Closed-loop Identification Revisited – Updated Version

Urban Forssell and Lennart Ljung

Department of Electrical Engineering
Linkping University, S-581 83 Linkping, Sweden
WWW: http://www.control.isy.liu.se
Email: ufo@isy.liu.se, ljung@isy.liu.se

1 April 1998

REGLERTEKNIK
AUTOMATIC CONTROL
LINKÖPING

Report no.: LiTH-ISY-R-2021
Submitted to Automatica

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Urban Forssell and Lennart Ljung

Division of Automatic Control, Department of Electrical Engineering, Linköping University, S-581 83 Linköping, Sweden. URL: http://www.control.isy.liu.se/.

Abstract

Identification of systems operating in closed loop has long been of prime interest in industrial applications. The problem offers many possibilities, and also some fallacies, and a wide variety of approaches have been suggested, many quite recently. The purpose of the current contribution is to place most of these approaches in a coherent framework, thereby showing their connections and display similarities and differences in the asymptotic properties of the resulting estimates. The common framework is created by the basic prediction error method, and it is shown that most of the common methods correspond to different parameterizations of the dynamics and noise models. The so called *indirect methods*, e.g., are indeed "direct" methods employing noise models that contain the regulator. The asymptotic properties of the estimates then follow from the general theory and take different forms as they are translated to the particular parameterizations. In the course of the analysis we also suggest a projection approach to closed-loop identification with the advantage of allowing approximation of the open loop dynamics in a given, and user-chosen frequency domain norm, even in the case of an unknown, non-linear regulator.

Key words: System identification; Closed-loop identification; Prediction error methods

⋆ This paper was not presented at any IFAC meeting. Corresponding author U. Forssell. Tel. +46-13-282226. Fax +46-13-282622. E-mail ufo@isy.liu.se.

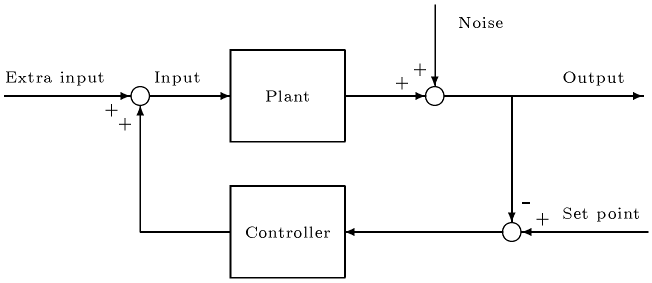

Fig. 1. A closed-loop system

1 Introduction

1.1 Motivation and Previous Work

System identification is a well established field with a number of approaches, that can broadly be classified into the *prediction error family*, e.g,, [22], the *subspace approaches*, e.g., [31], and the non-parametric *correlation and spectral analysis methods*, e.g., [5]. Of special interest is the situation when the data to be used has been collected under closed-loop operation, as in Fig. 1.

The fundamental problem with closed-loop data is the correlation between the unmeasurable noise and the input. It is clear that whenever the feedback controller is not identically zero, the input and the noise will be correlated. This is the reason why several methods that work in open loop fail when applied to closed-loop data. This is for example true for the subspace approach and the non-parametric methods, unless special measures are taken. Despite these problems, performing identification experiments under output feedback (i.e. in closed loop) may be necessary due to safety or economic reasons, or if the system contains inherent feedback mechanisms. Closed-loop experiments may also be advantageous in certain situations:

- In [13] the problem of optimal experiment design is studied. It is shown that if the model is to be used for minimum variance control design the identification experiment should be performed in closed-loop with the optimal minimum variance controller in the loop. In general it can be seen that optimal experiment design with variance constraints on the output leads to closed-loop solutions.
- In "identification for control" the objective is to achieve a model that is suited for robust control design (see, e.g., [7, 19, 33]). Thus one has to tailor the experiment and preprocessing of data so that the model is reliable in regions where the design process does not tolerate significant uncertainties. The use of closed-loop experiments has been a prominent feature in these approaches.

Historically, there has been a substantial interest in both special identification techniques for closed-loop data, and for analysis of existing methods when applied to such data. One of the earliest results was given by Akaike [1] who analyzed the effect of feedback loops in the system on correlation and spectral analysis. In the seventies there was a very active interest in questions concerning closed-loop identification, as summarized in the survey paper [15]. See also [3]. Up to this point much of the attention had been directed towards identifiability and accuracy problems. With the increasing interest "identification for control", the focus has shifted to the ability to shape the bias distribution so that control-relevant model approximations of the system are obtained. The surveys [12] and [29] cover most of the results along this line of research.

1.2 Scope and Outline

It is the purpose of the present paper to "revisit" the area of closed-loop identification, to put some of the new results and methods into perspective, and to give a status report of what can be done and what cannot. In the course of this exposé, some new results will also be generated.

We will exclusively deal with methods derived in the prediction error framework and most of the results will be given for the multi-input multi-output (MIMO) case. The leading idea in the paper will be to provide a unified framework for many closed-loop methods by treating them as different parameterizations of *the* prediction error method:

> *There is only one method. The different approaches are obtained by different parameterizations of the dynamics and noise models.*

Despite this we will often use the terminology "method" to distinguish between the different approaches and parameterizations. This has also been standard in the literature.

The organization of the paper is as follows. Next, in Section 2 we characterize the kinds of assumptions that can be made about the nature of the feedback. This leads to a classification of closed-loop identification methods into, so called, direct, indirect, and joint input-output methods. As we will show, these approaches can be viewed as variants of *the* prediction error method with the models parameterized in different ways. A consequence of this is that we may use all results for the statistical properties of the prediction error estimates known from the literature. In Section 3 the assumptions we will make regarding the data generating mechanism are formalized. This section also introduces the some of the notation that will be used in the paper.

Section 4 contains a brief review of the standard prediction error method as well as the basic statements on the asymptotic statistical properties of this method:

- Convergence and bias distribution of the limit transfer function estimate.
- Asymptotic variance of the transfer function estimates (as the model orders increase).
- Asymptotic variance and distribution of the parameter estimates.

The application of these basic results to the direct, indirect, and joint input-output approaches will be presented in some detail in Sections 5–8. All proofs will be given in the Appendix. The paper ends with a summarizing discussion in Section 9.

2 Approaches to Closed-loop Identification

2.1 A Classification of Approaches

In the literature several different types of closed-loop identification methods have been suggested. In general one may distinguish between methods that

(a) Assume no knowledge about the nature of the feedback mechanism, and do not use the reference signal $r(t)$ even if known.
(b) Assume the feedback to be known and typically of the form

$$u(t) = r(t) - K(q)y(t) \tag{1}$$

where $u(t)$ is the input, $y(t)$ the output, $r(t)$ an external reference signal, and $K(q)$ a linear time-invariant regulator. The symbol q denotes the usual shift operator, $q^{-1}y(t) = y(t-1)$, etc.
(c) Assume the regulator to be unknown, but of a certain structure (like (1)).

If the regulator indeed has the form (1), there is no major difference between (a), (b) and (c): The noise-free relation (1) can be exactly determined based on a fairly short data record, and then $r(t)$ carries no further information about the system, if $u(t)$ is measured. The problem in industrial practice is rather that no regulator has this simple, linear form: Various delimiters, anti-windup functions and other non-linearities will have the input deviate from (1), even if the regulator parameters (e.g. PID-coefficients) are known. This strongly disfavors the second approach.

In this paper we will use a classification of the different methods that is similar to the one in [15]. See also [26]. The basis for the classification is the different kinds of possible assumptions on the feedback listed above. The closed-loop identification methods correspondingly fall into the following main groups:

(1) *The Direct Approach*: Ignore the feedback and identify the open-loop system using measurements of the input $u(t)$ and the output $y(t)$.
(2) *The Indirect Approach*: Identify some closed-loop transfer function and determine the open-loop parameters using the knowledge of the controller.
(3) *The Joint Input-Output Approach*: Regard the input $u(t)$ and the output $y(t)$ jointly as the output from a system driven by the reference signal $r(t)$ and noise. Use some method to determine the open-loop parameters from an estimate of this system.

These categories are basically the same as those in [15], the only difference is that in the joint input-output approach we allow the joint system to have a measurable input $r(t)$ in addition to the unmeasurable noise $e(t)$. For the indirect approach it can be noted that most methods studied in the literature assume a linear regulator but the same ideas can also be applied if non-linear and/or time-varying controllers are used. The price is, of course, that the estimation problems then become much more involved.

In the closed-loop identification literature it has been common to classify the methods primarily based on how the final estimates are computed (e.g. directly or indirectly using multi-step estimation schemes), and then the main groupings have been into "direct" and "indirect" methods. This should not, however, be confused with the classification (1)-(3) which is based on the assumptions made on the feedback.

3 Technical Assumptions and Notation

The basis of all identification is the data set

$$Z^N = \{u(1), y(1), \ldots, u(N), y(N)\} \tag{2}$$

consisting of measured input-output signals $u(t)$ and $y(t)$, $t = 1, \ldots, N$. We will make the following assumptions regarding how this data set was generated.

