up-to-N-steps controllable set to Ω_I provided that if $x_k \in X_N^{\Omega_I}$, then there exists a sequence of control inputs $\{u_k, \ldots, u_{k+i-1}\} \in U^i$ for some $i \in \mathbb{Z}_{1:N}$ for which $x_{k+i} \in \Omega_I$ and $\{x_k, \ldots, x_{k+i-1}\} \in X^i$.

Notice that by Assumption 1 and Definition 1, it follows that $\Omega_I \subset X_N^{\Omega_I}$, a condition that can be linked to a sort of controllability assumption. However, since $X_N^{\Omega_I}$ depends exclusively on Ω_I and given that Ω_O is a predefined set, the condition $\Omega_O \subset X_N^{\Omega_I}$ may not be accomplished.

The control goals of finite-time convergence and ultimate boundedness to Ω_O imply that the closed-loop system with our controller achieves uniform practical asymptotic stability of a ball containing Ω_O, from initial states in $X_N^{\Omega_I}$. This stability property is usually defined in the following way:

Definition 2 (Uniform Practical Asymptotic Stability)**.** Given $\epsilon > 0$, a ball B_ϵ is said to be Uniformly Practically Asymptotically Stable for (1) in a set A if there exists a $\mathcal{KL}$-function $\beta(s,t)$ such that the solution of (1) from any initial state $x_0 \in A$ satisfies

$$|x_k| \leq \beta(|x_0|, k) + \epsilon,\ \forall k \geq 0.$$

Taking into account the relationship between uniform practical asymptotic stability, and finite-time convergence and ultimate boundedness, and considering space and readability reasons, we shall refer to these properties simply as practical stability.

Remark 1. It is common in applications for the target set to be taken as a control invariant region where the system can remain indefinitely. Assumption 1 relaxes this requirement by allowing Ω_O to be an arbitrary set containing an N-steps control inner set Ω_I that does not need to be invariant. While it can be proven that the existence of Ω_I implies the existence of at least one control invariant set inside Ω_O, this invariant set needs neither be computed nor be explicitly used. Obviously, in case a control invariant set inside the target set Ω_O is already known, one could take Ω_I to be this control invariant set and retrieve the classical MPC formulations.

Remark 2. The concepts of weak p-invariance [21, 30, 31] and (k,λ)-contractiveness [5, 18] are related to Assumption 1 in the sense that these concepts also require the state to return to a certain set after some time. However, a significant difference with those ideas is that here the state is expected not to leave the target region Ω_O rather than the constraint set X. In general, when properties like p-invariance or (k,λ)-contractiveness hold, nothing can be said about how far from the target set the state can go.

3. Proposed MPC Scheme

Our MPC scheme is designed starting from a preliminary stage cost function given by $\tilde{L} : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}_{\geq 0}$. From this original function, we define

$$\tilde{L}_{\max} \triangleq \sup_{x \in \Omega_O, u \in U} \tilde{L}(x,u), \tag{2}$$

$$\tilde{L}_{\min} \triangleq \inf_{x \in X \setminus \Omega_O, u \in U} \tilde{L}(x,u). \tag{3}$$

We shall assume that $\tilde{L}(x,u)$ and sets Ω_O and U are such that $\tilde{L}_{\max}$ is finite, and $\tilde{L}_{\min} > 0$.

In order to ensure the convergence of the state from the region $X_N^{\Omega_I}$ to Ω_O, we change the original stage cost function such that it becomes smaller inside Ω_O and, outside Ω_O, it becomes larger for predicted states further in the future. This is achieved by using two constants c and r, to yield a modified index-dependent stage cost function $L : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{N} \to \mathbb{R}_{\geq 0}$, as follows:

$$L(x, u, i) \triangleq \begin{cases} c\tilde{L}(x, u) & \text{if } x \in \Omega_O \\ (1 + (i-1)r)\tilde{L}(x, u) & \text{if } x \notin \Omega_O \end{cases}, \tag{4}$$

where $c \geq 0$ and $r \in [0, 1)$ are such that

$$cN\tilde{L}_{\max} \leq r\tilde{L}_{\min}, \tag{5}$$

and where $i \in \mathbb{Z}_{1:N}$ is the stage index, i.e., the number of samples from the beginning of the prediction horizon.

The idea behind the usage of c in Eq. (4), with a value that satisfies the condition stated in Eq. (5), is to favor having all the predicted states inside the target region Ω_O over having a single predicted state outside Ω_O. In addition, the term $(i-1)r$ results in a predicted state having a reduced stage cost when it approaches the beginning of the prediction horizon. That way, if the tail of the optimal input trajectory at time k is applied from time $k+1$, then the stage costs of the predicted states outside the target region are reduced from time k to time $k+1$. As it will become clear from the proof of Theorem 1, provided that this reduction is larger than the stage cost of the predicted states inside the target region, a condition that is related to Eq. (5), then the MPC cost function is reduced from time k to time $k+1$.

Given a feasible input sequence $\boldsymbol{u}_k = \{u_{k|k}, \ldots, u_{k+N-1|k}\}$ and a predicted state sequence $\boldsymbol{x}_k = \{x_{k|k}, \ldots, x_{k+N|k}\}$ computed as

$$x_{k|k} = x_k, \tag{6}$$

$$x_{k+i+1|k} = f(x_{k+i|k}, u_{k+i|k}) \tag{7}$$

for all $i \in \mathbb{Z}_{0:N-1}$, we define the MPC cost function as[1]

$$V(x_k, \boldsymbol{u}_k) \triangleq \sum_{i=1}^{N} L(x_{k+i|k}, u_{k+i-1|k}, i). \tag{8}$$

[1]This form of the cost function differs from other formulations in that the initial state $x_{k|k} = x_k$ is not taken into account (since it is the same for all possible predicted state evolutions). However, notice that by redefining $L(x, u, i)$, it is general and could include commonly used terms like a terminal cost.

Now, for a state $x_k \in X_N^{\Omega_I}$, consider the following optimal control problem:

$$P(x_k) \ : \ V^*(x_k) \triangleq \min_{\boldsymbol{u}_k \in U_N(x_k)} V(x_k, \boldsymbol{u}_k), \tag{9}$$

subject to Eqs. (6) and (7) and where $U_N(x_k)$ denotes the set of feasible control input sequences $\boldsymbol{u}_k$ that satisfy

$$u_{k+i|k} \in U, \ \forall i \in \mathbb{Z}_{0:N-1}, \tag{10}$$

$$x_{k+i|k} \in X, \ \forall i \in \mathbb{Z}_{1:N}, \tag{11}$$

$$x_{k+i|k} \in \Omega_I, \text{ for some } i \in \mathbb{Z}_{1:N}. \tag{12}$$

Note that the usual terminal constraint is replaced by the less restrictive condition (12) of "passing through" the set Ω_I at some step over the prediction horizon.

We shall assume that the optimal control problem (9) at each sampling time k has at least one solution[2] given by the optimal input sequence $\boldsymbol{u}_k^* \triangleq \{u_{k|k}^*, \dots, u_{k+N-1|k}^*\}$, that has associated an optimal state sequence $\boldsymbol{x}_k^* \triangleq \{x_{k|k}^*, \dots, x_{k+N|k}^*\}$ computed as

$$x_{k+i+1|k}^* = f(x_{k+i|k}^*, u_{k+i|k}^*). \tag{13}$$

Finally, following a receding horizon scheme, the first element of $\boldsymbol{u}_k^*$ is applied to the system while the rest is discarded. The control law is then given by $\kappa_{MPC}(x) \triangleq u_{k|k}^*$ and, under closed-loop operation, the system is described as

$$x_{k+1} = f(x_k, u_{k|k}^*) = f(x_k, \kappa_{MPC}(x_k)).$$

Remark 3. Even though the proposed stage cost given by Eq. (4) is index-dependent, the MPC cost function defined in Eq. (8) is not time-varying. The time argument of $L(x, u, i)$ is the stage index so, from the perspective of the MPC cost function, it is fixed and always goes from 1 to N.

Remark 4. There are different (and independent) sufficient conditions under which the optimization problem has at least one solution, i.e., it is not ill-posed. The following are two examples of these conditions:

- Finite control input set. In this case there is always at least an input sequence with the minimum cost.

[2]The results derived in Section 4 also hold when the solution is not unique.

- Original stage cost $\tilde{L}(x,u)$ and function $f(x,u)$ continuous, with compact U. In this case, even though continuity of the stage cost $L(x,u,i)$ is lost on the boundary of the target set Ω_O, lower semicontinuity is kept because $L(x,u,i)$ is lower in the closed part of the target region. Then, it can be proven that the MPC cost function itself is lower semicontinuous (on a compact set) and that guarantees (see [6], Theorem 3 from Section 6 in Chapter 4), that the minimization problem has at least one solution.

Remark 5. Regarding the numerical implementation of the strategy, especially considering the constraint (12), a first approach could be to solve N optimization problems with independent constraints for the state to belong to the inner set at all possible times over the prediction horizon. Then, the control sequence to be chosen would be the one that achieves the lowest cost among the N problems. Notice that a parallel implementation should not involve much more time for any reasonable horizon length. Furthermore, even though this may seem to require a much higher computational burden, it should be recalled that checking set membership constraints in our formulation would be less expensive given the simplicity of the inner and outer sets.

4. Stability Analysis

The following lemma is a preliminary result linking the bound $cN\tilde{L}_{\max}$ on the MPC cost function with the fact that all the predicted states are inside the target set Ω_O.

Lemma 1. *Consider a state $x_k \in X_N^{\Omega_I}$ and a feasible input sequence $\boldsymbol{u}_k$. Then, the cost function of Eq. (8) verifies $V(x_k, \boldsymbol{u}_k) \leq cN\tilde{L}_{\max}$ if and only if $x_{k+j|k} \in \Omega_O$, $\forall j \in \mathbb{Z}_{1:N}$.*

Proof. Suppose that for certain $j \in \mathbb{Z}_{1:N}$ it results $x_{k+j|k} \notin \Omega_O$. Then, from Eq. (4) the stage cost of that state would be

$$L_{k+j,k} = (1 + (j-1)r)\tilde{L}_{k+j,k} \geq \tilde{L}_{\min} > 0 \tag{14}$$

where we used, for simplicity, the notation

$$L_{k+j,k} \triangleq L(x_{k+j|k}, u_{k+j-1|k}, j), \tag{15}$$

$$\tilde{L}_{k+j,k} \triangleq \tilde{L}(x_{k+j|k}, u_{k+j-1|k}). \tag{16}$$

Then, using Eq. (5), we have for $r \in (0, 1)$ that

$$L_{k+j,k} \geq \tilde{L}_{\min} \geq \frac{cN\tilde{L}_{\max}}{r} > cN\tilde{L}_{\max}$$

contradicting that $V(x_k, \boldsymbol{u}_k) \leq cN\tilde{L}_{\max}$.

In case $r = 0$, according to Eq. (5) it results $cN\tilde{L}_{\max} = 0$ and then Eq. (14) also contradicts that $V(x_k, \boldsymbol{u}_k) \leq cN\tilde{L}_{\max}$.