Assumption 1 *The true system $\mathcal{S}$ is linear with p outputs and m inputs and given by*

$$\begin{aligned} y(t) &= G_0(q)u(t) + v(t) \\ v(t) &= H_0(q)e(t) \end{aligned} \tag{3}$$

where $\{e(t)\}$ $(p \times 1)$ is a zero-mean white noise process with covariance matrix

Λ_0, and bounded moments of order $4+\delta$, some $\delta > 0$, and $H_0(q)$ is an inversely stable, monic filter.

For some of the analytic treatment we shall assume that the input $\{u(t)\}$ is generated as

$$u(t) = r(t) - K(q)y(t) \tag{4}$$

where $K(q)$ is a linear regulator of appropriate dimensions and where the reference signal $\{r(t)\}$ is independent of $\{v(t)\}$.

This assumption of a linear feedback law is rather restrictive and in general we shall only assume that the input $u(t)$ satisfies the following milder condition (cf. [20], condition S3):

Assumption 2 *The input $u(t)$ is given by*

$$u(t) = k(t, y^t, u^{t-1}, r(t)) \tag{5}$$

where $y^t = [y(1), \ldots, y(t)]$, etc., and where where the reference signal $\{r(t)\}$ is a given quasi-stationary signal, independent of $\{v(t)\}$ and k is a given deterministic function such that the closed-loop system (3) and (5) is exponentially stable, which we define as follows: For each t, s; $t \geq s$ there exist random variables $\bar{y}_s(t), \bar{u}_s(t)$, independent of r^s and v^s but not independent of r^t and v^t, such that

$$\bar{E}\|y(t) - \bar{y}_s(t)\|^4 < C\lambda^{t-s} \tag{6}$$
$$\bar{E}\|u(t) - \bar{u}_s(t)\|^4 < C\lambda^{t-s} \tag{7}$$

for some $C < \infty$, $\lambda < 1$. In addition, k is such that either $G_0(q)$ or k contains a delay.

Here we have used the notation

$$\bar{E}f(t) = \lim_{N\to\infty} \frac{1}{N} \sum_{t=1}^{N} Ef(t) \tag{8}$$

The concept of quasi-stationarity is defined in, e.g., [22].

If the feedback is indeed linear and given by (4) then Assumption 2 means that the closed-loop system is asymptotically stable.

Let us now introduce some further notation for the linear feedback case. By combining the equations (3) and (4) we have that the closed-loop system is

$$y(t) = S_0(q)G_0(q)r(t) + S_0(q)v(t) \tag{9}$$

where $S_0(q)$ is the sensitivity function,

$$S_0(q) = (I + G_0(q)K(q))^{-1} \tag{10}$$

This is also called the output sensitivity function. With

$$G_{c,0}(q) = S_0(q)G_0(q) \text{ and } H_{c,0}(q) = S_0(q)H_0(q) \tag{11}$$

we can rewrite (9) as

$$\begin{aligned} y(t) &= G_{c,0}(q)r(t) + v_c(t) \\ v_c(t) &= H_{c,0}(q)e(t) \end{aligned} \tag{12}$$

In closed loop the input can be written as

$$u(t) = S_0^i(q)r(t) - S_0^i(q)K(q)v(t) \tag{13}$$

$$= S_0^i(q)r(t) - K(q)S_0(q)v(t) \tag{14}$$

The input sensitivity function $S_0^i(q)$ is defined as

$$S_0^i(q) = (I + K(q)G_0(q))^{-1} \tag{15}$$

The spectrum of the input is (cf. (14))

$$\Phi_u = S_0^i \Phi_r (S_0^i)^* + K S_0 \Phi_v S_0^* K^* \tag{16}$$

where Φ_r is the spectrum of the reference signal and $\Phi_v = H_0 \Lambda_0 H_0^*$ the noise spectrum. Superscript $*$ denotes complex conjugate transpose. Here we have suppressed the arguments ω and $e^{i\omega}$ which also will be done in the sequel whenever there is no risk of confusion. Similarly, we will also frequently suppress the arguments t and q for notational convenience. We shall denote the two terms in (16)

$$\Phi_u^r = S_0^i \Phi_r (S_0^i)^* \tag{17}$$

and

$$\Phi_u^e = K S_0 \Phi_v S_0^* K^* = S_0^i K \Phi_v K^* (S_0^i)^* \tag{18}$$

The cross spectrum between u and e is

$$\Phi_{ue} = -K S_0 H_0 \Lambda_0 = -S_0^i K H_0 \Lambda_0 \tag{19}$$

The cross spectrum between e and u will be denoted Φ_{eu}, $\Phi_{eu} = \Phi_{ue}^*$.

Occasionally we shall also consider the case where the regulator is linear as in (4) but contains an unknown additive disturbance d:

$$u(t) = r(t) - K(q)y(t) + d(t) \tag{20}$$

The disturbance d could for instance be due to imperfect knowledge of the true regulator: Suppose that the true regulator is given by

$$K^{true}(q) = K(q) + \Delta_K(q) \tag{21}$$

for some (unknown) function Δ_K. In this case the signal $d = -\Delta_K y$. Let Φ_{rd} (Φ_{dr}) denote the cross spectrum between r and d (d and r), whenever it exists.

4 Prediction Error Identification

In this section we shall review some basic results on prediction error methods, that will be used in the sequel. See Appendix A and [22] for more details.

4.1 The Method

We will work with a model structure $\mathcal{M}$ of the form

$$y(t) = G(q, \theta)u(t) + H(q, \theta)e(t) \tag{22}$$

G will be called the dynamics model and H the noise model. We will assume that either G (and the true system G_0) or the regulator k contains a delay and that H is monic. The parameter vector θ ranges over a set $D_{\mathcal{M}}$ which is assumed compact and connected. The one-step-ahead predictor for the model structure (22) is [22]

$$\hat{y}(t|\theta) = H^{-1}(q, \theta)G(q, \theta)u(t) + (I - H^{-1}(q, \theta))y(t) \tag{23}$$

The prediction errors are

$$\varepsilon(t, \theta) = y(t) - \hat{y}(t|\theta) = H^{-1}(q, \theta)(y(t) - G(q, \theta)u(t)) \tag{24}$$

Given the model (23) and measured data Z^N we determine the prediction error estimate through

$$\hat{\theta}_N = \arg\min_{\theta \in D_{\mathcal{M}}} V_N(\theta, Z^N) \tag{25}$$

$$V_N(\theta, Z^N) = \frac{1}{N}\sum_{t=1}^{N} \varepsilon_F^T(t, \theta)\Lambda^{-1}\varepsilon_F(t, \theta) \tag{26}$$

$$\varepsilon_F(t, \theta) = L(q, \theta)\varepsilon(t, \theta) \tag{27}$$

Here Λ is a symmetric, positive definite weighting matrix and L a (possibly parameter-dependent) monic prefilter that can be used to enhance certain frequency regions. It is easy to see that

$$\varepsilon_F(t,\theta) = L(q,\theta)H^{-1}(q,\theta)(y(t) - G(q,\theta)u(t)) \tag{28}$$

Thus the effect of the prefilter L can be included in the noise model and $L(q,\theta) = 1$ can be assumed without loss of generality. This will be done in the sequel.

We say that the true system is contained in the model set if, for some $\theta_0 \in D_{\mathcal{M}}$,

$$G(q,\theta_0) = G_0(q), \;\; H(q,\theta_0) = H_0(q) \tag{29}$$

This will also be written $\mathcal{S} \in \mathcal{M}$. The case when the true noise properties cannot be correctly described within the model set but where there exists a $\theta_0 \in D_{\mathcal{M}}$ such that

$$G(q,\theta_0) = G_0(q) \tag{30}$$

will be denoted $G_0 \in \mathcal{G}$.

4.2 Convergence

Define the average criterion $\bar{V}(\theta)$ as

$$\bar{V}(\theta) = \bar{E}\varepsilon^T(t,\theta)\Lambda^{-1}\varepsilon(t,\theta) \tag{31}$$

Then we have the following result (see, e.g., [20, 22]):

$$\hat{\theta}_N \to D_c = \arg\min_{\theta \in D_{\mathcal{M}}} \bar{V}(\theta), \quad \text{with probability (w. p.) 1 as } N \to \infty \tag{32}$$

In case the input-output data can be described by (3) we have the following characterization of D_c (G_θ is short for $G(q,\theta)$, etc.):

$$D_c = \arg\min_{\theta \in D_{\mathcal{M}}} \int_{-\pi}^{\pi} \operatorname{tr}\Big[\Lambda^{-1}H_\theta^{-1} \begin{bmatrix} (G_0 - G_\theta)^* \\ (H_0 - H_\theta)^* \end{bmatrix}^* \begin{bmatrix} \Phi_u & \Phi_{ue} \\ \Phi_{eu} & \Lambda_0 \end{bmatrix} \begin{bmatrix} (G_0 - G_\theta)^* \\ (H_0 - H_\theta)^* \end{bmatrix} H_\theta^{-*}\Big] d\omega \tag{33}$$

This is shown in Appendix A.1. Note that the result holds regardless of the nature of the regulator, as long as Assumptions 1 and 2 hold and the signals involved are quasistationary.

From (33) several conclusions regarding the consistency of the method can be drawn. First of all, suppose that the parameterization of G and H is sufficiently flexible so that $\mathcal{S} \in \mathcal{M}$. If this holds then the method will in general give consistent estimates of G_0 and H_0 if the experiment is informative [22], which means that the matrix

$$\Phi_{\chi_0} = \begin{bmatrix} \Phi_u & \Phi_{ue} \\ \Phi_{eu} & \Lambda_0 \end{bmatrix} \tag{34}$$

is positive definite for all frequencies. (Note that it will always be positive *semi*-definite since it is a spectral matrix.) Suppose for the moment that the regulator is linear and given by (4). Then we can factorize the matrix in (34) as

$$\Phi_{\chi_0} = \begin{bmatrix} I & \Phi_{ue}\Lambda_0^{-1} \\ 0 & I \end{bmatrix} \begin{bmatrix} \Phi_u^r & 0 \\ 0 & \Lambda_0 \end{bmatrix} \begin{bmatrix} I & 0 \\ \Lambda_0^{-1}\Phi_{eu} & I \end{bmatrix} \tag{35}$$

The left and right factors in (35) always have full rank, hence the condition becomes that Φ_u^r is positive definite for all frequencies (Λ_0 is assumed positive definite). This is true if and only if Φ_r is positive definite for all frequencies (which is the same as to say that the reference signal is persistently exciting [22]). In the last step we used the fact that the analytical function S_0^i (cf. (17)) can be zero at at most finitely many points. The conclusion is that for linear feedback we should use a persistently exciting, external reference signal, otherwise the experiment may not be informative.

The general condition is that there should not be a linear, time-invariant, and noise-free relationship between u and y. With an external reference signal this is automatically satisfied but it should also be clear that informative closed-loop experiments can also be guaranteed if we switch between different linear regulators or use a non-linear regulator. For a more detailed discussion on this see, e.g., [15] and [22].

4.3 Asymptotic Variance of Black Box Transfer Function Estimates

Consider the model (23). Introduce

$$T(q,\theta) = \mathrm{vec}\Big[G(q,\theta)\ H(q,\theta)\Big] \tag{36}$$

(The vec-operator stacks the columns of its argument on top of each other in a vector. A more formal definition is given in Appendix A.2.) Suppose that the vector θ can be decomposed so that

$$\theta = \Big[\theta_1,\ \theta_2,\ \ldots,\ \theta_n\Big]^T \quad \dim\theta_k = s \quad \dim\theta = n\cdot s \tag{37}$$

We shall call n the order of the model (23) and we allow n to tend to infinity as N tends to infinity. Suppose also that T in (36) has the following shift structure:

$$\frac{\partial}{\partial \theta_k} T(q, \theta) = q^{-k+1} \frac{\partial}{\partial \theta_1} T(q, \theta) \tag{38}$$

It should be noted that most polynomial-type model structures, including the ones studied in this paper, satisfy this shift structure. Thus (38) is a rather weak assumption.

More background material including further technical assumptions and additional notation can be found in Appendix A.2. For brevity reasons we here go directly to the main result ($\otimes$ denotes the Kronecker product):

$$\text{Cov vec}\left[\hat{T}_N(e^{i\omega})\right] \sim \frac{n}{N} \begin{bmatrix} \Phi_u(\omega) & \Phi_{ue}(\omega) \\ \Phi_{eu}(\omega) & \Lambda_0 \end{bmatrix}^{-T} \otimes \Phi_v(\omega) \tag{39}$$

The covariance matrix is thus proportional to the model order divided n by the number of data N. This holds asymptotically as both n and N tend to infinity. In open loop we have $\Phi_{ue} = 0$ and

$$\text{Cov vec}\left[\hat{G}_N(e^{i\omega})\right] \sim \frac{n}{N} \Phi_u(\omega)^{-T} \otimes \Phi_v(\omega) \tag{40}$$

$$\text{Cov vec}\left[\hat{H}_N(e^{i\omega})\right] \sim \frac{n}{N} \Lambda_0^{-1} \otimes \Phi_v(\omega) \tag{41}$$

Notice that (40) for the dynamics model holds also in case the noise model is fixed (e.g. $H(q, \theta) = I$).

4.4 Asymptotic Distribution of the Parameter Vector Estimates

If $\mathcal{S} \in \mathcal{M}$ then $\hat{\theta}_N \to \theta_0$ as $N \to \infty$ under reasonable conditions (e.g., $\Phi_{\chi_0} > 0$, see [22]). Then, if $\Lambda = \Lambda_0$,

$$\sqrt{N}(\hat{\theta}_N - \theta^*) \in AsN(0, P_\theta) \tag{42a}$$

$$P_\theta = \left[\bar{E}\psi(t, \theta_0)\Lambda_0^{-1}\psi^T(t, \theta_0)\right]^{-1} \tag{42b}$$

where ψ is the negative gradient of the prediction errors ε with respect to θ.

In this paper we will restrict to the SISO case when discussing the asymptotic distribution of the parameter vector estimates for notational convenience. For ease of reference we have in Appendix A.3 stated a variant of (42) as a theorem.

5 Closed-loop Identification in the Prediction Error Framework

5.1 The Direct Approach

The direct approach amounts to applying a prediction error method directly to input-output data, ignoring possible feedback. In general one works with models of the form (cf. (23))

$$\hat{y}(t|\theta) = H^{-1}(q,\theta)G(q,\theta)u(t) + (I - H^{-1}(q,\theta))y(t) \tag{43}$$

The direct method can thus be formulated as in (25)-(27). This coincides with the standard (open-loop) prediction error method [22, 26]. Since this method is well known we will not go into any further details here. Instead we turn to the indirect approach.

5.2 The Indirect Approach

5.2.1 General

Consider the linear feedback set-up (4). If the regulator K is known and r is measurable, we can use the indirect identification approach. It consists of two steps:

(1) Identify the closed-loop system from the reference signal r to the output y.
(2) Determine the open-loop system parameters from the closed-loop model obtained in step 1, using the knowledge of the regulator.

Instead of identifying the closed-loop system in the first step one can identify any closed-loop transfer function, for instance the sensitivity function. Here we will concentrate on methods in which the closed-loop system is identified. The model structure is

$$y(t) = G_c(q,\theta)r(t) + H_*(q)e(t) \tag{44}$$

Here $G_c(q,\theta)$ is a model of the closed-loop system. We have also included a fixed noise model H_* which is standard in the indirect method. Often $H_*(q) = 1$ is used, but we can also use H_* as a fixed prefilter to emphasize certain frequency ranges. The corresponding one-step-ahead predictor is

$$\hat{y}(t|\theta) = H_*^{-1}(q)G_c(q,\theta)r(t) + (I - H_*^{-1}(q))y(t) \tag{45}$$

Note that estimating θ in (45) is an "open-loop" problem since the noise and the reference signal are uncorrelated. This implies that we may use any identification method that works in open loop to find this estimate of the closed-loop system. For instance, we can use output error models with fixed noise models (prefilters) and still guarantee consistency (cf. Corollary 4 below).

Consider the closed-loop system (cf. (12))

$$y(t) = G_{c,0}(q)r(t) + v_c(t) \tag{46}$$

Suppose that we in the first step have obtained an estimate $\hat{G}_{c,N}(q) = G_c(q, \hat{\theta}_N)$ of $G_{c,0}(q)$. In the second step we then have to solve the equation

$$\hat{G}_{c,N}(q) = (I + \hat{G}_N(q)K(q))^{-1}\hat{G}_N(q) \tag{47}$$

using the knowledge of the regulator. The exact solution is

$$\hat{G}_N(q) = \hat{G}_{c,N}(q)(I - \hat{G}_{c,N}(q)K(q))^{-1} \tag{48}$$

Unfortunately this gives a high-order estimate $\hat{G}_N$ in general – typically the order of $\hat{G}_N$ will be equal to the sum of the orders of $\hat{G}_{c,N}$ and K. If we attempt to solve (47) with the additional constraint that $\hat{G}_N$ should be of a certain (low) order we end up with an over-determined system of equations which can be solved in many ways, for instance in a weighted least-squares sense. For methods, like the prediction error method, that allow arbitrary parameterizations $G_c(q, \theta)$ it is natural to let the parameters θ relate to properties of the open-loop system G, so that in the first step we should parameterize $G_c(q, \theta)$ as

$$G_c(q, \theta) = (I + G(q, \theta)K(q))^{-1}G(q, \theta) \tag{49}$$

This was apparently first suggested as an exercise in [22]. This parameterization has also been analyzed in [8].