Finally, from Eqs. (2), (4) and (8) it becomes straightforward that $x_{k+j|k} \in \Omega_O$, $\forall j \in \mathbb{Z}_{1:N}$ implies that $V(x_k, \boldsymbol{u}_k) \leq cN\tilde{L}_{\max}$ completing the proof. □

The following corollary to Lemma 1 establishes that when the optimal cost corresponding to a given state is smaller than the bound $cN\tilde{L}_{\max}$, then the next state lies inside Ω_O.

Corollary 1. *Given the MPC formulation of Eq.* (9) *with* $x_k \in X_N^{\Omega_I}$*, the condition* $V^*(x_k) \leq cN\tilde{L}_{\max}$ *implies* $x_{k+1} \in \Omega_O$*.*

Proof. Lemma 1 with $\boldsymbol{u}_k = \boldsymbol{u}_k^*$ establishes that the condition $V^*(x_k) \leq cN\tilde{L}_{\max}$ implies that $x^*_{k+1|k} \in \Omega_O$. The result then follows from the fact that $x_{k+1} = x^*_{k+1|k}$. □

The next theorem shows that when the optimal MPC cost function has a value that exceeds $cN\tilde{L}_{\max}$ at time $k+1$ (i.e., the optimal predicted state sequence has at least one entry outside the target set Ω_O), then the optimal value is smaller than the cost at time k, by a certain amount which admits a positive lower bound.

Theorem 1. *Consider the MPC formulation given by* (9)*. Then, for any* $x_k \in X_N^{\Omega_I}$ *the condition* $V^*(x_{k+1}) > cN\tilde{L}_{\max}$ *implies*

$$V^*(x_k) - V^*(x_{k+1}) \geq \epsilon \triangleq \frac{r\tilde{L}_{\min}}{N}. \tag{17}$$

In case $\Omega_I = \Omega_O$*, or provided that* $x^*_{k+1|k} \notin \Omega_O$*, it is also verified that*

$$V^*(x_k) - V^*(x_{k+1}) \geq \tilde{\epsilon} \triangleq (1 - r\frac{N-1}{N})\tilde{L}_{\min}. \tag{18}$$

The proof of this result, given below, is based on the following idea. After applying the first element of the optimal input sequence $\boldsymbol{u}_k^*$ at time k, a feasible strategy for the next step is to apply the same optimal input sequence (removing the first element $u^*_{k|k}$) until the state enters Ω_I and then use any input sequence that keeps the state inside Ω_O. Thus, the states outside Ω_O of the resulting predicted state sequence $\boldsymbol{x}_{k+1}$ will be already present in the previous optimal sequence $\boldsymbol{x}^*_k$. However the stage cost of those states outside Ω_O predicted at time $k+1$, will be smaller than the one of the same states outside Ω_O predicted at instant k, due to the factor r in Eq. (4). We suggest to consult [8] for further explanations and some illustrative examples.

Proof. Let $\boldsymbol{x}^*_k = \{x^*_{k|k}, \ldots, x^*_{k+N|k}\}$ be the optimal state sequence predicted at time k, computed according to Eq. (13).

Notice first that given the constraint of Eq. (12) there is at least one predicted state in Ω_I. Then, let $m \in \mathbb{N}$ denote the minimum time such that $x^*_{k+m|k} \in \Omega_I$.

Let $x_{k+1} = f(x_k, u^*_{k|k})$ be the state after applying the input $u_k = u^*_{k|k}$ and consider the following predicted state sequence:

$$x_{k+i+1|k+1} = f(x_{k+i|k+1}, u_{k+i|k+1})$$

with $x_{k+1|k+1} = x_{k+1}$ and where the sequence $\boldsymbol{u}_{k+1}$ verifies $u_{k+i|k+1} = u^*_{k+i|k}$ for all $i \in \mathbb{Z}_{1:m-1}$ and $u_{k+i|k+1}$ for all $i \in \mathbb{Z}_{m:N}$ is such that $x_{k+i+1|k+1} \in \Omega_O$ (which is feasible by Assumption 1, i.e., after passing through Ω_I, there exists a feasible input sequence that brings the state back to it keeping the trajectory inside Ω_O). Notice that with this control input sequence, it follows that $x_{k+i|k+1} = x^*_{k+i|k}$ for all $i \in \mathbb{Z}_{1:m}$ and this means that $x_{k+m|k+1} \in \Omega_I$. On the other hand, if $m < N$, the predicted states $x_{k+i|k+1}$ for $i \in \mathbb{Z}_{m+1:N}$ are in general different from $x^*_{k+i|k}$.

According to Eq. (8), the expression for $V^*(x_k)$ is

$$V^*(x_k) = \sum_{i=1}^{N} L(x^*_{k+i|k}, u^*_{k+i-1|k}, i) \tag{19}$$

while the cost associated to the sequence $\boldsymbol{x}_{k+1}$ is

$$V(x_{k+1}, \boldsymbol{u}_{k+1}) = \sum_{i=1}^{N} L(x_{k+i+1|k+1}, u_{k+i|k+1}, i). \tag{20}$$

Notice that, by optimality, $V(x_{k+1}, \boldsymbol{u}_{k+1}) \geq V^*(x_{k+1})$. Thus, the condition $V^*(x_{k+1}) > cN\tilde{L}_{\max}$ implies that $V(x_{k+1}, \boldsymbol{u}_{k+1}) > cN\tilde{L}_{\max}$.

Then, according to Lemma 1 there exists $j \geq 2$ such that $x_{k+j|k+1} \notin \Omega_O$. Moreover, since $\boldsymbol{u}_{k+1}$ was computed so that $x_{k+i|k+1} \in \Omega_O$ for all $i \geq m$, it results that $j < m$ (which also implies that $m > 2$). Since the predictions $x^*_{k+i|k}$ and $x_{k+i|k+1}$ coincide up to $i = m$, it results

$$x_{k+j|k+1} = x^*_{k+j|k} \notin \Omega_O \tag{21}$$

for some $j < m$.

Consider next the following sets of indices:

- $D_k = \{i \in \mathbb{Z}_{2:m} \mid x^*_{k+i|k} \notin \Omega_O\}$ and
- $\overline{D}_k = \mathbb{Z}_{2:m} \setminus D_k$.

Then, $V^*(x_k)$ in Eq. (19) can be rewritten as:

$$V^*(x_k) = L^*_{k+1,k} + \sum_{i \in D_k} (1 + (i-1)r)\tilde{L}^*_{k+i,k} \\ + \sum_{i \in \overline{D}_k} c\tilde{L}^*_{k+i,k} + \sum_{i=m+1}^{N} L^*_{k+i,k} \tag{22}$$

with the last term being null in case $m = N$ and where we used notations like those from Eqs. (15) and (16) but for optimal sequences $\boldsymbol{x}^*_k$ and $\boldsymbol{u}^*_k$.

Similarly, $V(x_{k+1}, \boldsymbol{u}_{k+1})$ in Eq. (20) can be rewritten as:

$$V(x_{k+1}, \boldsymbol{u}_{k+1}) = \sum_{i \in D_k} (1 + (i-2)r)\tilde{L}_{k+i,k+1} \\ + \sum_{i \in \overline{D}_k} c\tilde{L}_{k+i,k+1} + \sum_{i=m+1}^{N+1} L_{k+i,k+1}. \tag{23}$$

Then, taking into account that $x^*_{k+j+1|k} = x_{k+j+1|k+1}$ and $u^*_{k+j|k} = u_{k+j|k+1}$ for all $j \in \mathbb{Z}_{1:m-1}$, it results that $\tilde{L}^*_{k+j,k} = \tilde{L}_{k+j,k+1}$, and subtracting Eq. (23) from Eq. (22) we obtain

$$V^*(x_k) - V(x_{k+1}, \boldsymbol{u}_{k+1}) = L^*_{k+1,k} + \sum_{i \in D_k} r\tilde{L}^*_{k+i,k} \\ + \sum_{i=m+1}^{N} L^*_{k+i,k} - \sum_{i=m+1}^{N+1} L_{k+i,k+1}. \tag{24}$$

Recalling that at least one state $x^*_{k+i|k} \notin \Omega_O$ with $i < m$ and since $r < 1$, then,

$$L^*_{k+1,k} + \sum_{i \in D_k} r\tilde{L}^*_{k+i,k} \geq r\tilde{L}_{\min}. \tag{25}$$

The term $\sum_{i=m+1}^{N} L^*_{k+i,k}$ is non-negative and, considering that $x_{k+i|k+1} \in \Omega_O$ for all $i > m$, it follows that

$$\sum_{i=m+1}^{N+1} L_{k+i,k+1} \leq (N-1)c\tilde{L}_{\max}. \tag{26}$$

Then, from Eqs. (24), (25) and (26), it results that

$$\begin{aligned} V^*(x_k) - V(x_{k+1}, \boldsymbol{u}_{k+1}) &\geq r\tilde{L}_{\min} - (N-1)c\tilde{L}_{\max} \\ &\geq \frac{N-1}{N}(r\tilde{L}_{\min} - Nc\tilde{L}_{\max}) + \frac{r\tilde{L}_{\min}}{N}. \end{aligned}$$

From Eq. (5), we have $Nc\tilde{L}_{\max} \leq r\tilde{L}_{\min}$ and then, recalling that $V(x_{k+1}, \boldsymbol{u}_{k+1}) \geq V^*(x_{k+1})$, we obtain

$$V^*(x_k) - V^*(x_{k+1}) \geq \frac{r\tilde{L}_{\min}}{N},$$

showing that Eq. (17) holds.

In case $\Omega_I = \Omega_O$, the fact that $m > 1$ implies that $x^*_{k+1|k} \notin \Omega_I$ which implies that $x^*_{k+1|k} \notin \Omega_O$. Then, in this case and whenever $x^*_{k+1|k} \notin \Omega_O$ Eq. (25) can be replaced by

$$L^*_{k+1,k} + \sum_{i \in D_k} r\tilde{L}^*_{k+i,k} \geq \tilde{L}_{\min}, \tag{27}$$

and following the same procedure as before, from Eqs. (26) and (27) we conclude that Eq. (18) holds completing the proof. □

Remark 6. In the proof of Theorem 1, before Eq. (21), it is stated that the condition $V^*(x_{k+1}) > cN\tilde{L}_{\max}$ requires $m > 2$ implying that the horizon length is $N > 2$. This does not invalidate the conclusions of the theorem for $N = 1$ or $N = 2$, it only implies that the condition $V^*(x_{k+1}) > cN\tilde{L}_{\max}$ given in the statement is only possible for $N > 2$ (i.e. for $N = 1$ or $N = 2$ the cost at $k+1$ is always less than $cN\tilde{L}_{\max}$). Thus, the results that are derived next based on this theorem also hold for any $N \geq 1$.

The following corollary shows that when the optimal cost becomes smaller than the value $cN\tilde{L}_{\max}$, it remains forever bounded by that value.