The choice (49) will of course have the effect that the second step in the indirect method becomes superfluous, since we directly estimate the open-loop parameters. The choice of parameterization may thus be important for numerical and algebraic issues, but it does not affect the statistical properties of the estimated transfer function:

> *As long as the parameterization describes the same set of G, the resulting transfer function $\hat{G}$ will be the same, regardless of the parameterizations.*

5.2.2 The Dual-Youla Parameterization

A nice and interesting idea is to use the so called dual-Youla parameterization that parameterizes all systems that are stabilized by a certain regulator K

(see, e.g., [32]). To present the idea, the concept of coprime factorizations of transfer functions is required: A pair of stable transfer functions $N, D \in \mathbb{R}H_\infty$ is a right coprime factorization (rcf) of G if $G = ND^{-1}$ and there exist stable transfer functions $X, Y \in \mathbb{R}H_\infty$ such that $XN + YD = I$. The dual-Youla parameterization now works as follows. Let G_{nom} with rcf(N, D) be any system that is stabilized by K with rcf(X, Y). Then, as R ranges over all stable transfer functions, the set

$$\left\{G : G(q,\theta) = (N(q) + Y(q)R(q,\theta))(D(q) - X(q)R(q,\theta))^{-1}\right\} \tag{50}$$

describes all systems that are stabilized by K. The unique value of R that corresponds to the true plant G_0 is given by

$$R_0(q) = Y^{-1}(q)(I + G_0(q)K(q))^{-1}(G_0(q) - G_{nom}(q))D(q) \tag{51}$$

This idea can now be used for identification (see, e.g., [16], [17], and [6]): Given an estimate $\hat{R}_N$ of R_0 we can compute an estimate of G_0 as

$$\hat{G}_N(q) = (N(q) + Y(q)\hat{R}_N(q))(D(q) - X(q)\hat{R}_N(q))^{-1} \tag{52}$$

Using the dual-Youla parameterization we can write

$$\begin{aligned} G_c(q,\theta) &= (N(q) + Y(q)R(q,\theta))(D(q) + X(q)Y^{-1}(q)N(q))^{-1} && (53)\\ &\triangleq (N(q) + Y(q)R(q,\theta))M(q) && (54)\end{aligned}$$

With this parameterization the identification problem

$$y(t) = G_c(q,\theta)r(t) + v_c(t) \tag{55}$$

becomes

$$z(t) = R(q,\theta)x(t) + \bar{v}_c(t) \tag{56}$$

where

$$\begin{aligned} z(t) &= Y^{-1}(q)(y(t) - N(q)M(q)r(t)) && (57)\\ x(t) &= M(q)r(t) && (58)\\ \bar{v}_c(t) &= Y^{-1}(q)v_c(t) && (59)\end{aligned}$$

Thus the dual-Youla method is a special parameterization of the general indirect method. This means, especially, that the statistical properties of the resulting estimates for the indirect method remain unaffected for the dual-Youla method. The main advantage of this method is of course that the obtained estimate $\hat{G}_N$ is guaranteed to be stabilized by K, which clearly is a nice feature. A drawback is that this method typically will give high-order estimates — typically the order will be equal to the sum of the orders of G_{nom} and $\hat{R}_N$.

In this paper we will use (49) as the generic indirect method. Before turning to the joint input-output approach, let us pause and study an interesting variant of the parameterization idea used in (49) which will provide useful insights into the connection between the direct and indirect methods.

5.3 A Formal Connection Between Direct and Indirect Methods

The noise model H in a linear dynamics model structure has often turned out to be a key to interpretation of different "methods". The distinction between the models/"methods" ARX, ARMAX, output error, Box-Jenkins, etc., is entirely explained by the choice of the noise model. Also the practically important feature of prefiltering is equivalent to changing the noise model. Even the choice between minimizing one- or k-step prediction errors can be seen as a noise model issue. See, e.g., [22] for all this.

Therefore it should not come as a surprise that also the distinction between the fundamental approaches of direct and indirect identification can be seen as a choice of noise model.

The idea is to parameterize G as $G(q,\theta)$ and H as

$$H(q,\theta) = (I + G(q,\theta)K(q))H_1(q,\theta) \tag{60}$$

We thus link the noise model to the dynamics model. There is nothing strange with that: So do ARX and ARMAX models. Although this parameterization is perfectly valid, it must still be pointed out that the choice (60) is a highly specialized one using the knowledge of K. Also note that this particular parameterization scales H_1 with the inverse model sensitivity function. (Similar parameterizations have been suggested in, e.g., [4, 9, 18].)

Now, the predictor for

$$y(t) = G(q,\theta)u(t) + H(q,\theta)e(t) \tag{61}$$

is

$$\hat{y}(t|\theta) = H^{-1}(q,\theta)G(q,\theta)u(t) + (I - H^{-1}(q,\theta))y(t) \tag{62}$$

Using $u = r - Ky$ and inserting (60) we get

$$\hat{y}(t|\theta) = H_1^{-1}(q,\theta)(I + G(q,\theta)K(q))^{-1}G(q,\theta)r(t) + (I - H_1^{-1}(q,\theta))y(t) \tag{63}$$

But this is exactly the predictor also for the closed-loop model structure

$$y(t) = (I + G(q,\theta)K(q))^{-1}G(q,\theta)r(t) + H_1(q,\theta)e(t) \tag{64}$$

and hence the two approaches are equivalent. We formulate this result as a lemma:

Lemma 3 *Suppose that the input is generated as in* (4) *and that both u and r are measurable and that the linear regulator K is known. Then, applying a prediction error method to* (61) *with H parameterized as in* (60)*, or to* (64) *gives identical estimates $\hat{\theta}_N$. This holds regardless of the parameterization of G and H_1.*

Among other things, this shows that we can use any theory developed for the direct approach (allowing for feedback) to evaluate properties of the indirect approach, and vice versa. It can also be noted that the particular choice of noise model (60) is the answer to the question how H should be parameterized in the direct method in order to avoid the bias in the G-estimate in the case of closed-loop data, even if the true noise characteristics is not correctly modeled. This is shown in [23].

5.4 The Joint Input-output Approach

The third main approach to closed-loop identification is the so called joint input-output approach. The basic assumption in this approach is that the input is generated using a regulator of a certain form, e.g., (4). Exact knowledge of the regulator parameters is not required — an advantage over the indirect method where this is a necessity.

Suppose that the regulator is linear and of the form (20). The output y and input u then obey

$$\begin{bmatrix} y(t) \\ u(t) \end{bmatrix} = \begin{bmatrix} G_{c,0}(q) \\ S_0^i(q) \end{bmatrix} r(t) + \begin{bmatrix} S_0(q)H_0(q) & G_0(q)S_0^i(q) \\ -K(q)S_0(q)H_0(q) & S_0^i(q) \end{bmatrix} \begin{bmatrix} e(t) \\ d(t) \end{bmatrix} \tag{65}$$

$$\triangleq \mathsf{G}_0(q)r(t) + \mathsf{H}_0(q) \begin{bmatrix} e(t) \\ d(t) \end{bmatrix} \tag{66}$$

Consider the parameterized model structure

$$\begin{bmatrix} y(t) \\ u(t) \end{bmatrix} = \mathsf{G}(q,\theta)r(t) + \mathsf{H}(q,\theta) \begin{bmatrix} e(t) \\ d(t) \end{bmatrix} \tag{67}$$

$$= \begin{bmatrix} G_{yr}(q,\theta) \\ G_{ur}(q,\theta) \end{bmatrix} r(t) + \begin{bmatrix} H_{ye}(q,\theta) & H_{yd}(q,\theta) \\ H_{ue}(q,\theta) & H_{ud}(q,\theta) \end{bmatrix} \begin{bmatrix} e(t) \\ d(t) \end{bmatrix} \tag{68}$$

where the parameterizations of the indicated transfer functions, for the time being, are not further specified. Different parameterization will lead to different methods, as we shall see. Previously we have used a slightly different notation, e.g., $G_{yr}(q,\theta) = G_c(q,\theta)$. This will also be done in the sequel but for the moment we will use the "generic" model structure (68) in order not to obscure the presentation with too many parameterization details.

The basic idea in the joint input-output approach is to compute estimates of the open-loop system using estimates of the different transfer functions in (68). We can for instance use

$$\hat{G}^N_{yu}(q) = \hat{G}^N_{yr}(q)(\hat{G}^N_{ur}(q))^{-1} \tag{69}$$

The rationale behind this choice is the relation $G_0 = G_{c,0}(S^i_0)^{-1}$ (cf. (65)). We may also include a prefilter, F_r, for r in the model (68), so that instead of using r directly, $x = F_r r$ is used. The open-loop estimate would then be computed as

$$\hat{G}^N_{yu}(q) = \hat{G}^N_{yx}(q)(\hat{G}^N_{ux}(q))^{-1} \tag{70}$$

where $\hat{G}^N_{yx}(q)$ and $\hat{G}^N_{ux}(q)$ are the estimated transfer functions from x to y and u, respectively. In the ideal case the use of the prefilter F_r will not affect the results since $G_0 = G_{c,0}F_r^{-1}(S^i_0 F_r^{-1})^{-1}$ regardless of F_r, but in practice, with noisy data, the filter F_r can be used to improve the quality of the estimates. This idea really goes back to Akaike [1] who showed that spectral analysis of closed-loop data should be performed as follows: Compute the spectral estimates (SISO)

$$\hat{G}^N_{yx} = \frac{\hat{\Phi}^N_{yx}}{\hat{\Phi}^N_{x}} \quad \text{and} \quad \hat{G}^N_{ux} = \frac{\hat{\Phi}^N_{ux}}{\hat{\Phi}^N_{x}} \tag{71}$$

where the signal x is correlated with y and u but uncorrelated with the noise e — a standard choice is $x = r$. The open-loop system may now be estimated as

$$\hat{G}^N_{yu} = \frac{\hat{G}^N_{yx}}{\hat{G}^N_{ux}} = \frac{\hat{\Phi}^N_{yx}}{\hat{\Phi}^N_{ux}} \tag{72}$$

In [30] a parametric variant of this idea was presented. This will be briefly discussed in Section 5.4.1 below. A problem when using parametric methods is that the resulting open-loop estimate will typically be of high-order: from (70) it follows that the order of $\hat{G}_N$ will generically be equal to the sum of the orders of the factors $\hat{G}_{yx}(q)$ and $\hat{G}_{ux}(q)$. This problem is similar to the one we are faced with in the indirect method, where we noted that solving for the open-loop estimate in (47) typically gives high-order estimates. However in the joint input-output method (70) this can be circumvented, at least in the

SISO case, by parameterizing the factors $G_{yx}(q,\theta)$ and $G_{ux}(q,\theta)$ in a common-denominator form. The final estimate will then be the ratio of the numerator polynomials in the original models.