Corollary 2. *Consider the MPC formulation given by* (9)*. If* $x_k \in X_N^{\Omega_I}$ *and* $V^*(x_k) \le cN\tilde{L}_{\max}$*, then* $V^*(x_j) \le cN\tilde{L}_{\max}$*,* $\forall j \ge k$*.*

Proof. From Theorem 1 we know that either $V^*(x_{k+1}) \le cN\tilde{L}_{\max}$ or the cost reduction of Eq. (17) holds. In both possible cases, the condition $V^*(x_k) \le cN\tilde{L}_{\max}$ implies that $V^*(x_{k+1}) \le cN\tilde{L}_{\max}$.

That way, $V^*(x_k) \le cN\tilde{L}_{\max}$ implies $V^*(x_{k+1}) \le cN\tilde{L}_{\max}$ and using induction it implies that $V^*(x_j) \le cN\tilde{L}_{\max}$ for all $j \ge k$. □

An immediate consequence of Corollaries 1 and 2 is that when the optimal cost is smaller than certain bound, the state remains forever in Ω_O.

Corollary 3. *Consider the MPC formulation given by* (9)*. If* $x_k \in X_N^{\Omega_I}$ *and* $V^*(x_k) \le cN\tilde{L}_{\max}$*, then* $x_j \in \Omega_O$*,* $\forall j > k$*.*

Finally, the next corollary shows that the state arrives in finite-time at the target set.

Corollary 4. *Consider the MPC formulation given by* (9) *with* $\Omega_O = \Omega_I$ *or* $r > 0$*. If* $x_k \in X_N^{\Omega_I}$ *and* $V^*(x_k) > cN\tilde{L}_{\max}$*, then* $x_{k+i} \in \Omega_O$ *for all* $i > M$*, where*

$$M \triangleq \left\lceil \frac{V^*(x_k) - cN\tilde{L}_{\max}}{\alpha} \right\rceil \tag{28}$$

with $\alpha = \tilde{\epsilon}$ *(Eq.* (18)*) provided that* $\Omega_I = \Omega_O$*, even if* $r = 0$*, or* $\alpha = \epsilon$ *(Eq.* (17)*) in any other case with* $r > 0$*.*

Proof. Suppose that $V^*(x_{k+M}) > cN\tilde{L}_{\max}$. Then, according to Corollary 2, $V^*(x_{k+i}) > cN\tilde{L}_{\max}$ for all $i \in \mathbb{Z}_{0:M-1}$. From Theorem 1, the condition $V^*(x_{k+i+1}) > cN\tilde{L}_{\max}$ implies

$$V^*(x_{k+i}) - V^*(x_{k+i+1}) \ge \alpha.$$

Summing up left and right-hand sides of this last inequality for $i = 0, \dots, M-1$, we obtain that

$$V^*(x_k) - V^*(x_{k+M}) \ge M\alpha \ge V^*(x_k) - cN\tilde{L}_{\max}$$

which implies that

$$V^*(x_{k+M}) \leq cN\tilde{L}_{\max},$$

contradicting the initial assumption and showing that $V^*(x_{k+M}) \leq cN\tilde{L}_{\max}$. Then, according to Corollary 3 this condition implies that $x_{k+i} \in \Omega_O$ for all $i > M$ concluding the proof. □

To summarize the results, we have first given in Lemma 1 an equivalent condition between the MPC cost function being lower than or equal to certain bound and the fact that all predicted states are in the target set. Then Theorem 1 establishes that when the MPC cost function is greater than such bound, a cost reduction with a positive lower bound is always achieved. Finally, Corollaries 2 and 3 show that the state of the closed-loop system is ultimately bounded to the target region if it arrives there and in Corollary 4 it is proven the finite-time convergence to Ω_O.

5. Controller Design

The aim of this section is to provide a procedure that allows us to effectively design a controller based on the proposed strategy. While it may be possible to jointly define different elements that compose our formulation, we present a possible sequence of sorted steps to do so in Algorithm 1. It is assumed that the target set Ω_O has already been defined according to some prespecified control goals and that an N-steps control inner set Ω_I is known as well as the corresponding N.

Algorithm 1. Sequential procedure to design the controller.

1. Define an arbitrary stage cost function $\tilde{L}(x, u) \geq 0$ according to the particular control problem to address.

2. Find $\tilde{L}_{\max}$ finite and $\tilde{L}_{\min} > 0$ considering Eqs. (2) and (3).

3. Choose a value for r in the interval $[0, 1)$.

4. Select $c = \frac{r\tilde{L}_{\min}}{N\tilde{L}_{\max}}$ in accordance with Eq. (5), or any non-negative value lower than that.

While many types of functions for $\tilde{L}(x, u)$ may be useful, a quadratic one with its minimum for some state inside Ω_O is an intuitive option that will always be appropriate for our scheme. For these functions, the existence of solution to (9) can be guaranteed (see Remark 4) and the values of $\tilde{L}_{\max}$ and $\tilde{L}_{\min}$ can be easily found with standard tools to solve quadratic programming problems. Furthermore, a quadratic function $\tilde{L}(x,u)$ would allow these constants to satisfy the required conditions of being $\tilde{L}_{\max}$ finite and $\tilde{L}_{\min} > 0$.

6. Discussion

In this section, we comment on certain aspects and make observations regarding the ideas proposed in this work.

1. *Region of attraction:* Since it is desirable for $X_N^{\Omega_I}$ (and so Ω_I) to be as large as possible (it is the region from which convergence is ensured) with the target region Ω_O as small as possible, the best case occurs when $\Omega_O = \Omega_I$ which implies that Ω_I is a control invariant set. If it is not possible to find that set (or it is too complex to be used in practice), having Ω_I close to the given target region Ω_O ensures a larger region of attraction.

2. *Relation to invariant sets:* Properties of the inner sets presented in Assumption 1 imply an equivalence between the existence of control invariant and inner sets in Ω_O. In fact, Ω_I is always inside a control invariant set contained in the target region. However, the computation of such invariant set is not required. It suffices with the knowledge of Ω_O and Ω_I such that the assumption is verified. It is important to emphasize that the manipulation of these sets is easier than that of invariant sets. This is more evident for nonlinear systems (for which invariant sets are in general excessively complex), but even for the linear case, sets Ω_O and Ω_I can be simpler (especially when closeness between Ω_O and Ω_I can be relaxed).

3. *Maximal inner set:* Given certain target region, the corresponding maximal inner set, i.e. the union of all inner sets in there, is the 1-step control inner set that coincides with the maximal control invariant set within that target set. This may be useful if it is needed to estimate the size of some Ω_I.

4. *Regarding r:* Constant r is arbitrary. A larger value (close to 1) ensures a faster convergence but it distorts the original stage cost function. Yet, a small value of r according to Eq. (5) may enforce to use a small value for constant c.

5. *Regarding c:* It is always possible to use $c = 0$. However, using a larger value allows keeping the system under control even inside the target region, to bring the state as close as possible to the one with the smallest cost. Moreover, the closer this parameter is to one, the more equally states are considered in the cost function regardless of whether or not they are in the target set. Notice that, independently of what has been said here and in the previous item, whenever Eq. (5) with $r \in [0, 1)$ and $c \geq 0$ is satisfied, stability related properties claimed for the proposed scheme will hold.

6. *Prediction horizon:* It has been assumed that the length of the prediction horizon is N while the inner set is an N-steps Control Inner Set. Taking into account that an N-steps Control Inner Set is also an H-steps Control Inner Set, for any $H \in \mathbb{N}$ such that $H > N$, the prediction horizon in the optimal control problem (9) could be defined as H, i.e. longer than N. Clearly, H should be considered in the entire formulation (for Eq. (5), constraints (10), (11) and (12), etc.) which means a higher computational load but, assuming that an N-steps Control Inner Set is already known, this may be useful to enlarge the domain of attraction. A possibility to limit the numerical burden increase would be to use H in the formulation until the state arrives at the up-to-N-steps controllable set to Ω_I. From that time onwards, we could change the prediction horizon to N.

7. *Relation and differences with* [29]*:* Using $c = r = 0$ with $\Omega_I = \Omega_O$ being a control invariant set, finite-time convergence is ensured in a scheme similar to that of [29] until the trajectory enters the invariant region. The only difference under this set up would be that our "passing through" condition is a relaxation of the terminal constraint of [29]. Once the state is in the target set our controller will keep it there without any modification (even if $c = r = 0$) while for [29] it is necessary to change the control law for another one under which the target region is control invariant. As a difference, then, our strategy releases us from designing this other controller and moreover, if c and r are not zero, it will not only maintain the state in the target set but will also make it be as close as possible to the one with the least cost. In spite of this, we do not ensure asymptotic stability to any state inside Ω_O. [29] does ensure it but because it assumes that the local controller is able to do so. Finally, going beyond the case with $\Omega_I = \Omega_O$, the most important difference between our formulation and [29] is that we do not require an invariant set but a less restrictive

pair of inner and outer sets satisfying Assumption 1.

8. *Relation and differences with* [4]*:* Even with $c = r = 0$ and $\Omega_I = \Omega_O$ being a control invariant set, our proposed scheme considerably differs with [4]. Under these conditions both controllers have zero stage cost in the target region; however, in [4] this is not achieved with scaling as in our case but directly defining the stage cost by means of Euclidean distances to sets (considered zero within those sets). In addition, [4] requires the target region to be contractive and not just invariant and it imposes a terminal constraint on the state to belong to it which is more restrictive than our "passing through" condition. Another difference is that our controller keeps the state inside the target region (solutions are ultimately bounded) while the strategy of [4] may not, because it does not change the control law as in [29] but continues using the same MPC scheme with zero stage cost in the target set.

9. *Contributions:* To explain why the contributions presented in Section 1 are independent we begin by considering only (C1). If we use the pair of inner-outer sets (none of them being invariant) with zero cost in the target region ($c = 0$ but $r > 0$ because of the hypothesis of Corollary 4), we keep practical stability without using invariant sets but the ability to control the state once it arrives at the outer set is lost. On the other hand and considering (C2), if $\Omega_O = \Omega_I$ which means they are an invariant set, and the cost is reduced but not nulled there ($c > 0$), the MPC scheme continues controlling in the target region and it still maintains practical stability but it is necessary to deal with (compute and handle) invariant sets.

7. Simulation Example

In this section we present an example that illustrates the proposed design methodology and its advantages. The chosen system is a nonlinear discrete-time model of an inverted pendulum given by the difference equations

$$\begin{cases} \theta_{k+1} = \theta_k + a_1\,\omega_k \\ \omega_{k+1} = a_2 \sin\theta_k + a_3\,\omega_k + a_4\,\tau_k \\ \tau_{k+1} = \tau_k + a_5\,u_k \end{cases}, \tag{29}$$

where in the state $x_k = [\theta_k \; \omega_k \; \tau_k]^T \in X \subseteq \mathbb{R}^3$, θ_k is the angle with respect to the vertical axis, ω_k represents the angular speed and τ_k is the applied torque. The input $u_k \in U = \{-1, 0, 1\}$ defines the change in τ_k and belongs to a finite set that is limited to keep the applied torque or to modify it by a fixed amount ($\pm a_5$) after each sampling period. Parameters are $a_1 = 0.01$, $a_2 = 0.4$, $a_3 = 9.99 \cdot 10^{-1}$, $a_4 = 1.2$ and $a_5 = 5 \cdot 10^{-3}$.