Another way of avoiding this problem is to consider parameterizations of the form

$$G_{yx}(q,\theta) = G_{uy}(q,\rho)G_{ux}(q,\beta) \tag{73a}$$
$$G_{ux}(q,\theta) = G_{ux}(q,\beta) \tag{73b}$$

This way we will have control over the order of the final estimate through the factor $G_{uy}(q,\rho)$. If we disregard the correlation between the noise sources affecting y and u we may first estimate the β-parameters using u and r and then estimate the ρ-parameters using y and r, keeping β-parameters fixed to their estimated values. Such ideas will be studied in Sections 5.4.2-5.4.3 below.

We note that also $\mathsf{H}_0(q)$ contains all the necessary information about the open-loop system so that we can compute consistent estimates of G_0 even when no reference signal is used ($r = 0$). As an example we have that $\hat{G}^N_{yu} = \hat{H}^N_{yd}(\hat{H}^N_{ud})^{-1}$ is a consistent estimate of G_0. Such methods were studied in [15]. See also [3] and [26].

5.4.1 The Coprime Factor Identification Scheme

Consider the method (70). Recall that this method gives consistent estimates of G_0 regardless of the prefilter F_r (F_r is assumed stable). Can this freedom in the choice of prefilter F_r be utilized to give a better finite sample behavior? In [30] it is suggested to choose F_r so as to make $\hat{G}_{yx}(q)$ and $\hat{G}_{ux}(q)$ normalized coprime. The main advantage with normalized coprime factors is that they form a decomposition of the open-loop estimate $\hat{G}_N$ in minimal order, stable factors. There is a problem, though, and that is that the proper prefilter F_r that would make $\hat{G}_{yx}(q)$ and $\hat{G}_{ux}(q)$ normalized coprime is not known a priori. To cope with this problem, an iterative procedure is proposed in [30] in which the prefilter $F_r^{(i)}$ at step i is updated using the current models $\hat{G}^{(i)}_{yx}(q)$ and $\hat{G}^{(i)}_{ux}(q)$ giving $F_r^{(i+1)}$, and so on. The hope is, of course, that the iterations lead to normalized coprime factors $\hat{G}_{yx}(q)$ and $\hat{G}_{ux}(q)$.

5.4.2 The Two-stage Method

The next joint input-output method we will study is the two-stage method [28]. It is usually presented using the following two steps (cf. (73)):

(1) Identify the sensitivity function S_0^i using, e.g., an output error model

$$\hat{u}(t|\theta) = S^i(q,\theta)r(t) \tag{74}$$

(2) Construct the signal $\hat{u} = \hat{S}_N^i r$ and identify the open-loop system using the output error model

$$\hat{y}(t|\theta) = G(q,\theta)\hat{u}(t) = G(q,\theta)\hat{S}_N^i(q)r(t) \tag{75}$$

possibly using a fixed prefilter

Note that in the first step a high-order model of S_0 can be used since we in the second step can control the open-loop model order independently. Hence it should be possible to obtain very good estimates of the true sensitivity function in the first step, especially if the noise level is low. Ideally $\hat{S}_N^i \to S_0^i$ as $N \to \infty$ and $\hat{u}$ will be the noise free part of the input signal. Thus in the ideal case, the second step will be an "open-loop" problem so that an output error model with fixed noise model (prefilter) can be used without loosing consistency. See, e.g., Corollary 4 below. This result requires that the disturbance term d in (66) is uncorrelated with r.

5.4.3 The Projection Method

We will now present another method for closed-loop identification that is inspired by Akaike's idea (71)-(72) which may be interpreted as a way to correlate out the noise using the reference signal as instrumental variable. In form it will be similar to the two-stage method but the motivation for the methods will be quite different. Moreover, as we shall see the feedback need not be linear for this method to give consistent estimates. The method will be referred to as the projection method [10, 11].

This method uses the same two steps as the two-stage method. The only difference to the two-stage method is that in the first step one should use a doubly infinite, non-causal FIR filter instead. The model can be written

$$\hat{u}(t|\theta) = S^i(q,\theta)r(t) = \sum_{k=-M}^{M} s_k^i r(t-k), \quad M \to \infty, \ M = o(N) \tag{76}$$

This may be viewed as a "projection" of the input u onto the reference signal r and will result in a partitioning of the input u into two asymptotically uncorrelated parts:

$$u(t) = \hat{u}(t) + \tilde{u}(t) \tag{77}$$

where

$$\hat{u}(t) = \hat{S}_N^i(q)r(t) \tag{78}$$
$$\tilde{u}(t) = u(t) - \hat{u}(t) \tag{79}$$

We say *asymptotically* uncorrelated because $\hat{u}$ will always depend on e since u does and $S^i(q,\theta)$ is estimated using u. However, as this is a second order effect it will be neglected.

The advantage over the two-stage method is that the projection method gives consistent estimates of the open-loop system *regardless of the feedback*, even with a fixed prefilter (cf. Corollary 4 below). A consequence of this is that with the projection method we can use a fixed prefilter to shape the bias distribution of the G-estimate at will, just as in the open-loop case with output error models.

Further comments on the projection method:

- Here we chose to perform the projection using a non-causal FIR filter but this step may also be performed non-parametrically as in Akaike's cross-spectral method (71)-(72).
- In practice M can be chosen rather small. Good results are often obtained even with very modest values of M. This is clearly illustrated in Example 5 below.
- Finally, it would also be possible to project both the input u and the output y onto r in the first step. This is in fact what is done in (71)-(72).

5.5 Unifying Framework for All Joint Input-Output Methods

Consider the joint system (66) and assume, for the moment, that d is white noise with covariance matrix Λ_d independent of e. The maximum likelihood estimates of G_0 and H_0 are computed as

$$\min_{\theta \in D_{\mathcal{M}}} \frac{1}{N} \sum_{t=1}^{N} \begin{bmatrix} y(t) - \hat{y}(t|\theta) \\ u(t) - \hat{u}(t|\theta) \end{bmatrix}^T \begin{bmatrix} \Lambda_0 & 0 \\ 0 & \Lambda_d \end{bmatrix}^{-1} \begin{bmatrix} y(t) - \hat{y}(t|\theta) \\ u(t) - \hat{u}(t|\theta) \end{bmatrix} \tag{80}$$

where

$$\begin{bmatrix} \hat{y}(t|\theta) \\ \hat{u}(t|\theta) \end{bmatrix} = \mathsf{H}^{-1}(q,\theta)\mathsf{G}(q,\theta)r(t) + (I - \mathsf{H}^{-1}(q,\theta)) \begin{bmatrix} y(t) \\ u(t) \end{bmatrix} \tag{81}$$

The parameterizations of G and H can be arbitrary. Consider the system (66). This system was obtained using the assumption that the noise e affect the open-loop system only and the disturbance d affect the regulator only. The natural way to parameterize H in order to reflect these assumptions in the model is

$$\mathsf{H}(q,\theta) = \begin{bmatrix} (I+G(q,\theta)K(q,\theta))^{-1}H(q,\theta) & G(q,\theta)(I+K(q,\theta)G(q,\theta))^{-1} \\ -K(q,\theta)(I+G(q,\theta)K(q,\theta))^{-1}H(q,\theta) & (I+K(q,\theta)G(q,\theta))^{-1} \end{bmatrix} \tag{82}$$

The inverse of H is

$$\mathsf{H}^{-1}(q,\theta) = \begin{bmatrix} H^{-1}(q,\theta) & -H^{-1}(q,\theta)G(q,\theta) \\ K(q,\theta) & I \end{bmatrix} \tag{83}$$

Thus, with G parameterized as

$$\mathsf{G}(q,\theta) = \begin{bmatrix} G(q,\theta)(I+K(q,\theta)G(q,\theta))^{-1} \\ (I+K(q,\theta)G(q,\theta))^{-1} \end{bmatrix} \tag{84}$$

we get

$$\begin{bmatrix} \hat{y}(t|\theta) \\ \hat{u}(t|\theta) \end{bmatrix} = \begin{bmatrix} 0 \\ I \end{bmatrix} r(t) + \begin{bmatrix} I - H^{-1}(q,\theta) & H^{-1}(q,\theta)G(q,\theta) \\ -K(q,\theta) & 0 \end{bmatrix} \begin{bmatrix} y(t) \\ u(t) \end{bmatrix} \tag{85}$$

or

$$\hat{y}(t|\theta) = (I - H^{-1}(q,\theta))y(t) + H^{-1}(q,\theta)G(q,\theta)u(t) \tag{86}$$
$$\hat{u}(t|\theta) = r(t) - K(q,\theta)y(t) \tag{87}$$

The predictor (86) is the same as for the direct method (cf. (23)), while (87) is the natural predictor for estimating the regulator K. The maximum likelihood estimate becomes

$$\min_{\theta \in D_{\mathcal{M}}} \frac{1}{N} \Bigg\{ \sum_{t=1}^{N} (y(t) - \hat{y}(t|\theta))^T \Lambda_0^{-1} (y(t) - \hat{y}(t|\theta)) + \sum_{t=1}^{N} (u(t) - \hat{u}(t|\theta))^T \Lambda_d^{-1} (u(t) - \hat{u}(t|\theta)) \Bigg\} \tag{88}$$

We may thus view the joint input-output method as a combination of direct identification of the open-loop system and a direct identification of the regulator. Note that this holds even in the case where $r(t) = 0$. If the parameterization of the regulator K is independent of the one of the system the two terms in (88) can be minimized separately which decouples the two identification problems.