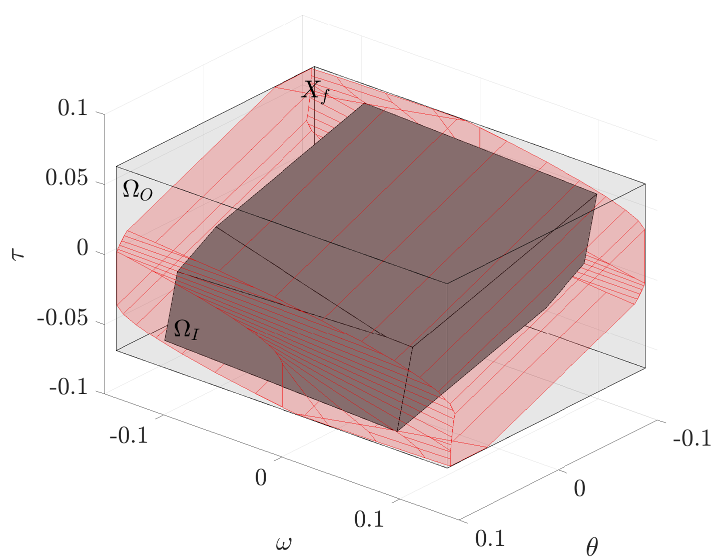


Figure 1: Sets Ω_I, Ω_O and X_f for the example.

The goal of the controller is to ensure convergence to the target set $\Omega_O = \{x \in X : |\theta| \leq 0.1, |\omega| \leq 0.14, |\tau| \leq 0.066\}$ which is shown in Figure 1. With this aim, a classical ingredient to consider for a stabilizing MPC formulation would be a terminal constraint to an invariant set in Ω_O. However, the nonlinearity of the system and its finite control set make the computation of this invariant set not only difficult but practically useless in the sense that even if it could be obtained, its representation complexity would make it not suitable for the online computation of set membership conditions involved in MPC. To illustrate this situation, Figure 1 also depicts the invariant set X_f obtained by [14] for the linearization of system (29) with a continuous input set given by $[-1, 1]$. Beyond the fact that it can be verified X_f is not invariant for the nonlinear system with finite-alphabet input, this set has 70 faces which even under these simplifications implies an extremely complex mathematical description.

On the other hand, a considerably simpler control inner set can be represented by just eight faces as it is also shown in Figure 1. This polyhedral set is given by $\Omega_I = \{x \in X : Fx \leq g\}$ with

$$F = \begin{bmatrix} 95.99 & 37.59 & 265.9 \\ -95.99 & -37.59 & -265.9 \\ 10.19 & 91.69 & 0 \\ -10.19 & -91.69 & 0 \\ 100.6 & 13.66 & 24.52 \\ -100.6 & -13.66 & -24.52 \\ 5.765 & 1.094 & 9.294 \\ -5.765 & -1.094 & -9.294 \end{bmatrix}, \quad g = \begin{bmatrix} 6.772 \\ 6.772 \\ 8.151 \\ 8.151 \\ 7.445 \\ 7.445 \\ 0.386 \\ 0.386 \end{bmatrix}.$$

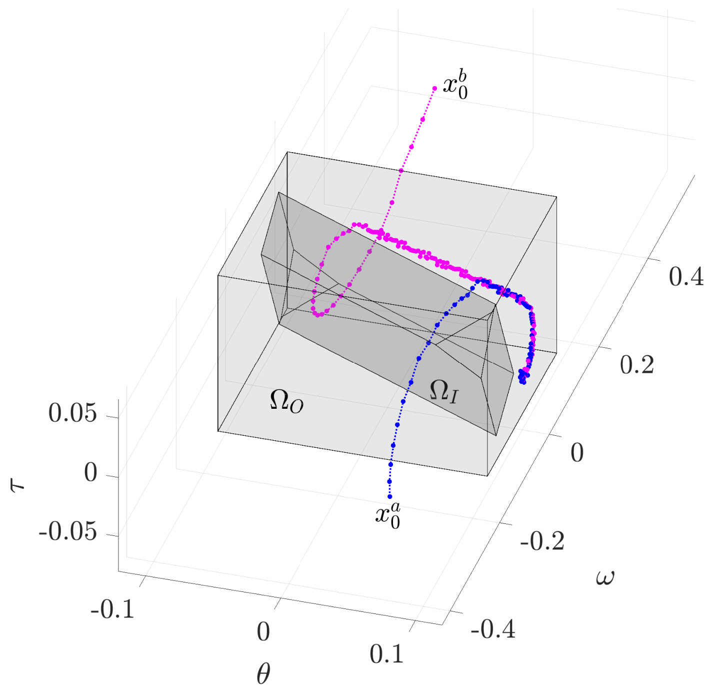
It can be computationally verified that Assumption 1 holds for Ω_O and Ω_I, being Ω_I a 9-steps control inner set. Taking this into account, we shall define $N = 9$ as the horizon length of our MPC strategy.