Let us return to (66). The natural output-error predictor for the joint system is

$$\begin{bmatrix} \hat{y}(t|\theta) \\ \hat{u}(t|\theta) \end{bmatrix} = \begin{bmatrix} G(q,\theta) \\ I \end{bmatrix} (I + K(q,\theta)G(q,\theta))^{-1} r(t) \tag{89}$$

According to standard open-loop prediction error theory this will give consistent estimates of G_0 and K independently of Λ_0 and Λ_d, as long as the parameterization of G and K is sufficiently flexible. See Corollary 4 below. With $S^i(q,\theta) = (I + K(q,\theta)G(q,\theta))^{-1}$ the model (89) can be written

$$\begin{bmatrix} \hat{y}(t|\theta) \\ \hat{u}(t|\theta) \end{bmatrix} = \begin{bmatrix} G(q,\theta) \\ I \end{bmatrix} S^i(q,\theta) r(t) \tag{90}$$

Consistency can be guaranteed if the parameterization of $S^i(q,\theta)$ contains the true input sensitivity function $S_0^i(q)$ (and similarly that $G(q,\theta) = G_0(q)$ for some $\theta \in D_{\mathcal{M}}$). See, e.g., Corollary 4 below. If

$$G(q,\theta) = G(q,\rho), \quad S^i(q,\theta) = S^i(q,\beta), \quad \theta = \begin{bmatrix} \rho \\ \beta \end{bmatrix} \tag{91}$$

the maximum likelihood estimate becomes

$$\begin{aligned} \min_{\theta \in D_{\mathcal{M}}} \frac{1}{N} \Bigg\{ &\sum_{t=1}^{N} (y(t) - G(q,\rho)S^i(q,\beta)r(t))^T \Lambda_0^{-1} (y(t) - G(q,\rho)S^i(q,\beta)r(t)) \\ &+ \sum_{t=1}^{N} (u(t) - S^i(q,\beta)r(t))^T \Lambda_d^{-1} (u(t) - S^i(q,\beta)r(t)) \Bigg\} \end{aligned} \tag{92}$$

Now, if $\Lambda_0 = \lambda_0 \cdot I$ and $\Lambda_d = \lambda_d \cdot I$, $\lambda_d \to 0$ then the maximum likelihood estimate will be identical to the one obtained with the two-stage or projection methods. This is true because for small λ_d the β-parameters will minimize

$$\frac{1}{N} \sum_{t=1}^{N} (u(t) - S^i(q,\beta)r(t))^T (u(t) - S^i(q,\beta)r(t)) \tag{93}$$

regardless of the ρ-parameters, which then will minimize

$$\frac{1}{N} \sum_{t=1}^{N} (y(t) - G(q,\rho)S^i(q,\beta)r(t))^T (y(t) - G(q,\rho)S^i(q,\beta)r(t)) \tag{94}$$

The weighting matrices $\Lambda_0 = \lambda_0 \cdot I$ and $\Lambda_d = \lambda_d \cdot I$ may be included in the noise models (prefilters). Thus the two-stage method (and the projection method) may be viewed as special cases of the general joint input-output approach corresponding to special choices of the noise models. In particular, this means that any result that holds for the joint input-output approach without constraints on the noise models, holds for the two-stage and projection methods as well. We will for instance use this fact in Corollary 6 below.

6 Convergence Results for the Closed-loop Identification Methods

Let us now apply the result of Theorem A.1 to the special case of closed-loop identification. In the following we will suppress the arguments ω, $e^{i\omega}$, and θ. Thus we write G_0 as short for $G_0(e^{i\omega})$ and G_θ as short for $G(e^{i\omega},\theta)$, etc. The subscript θ is included to emphasize the parameter dependence.

Corollary 4 *Consider the situation in Theorem A.1. Then, for*

(1) the direct approach, with a model structure

$$y(t) = G(q,\theta)u(t) + H(q,\theta)e(t) \tag{95}$$

where $G(q,\theta)$ is such that either $G_0(q)$ and $G(q,\theta)$ or the controller k in (5) contains a delay, we have that

$$\hat{\theta}_N \to D_c = \arg\min_{\theta\in D_{\mathcal{M}}} \int_{-\pi}^{\pi} \operatorname{tr}\Big[\Lambda^{-1}\Big\{H_\theta^{-1}[(G_0+B_\theta-G_\theta)\Phi_u(G_0+B_\theta-G_\theta)^* + (H_0-H_\theta)(\Lambda_0-\Phi_{eu}\Phi_u^{-1}\Phi_{ue})(H_0-H_\theta)^*]H_\theta^{-*}\Big\}\Big]d\omega \quad w.\ p.\ 1\ as\ N\to\infty \tag{96}$$

where

$$B_\theta = (H_0 - H_\theta)\Phi_{eu}\Phi_u^{-1} \tag{97}$$

(Φ_{eu} is the cross spectrum between e and u.)

(2) the indirect approach, if the model structure is

$$y(t) = G_c(q,\theta)r(t) + H_*(q)e(t) \tag{98}$$

and the input is given by (20), we have that

$$\hat{\theta}_N \to D_c = \arg\min_{\theta\in D_{\mathcal{M}}} \int_{-\pi}^{\pi} \operatorname{tr}\Big[\Lambda^{-1}H_*^{-1}(G_{c,0}\cdot D - G_{c,\theta})\Phi_r \cdot (G_{c,0}\cdot D - G_{c,\theta})^*(H_*)^{-*}\Big]d\omega \quad w.\ p.\ 1\ as\ N\to\infty \tag{99}$$

where

$$D = (I + \Phi_{dr}\Phi_r^{-1}) \tag{100}$$

(Φ_{dr} is the cross spectrum between d and r.)

(3) the joint input-output approach,

(a) if the model structure is

$$\begin{bmatrix} y(t) \\ u(t) \end{bmatrix} = \begin{bmatrix} G_c(q,\theta) \\ S^i(q,\theta) \end{bmatrix} r(t) + \begin{bmatrix} H_1(q) & 0 \\ 0 & H_2(q) \end{bmatrix} \begin{bmatrix} e(t) \\ d(t) \end{bmatrix} \tag{101}$$

and the input is given by (20), *we have that*

$$\hat{\theta}_N \to D_c = \arg\min_{\theta \in D_{\mathcal{M}}} \int_{-\pi}^{\pi} \mathrm{tr}\Big[\Lambda^{-1}\Big\{H_1^{-1}(G_{c,0}\cdot D - G_{c,\theta})\Phi_r \cdot \\ \cdot (G_{c,0}\cdot D - G_{c,\theta})^* H_1^{-*} + H_2^{-1}(S_0^i \cdot D - S_\theta^i)\Phi_r(S_0^i\cdot D - S_\theta^i)^* H_2^{-*}\Big\}\Big] d\omega \\ \textit{w. p. } 1 \textit{ as } N\to\infty \tag{102}$$

where D is given by (100).

(b) if the model structure is

$$y(t) = G(q,\theta)\hat{u}(t) + H_*(q)e(t) = G(q,\theta)\hat{S}_N^i(q)r(t) + H_*(q)e(t) \tag{103}$$

and the input is given by (20), *we have that*

$$\hat{\theta}_N \to D_c = \arg\min_{\theta \in D_{\mathcal{M}}} \int_{-\pi}^{\pi} \mathrm{tr}\Big[\Lambda^{-1}H_*^{-1}(G_{c,0}\cdot D - G_\theta\hat{\bar{S}})\Phi_r \cdot \\ \cdot (G_{c,0}\cdot D - G_\theta\hat{\bar{S}})^*(H_*)^{-*}\Big] d\omega \quad \textit{w. p. } 1 \textit{ as } N\to\infty \tag{104}$$

where D is given by (100).

The proof is given in Appendix B.1.

Remarks:

(1) Let us first discuss the result for the direct approach, expression (96). If the parameterization of the model G and the noise model H is flexible enough so that for some $\theta_0 \in D_{\mathcal{M}}$, $G(q,\theta_0) = G_0(q)$ and $H(q,\theta_0) = H_0(q)$ (i.e. $\mathcal{S} \in \mathcal{M}$) then $\bar{V}(\theta_0) = \Lambda_0$ so $D_c = \{\theta_0\}$ (provided this is a unique minimum) under reasonable conditions. See, e. g., the discussion following Eq. (33) and Theorem 8.3 in [22].

(2) If the system operates in open loop, so that $\Phi_{ue} = 0$, then the bias term $B_\theta = 0$ regardless of the noise model H and the limit model will be the best possible approximation of G_0 (in a frequency weighting norm that depends on H and Φ_u). A consequence of this is that in open loop we can use fixed noise models and still get consistent estimates of G_0 provided that $G(q,\theta_0) = G_0(q)$ for some $\theta \in D_{\mathcal{M}}$ and certain identifiability conditions hold. See, e.g., Theorem 8.4 in [22].

(3) A straightforward upper bound on $\bar{\sigma}(B_\theta)$ (the maximal singular value of B_θ) is

$$\bar{\sigma}(B_\theta) \leq \bar{\sigma}(H_0 - H_\theta) \cdot \bar{\sigma}(\Phi_{eu}) \cdot \frac{1}{\underline{\sigma}(\Phi_u)} \tag{105}$$

($\underline{\sigma}(\Phi_u)$ denotes the minimal singular values of Φ_u.) If the feedback is linear and given by (4) we get the following alternative upper bound:

$$\bar{\sigma}(B_\theta) \leq \bar{\sigma}(H_0 - H_\theta) \cdot \sqrt{\frac{\bar{\sigma}(\Lambda_0)}{\underline{\sigma}(\Phi_u)}} \cdot \sqrt{\frac{\bar{\sigma}(\Phi_u^e)}{\underline{\sigma}(\Phi_u)}} \tag{106}$$

It follows that the bias inclination in the G-estimate will be small in frequency ranges where either (or all) of the following holds:
- The noise model is good ($\bar{\sigma}(H_0 - H_\theta)$ is small).
- The feedback noise contribution to the input spectrum ($\bar{\sigma}(\Phi_u^e)/\underline{\sigma}(\Phi_u)$) is small.
- The signal to noise ratio is high ($\bar{\sigma}(\Lambda_0)/\underline{\sigma}(\Phi_u)$ is small).

In particular, if a reasonable flexible, independently parameterized noise model is used the bias-inclination of the G-estimate can be small.

(4) Consider now the indirect approach and especially expression (99). If the disturbance signal d in (20) is uncorrelated with r ($\Phi_{dr} = 0$) then it is possible to obtain consistent estimates of G_0 even with a fixed noise model. Suppose that G_c is parameterized as in (49). Then

$$G_{c,0} - G_{c,\theta} = G_{c,0} - S_\theta G_\theta = S_0(G_0 - G_\theta)S_\theta^i \tag{107}$$

Thus with under-modeling the resulting G-estimate will try to minimize the mismatch between G_0 and G and at the same time try to minimize the model sensitivity function S^i. There will thus be a "bias-pull" towards transfer functions that give a small sensitivity for the given regulator, but unlike (96) it is not easy to quantify this bias component.

(5) If d is correlated with r ($\Phi_{dr} \neq 0$) then the multiplicative factor $D \neq I$ (cf. (100)) and the G-estimate will try to minimize

$$G_{c,0} \cdot D - G_\theta(I + KG_\theta)^{-1} \tag{108}$$

which can lead to an arbitrarily bad open-loop estimate depending on D.

(6) Let us now turn to the joint input-output approach. From expression (102) we conclude that, if the parameterizations of G_c and S^i are sufficiently flexible, then $\hat{G}_N = \hat{G}_{c,N}(\hat{S}_N^i)^{-1} \to G_{c,0}D(S_0^i D)^{-1} = G_0$ regardless of D. Thus for *linear* feedback this approach can give consistent estimates, even if the regulator is unknown.

(7) Finally let us comment on expression (104). This holds for both the two-stage and projection methods. Here we see that any disturbance d that is correlated with r will deteriorate the G-estimate in the two-stage method.

The error is quantified by the multiplicative factor D in (104). With the projection method, on the other hand, $D = I$ since $\Phi_{dr} = 0$ (by construction). Hence with this method it is possible to obtain consistent estimates of G_0 with fixed noise models (prefilters), regardless of the feedback. The difference in the applicability of the two-stage method and the projection method will be illustrated in Example 5 below.

The results in Corollary 4 will now be illustrated by means of a small simulation study.

Example 5 *Consider the closed-loop system in Fig. 2. Here*

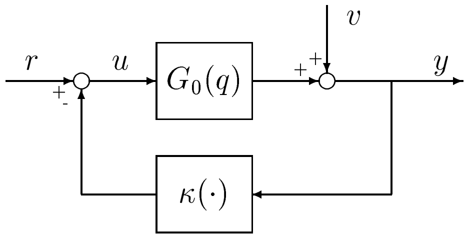

Fig. 2. Closed-loop system with non-linear feedback.

$$G_0(q) = q^{-1} - 0.8q^{-2} \tag{109}$$

The feedback controller is

$$\kappa(x) = \begin{cases} 0.25x + 4, & \text{if } x > 0 \\ 0.25x - 4, & \text{if } x < 0 \end{cases} \tag{110}$$

Thus, apart from just a scaling, the controller consists of a columbic friction element.

In order to identify G a simulation was made with no noise present, i.e. with $v \equiv 0$, and using a white noise reference signal. 5000 data samples were collected. Thus the only errors in the models should be bias errors.

Next, the system was identified using the direct, indirect, two-stage, and projection methods. For the direct method the model was

$$G(q, \theta) = b_1 q^{-1} + b_2 q^{-2} \tag{111}$$

For the indirect method we used (cf. (49))

$$G_c(q, \theta) = \frac{b_1 q^{-1} + b_2 q^{-2}}{1 + 0.25(b_1 q^{-1} + b_2 q^{-2})} \tag{112}$$

The following models were employed in the first step of the two-stage method:

(1) An 11th order FIR model:

$$S(q, \beta) = b_0 + b_1 q^{-1} + \cdots + b_{10} q^{-10} \tag{113}$$

(2) An 11th order ARX model:

$$S(q, \beta) = \frac{b_0 + b_1 q^{-1} + \cdots + b_5 q^{-5}}{1 + a_1 q^{-1} + \cdots + a_5 q^{-5}} \tag{114}$$

In the first step of the projection method the model was

$$S(q, \beta) = \sum_{k=-5}^{5} s_k q^{-k} \tag{115}$$

That is, an 11th order non-causal FIR model. In the second step of the two-stage and projection methods the model was chosen as

$$G(q, \rho) = b_1 q^{-1} + b_2 q^{-2} \tag{116}$$

The identification results are summarized in Table 1: Here we can note that

Table 1. Summary of identification results.

	b_1	b_2
True system	1.0000	−0.8000
Direct	1.0000	−0.8000
Indirect	−1.8060	−4.8793
Two-stage (FIR)	0.9595	−0.8258
Two-stage (ARX)	0.8857	−0.6906
Projection	0.9985	−0.7965

the direct method retrieves the true system correctly while the indirect method gives completely useless results. This is not surprising since we did not take the non-linear element κ into account when applying the indirect method. The two variants of the two-stage method give biased results but with the projection method the estimate is almost perfect. Bode plots of the resulting model errors are shown in Fig. 3 together with a plot of G_0. Here we chose to use 11th order models in the first step of the two-stage and projection methods; other choices give similar results as long as the model order is not too small (larger than 5 say). Note especially that the projection method in this example works fine with a small number of parameters in the non-causal FIR model. For linear systems the two-stage method applied with a high-order FIR model in the first step usually gives very good results. However, as this example shows, the non-causal terms in the FIR model in the projection method can not be excluded in general.

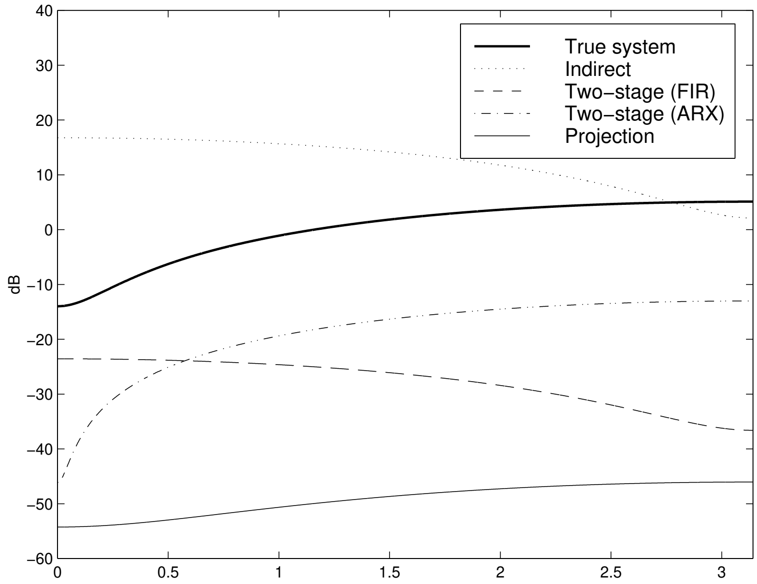

Fig. 3. Bode plot of the true system and the errors in the estimated models.