The numerical verification was done using a dense cloud of points in Ω_I and checking that, for each of those states, there exists an input sequence, of length less or equal than N, that drives it to Ω_I' without leaving Ω_O', where these sets are reduced versions of Ω_I and Ω_O, respectively. We adopted a minimum distance ϵ between their boundaries such that $\Omega_I' \oplus B_\epsilon \subseteq \Omega_I$ and $\Omega_O' \oplus B_\epsilon \subseteq \Omega_O$, where $\oplus$ is the Minkowski set addition. Then, we verified that any point in Ω_I has its closest point in the cloud at a distance less than ϵ/K_L, with K_L being an upper bound on the Lipschitz constants of the up to N compositions of the system function $f(x, u)$ for each possible input sequence. That way, the fact that all the points of the cloud can be driven to Ω_I' without leaving Ω_O' implies that all the points in Ω_I can be driven to Ω_I without leaving Ω_O.

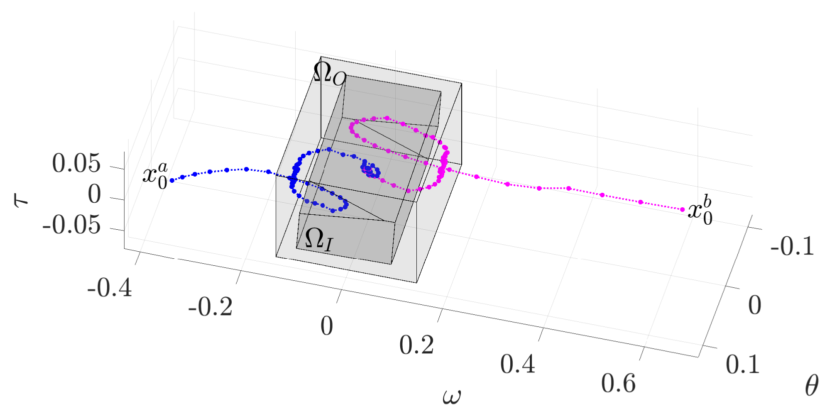
In order to design the controller, we follow the procedure given in Algorithm 1. We shall suppose that the original stage cost function to minimize was $\tilde{L}(x) = \tilde{L}([\theta\,\omega\,\tau]^T) = \theta^2 + 10^{-4}\,\omega^2 + 1.44 \cdot 10^{-4}\,\tau^2$, which mainly penalizes the error in θ. Then, according to Eqs. (2) and (3) it can be obtained, by solving two quadratic programming problems, that $\tilde{L}_{\min} = 6.28 \cdot 10^{-7}$ and $\tilde{L}_{\max} = 10^{-2}$. To finish defining all the elements of the stage cost function (4) (and the optimization problem (9)) we select $r = 0.1$ and obtain $c \leq 6.98 \cdot 10^{-7}$ from Eq. (5).

To verify that our MPC formulation works as expected, we simulated the system from initial conditions $x_0^a = [0.068 \;\; -0.36 \;\; -7.3 \cdot 10^{-3}]^T$ and

$x_0^b = [-0.073 \;\; 0.59 \;\; -0.043]^T$, with two different values of the parameter c: 0 and $6 \cdot 10^{-7}$. The trajectories obtained are depicted in Figure 2.



(a) State trajectories with $c = 0$.



(b) State trajectories with $c = 6 \cdot 10^{-7}$.

Figure 2: Application of our MPC formulation.

It can be seen that all state evolutions converge in finite-time to the target region Ω_O remaining thereafter in its interior. As shown in Figure 2a, when $c = 0$ the proposed controller guarantees this behavior without the need for Ω_O to be invariant. Notice that it is neither required to switch to another controller inside this region. These two features are significant differences with respect to dual-mode MPC approaches. Another aspect to comment on is that, interestingly for this case, the trajectories of the closed-loop system do not keep effectively passing through Ω_I although the optimal control sequences computed at each sampling instant guarantee that it is feasible in $N = 9$ or less steps. Finally, it is also important to mention here that having more than one solution to the optimization problem (9) does not affect practical stability. It can be inferred from the proofs given in Section 4 that any solution would keep this property and that is exemplified with these numerical simulations when $c = 0$.

The case with $c = 6 \cdot 10^{-7} > 0$ corresponding to Figure 2b shows how scaling the cost inside the target region allows the scheme to also control the system inside the target region. This means that the trajectories not only enter Ω_O but are also driven closer to the origin whenever this is possible.

8. Conclusions

We have presented a simple procedure to modify the cost function of an MPC scheme such that practical stability, with finite-time convergence and ultimate boundedness to a target set, is ensured. Given a stage cost function $\tilde{L}(x, u)$ and some target region Ω_O with an N-steps control inner set Ω_I in it, the proposed MPC formulation uses a modified stage cost function that considers whether the predicted states are in Ω_O or not. The optimal control problem is then constrained by a condition that the predicted state "passes through" Ω_I at some step in the prediction horizon. This condition is less restrictive than the usual state terminal restriction to belong to a control invariant set, required by most stabilizing MPC strategies. With this formulation we have given a theoretical upper bound for the convergence time to the target set and we have proved that the state is ultimately bounded there.

Being the assumptions mild, the presented results are general and hold for systems of general classes and also for non-convex sets. Furthermore, the fact that neither computing nor handling complicated invariant sets is needed for the proposed methodology makes it particularly interesting for switched and nonlinear systems. To illustrate this fact, a simulation example of a nonlinear model with a finite input set was presented. There, it was also shown how the MPC scheme can keep the system controlled when the state is in the target region without changing to a different control strategy.

Current work is aimed at developing systematic procedures to find inner sets Ω_I verifying Assumption 1 for a given target set Ω_O and a horizon length N, and at finding efficient solutions to the MPC optimization problem for specific types of systems and constraints. Future work includes the implementation of the proposed strategy for different applications, including autonomous navigation and switched systems.

9. Acknowledgments

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