7 Asymptotic Variance of Black Box Transfer Function Estimate in the Case of Linear Feedback

Consider now the case where the system to be identified operates in closed loop with a linear regulator given by (4) controlling the system. Then, for the direct, the indirect, and the joint input-output approach we have that

$$\operatorname{Cov} \operatorname{vec}\left[\hat{G}_N(e^{i\omega})\right] \sim \frac{n}{N}(\Phi_u^r(\omega))^{-T} \otimes \Phi_v(\omega) \tag{117}$$

as the model order n as well as the number of data N tends to infinity. This follows from

Corollary 6 *Suppose that the input is given by* (4)*. Then*

$$\sqrt{N} \operatorname{vec}\left[\hat{G}_N(e^{i\omega}, n, \delta) - G_n^*(e^{i\omega})\right] \in AsN(0, P(\omega, n, \delta)) \quad \text{as } N \to \infty \text{ for fixed } n,\ \delta \tag{118}$$

where

$$\lim_{\delta \to 0} \lim_{n \to \infty} \frac{1}{n} P(\omega, n, \delta) = (\Phi_u^r(\omega))^{-T} \otimes \Phi_v(\omega) \tag{119}$$

holds for

(1) the direct approach, under the conditions in Theorem A.2.
(2) the indirect approach, under the conditions in Corollary A.3.
(3) the joint input-output approach, under the conditions in Corollary A.3.

The proof is given in Appendix B.2. The variable δ is explained in Appendix A.2.

Remarks:

(1) The difference between (117) (or (119)) and the corresponding open-loop expression (40) (or (A.24)) is thus that Φ_u^r (i.e., that part of the input that originates from the reference signal r) replaces Φ_u.

(2) The expression (117) – which also is the asymptotic Cramer-Rao lower limit – tells us precisely "the value of information" of closed-loop experiments. It is the noise-to-signal ratio (where "signal" is what derives from the injected reference) that determines how well the open-loop transfer function can be estimated. From this perspective, that part of the input that originates from the feedback noise has no information value when estimating G. Since this property is, so to say, inherent in the problem, it should come as no surprise that all approaches give the same asymptotic variance.

(3) The results in (A.2), (40), and (117) are independent of the parameterization and an important assumption is that the model order tends to infinity. It is then quite natural that the differences between the methods disappear, since they may be viewed as different parameterizations of the same prediction error method. When we fix the model order the results will be different, though. This will be illustrated in the next section. Let us also mention that if we in the direct method use a fixed noise model, equal to the true one, $H(q, \theta) = H_*(q) = H_0(q)$, then as the model order tends to infinity we have that

$$\text{Cov vec}\left[\hat{G}_N(e^{i\omega}, n)\right] \sim \frac{n}{N}(\Phi_u(\omega))^{-T} \otimes \Phi_v(\omega) \tag{120}$$

just as in the open-loop case. This is true since the covariance matrix is determined entirely by the second order properties (the spectrum) of the input, and it is immaterial whether this spectrum is a result of open-loop or closed-loop operation.

8 Asymptotic Distribution of Parameter Vector Estimates in the Case of Closed-loop Data

8.1 The Direct and Indirect Approaches

The following corollary to Theorem A.4 in Appendix A.3 states what the asymptotic distribution of the estimate $\hat{\rho}_N$ will be for the direct and indirect approaches in their standard formulations. The result will be given for the SISO case only. In the following G'_ρ will be used as short for $(d/d\rho)G(e^{i\omega},\rho)|_{\rho=\rho_0}$, etc.

Corollary 7 *Assume that the conditions in Theorem A.4 hold and, in addition, that the data set Z^∞ is informative enough with respect to the chosen model structure. Then for*

(1) the direct approach, with a model structure

$$y(t) = G(q,\rho)u(t) + H(q,\beta)e(t), \quad \theta = \begin{bmatrix} \rho \\ \beta \end{bmatrix} \tag{121}$$

satisfying $\mathcal{S} \in \mathcal{M}$ and that is globally identifiable at θ_0, we have that

$$\sqrt{N}(\hat{\rho}_N - \rho_0) \in AsN(0, P_\rho^D) \tag{122a}$$

where

$$P_\rho^D = \lambda_0 (R_\rho^r + \Delta)^{-1} \tag{122b}$$
$$R_\rho^r = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{\Phi_u^r}{|H_0|^2} G'_\rho G'^*_\rho d\omega \tag{122c}$$
$$\Delta = R_\rho^e - R_{\rho\beta}^e (R_\beta^e)^{-1} R_{\beta\rho}^e \tag{122d}$$
$$R_\rho^e = \frac{1}{2\pi}\int_{-\pi}^{\pi} \lambda_0 |K|^2 |S_0|^2 G'_\rho G'^*_\rho d\omega \tag{122e}$$
$$R_{\rho\beta}^e = -\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{\lambda_0}{|H_0|^2} K S_0 H_0 G'_\rho H'^*_\beta d\omega \tag{122f}$$
$$R_\beta^e = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{\lambda_0}{|H_0|^2} H'_\beta H'^*_\beta d\omega \tag{122g}$$

(2) the indirect approach,
(a) with a model structure,

$$y(t) = \frac{G(q,\rho)}{1+G(q,\rho)K(q)} r(t) + H_c(q,\beta)e(t), \quad \theta = \begin{bmatrix} \rho \\ \beta \end{bmatrix} \tag{123}$$

satisfying $\mathcal{S} \in \mathcal{M}$ and that is globally identifiable at θ_0, we have that

$$\sqrt{N}(\hat{\rho}_N - \rho_0) \in AsN(0, P_\rho^I) \tag{124a}$$

where

$$P_\rho^I = \lambda_0 (R_\rho^r)^{-1} \tag{124b}$$

with R_ρ^r given by (122c)

(b) with a model structure,

$$y(t) = \frac{G(q,\rho)}{1+G(q,\rho)K(q)}r(t) + H_*(q)e(t) \tag{125}$$

satisfying $G_0 \in \mathcal{G}$ and that is globally identifiable at ρ_0, we have that

$$\sqrt{N}(\hat{\rho}_N - \rho_0) \in AsN(0, \bar{P}^I_\rho) \tag{126a}$$

where

$$\bar{P}^I_\rho = (R)^{-1}Q(R)^{-1} \tag{126b}$$
$$R = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{|S_0|^2\Phi^r_u}{|H_*|^2}G'_\rho G'^*_\rho d\omega \tag{126c}$$
$$Q = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{|S_0|^4\Phi_v\Phi^r_u}{|H_*|^4}G'_\rho G'^*_\rho d\omega \tag{126d}$$

The proof is given in Appendix B.3.

A number of comments can be made regarding the Δ-term in (122). We collect these in

Lemma 8 *Consider the situation in Corollary 7. For all choices of noise model $H(q,\beta)$ such that $\mathcal{S} \in \mathcal{M}$ we have that*

$$0 \le \Delta \le R^e_\rho \tag{127}$$

Furthermore,

$$\Delta \to 0 \tag{128}$$

as the order of the noise model tends to infinity, and

$$\Delta = R^e_\rho \tag{129}$$

if the noise model is fixed and equal to the true one, that is if $H(q,\beta) = H_*(q) = H_0(q)$.

The proof is given in Appendix B.4.

Further remarks:

(1) An important observation regarding the result (122) is that the term Δ is entirely due to the noise part of the input spectrum and since $\Delta \ge 0$ this contribution has a positive effect on the accuracy, contrarily to what one might have guessed. We conclude that in the direct method the noise in the loop is utilized in reducing the variance. For the indirect methods this contribution is zero.

(2) From (122) it is also clear that — from the accuracy point of view — the worst-case experimental conditions with respect to the reference signal is when there is *no* external reference signal present, that is when $\Phi_r = 0$. In this case

$$Cov\ \hat{\rho}_N \sim \frac{\lambda_0}{N}\Delta^{-1} \tag{130}$$

Thus Δ characterizes the lower limit of achievable accuracy for the direct method. Now, if Δ is non-singular, (122) says that we can consistently estimate the system parameters even though no reference signal is present. The exact conditions for this to happen are given in [24] for some common special cases. However, even if Δ is singular it will have a beneficial effect on the variance of the estimates, according to (122). Only when $\Delta = 0$ there is no positive effect from the noise source on the accuracy of the estimates. According to Lemma 8 this would be the case when the order of the noise model tends to infinity. At the other extreme we have that a fixed (and correct) noise model will make Δ as "large" as it can be.

This puts the finger on the value of information in the noise source e for estimating the dynamics:

> *It is the knowledge/assumption of a constrained noise model that improves the estimate of G.*

This also explains the difference between (117) (which assumes the noise model order to tend to infinity) and (120) (which assumes a fixed and correct noise model).

(3) For the indirect approach, note that for all H_*,

$$P_\rho^{\mathrm{I}} \leq \bar{P}_\rho^{\mathrm{I}} \tag{131}$$

with equality for $H_* = H_{c,0} = S_0 H_0$ in (125). By comparing (124) with (122) we also see that the covariance matrix for the indirect method will always be larger than the one for the direct. The difference is the term Δ that is missing in (124). We thus have the following ranking:

$$P_\rho^{\mathrm{D}} \leq P_\rho^{\mathrm{I}} \leq \bar{P}_\rho^{\mathrm{I}} \tag{132}$$

8.2 Further Results for the Indirect Approach

In Corollary 7 we considered two variants of the indirect approach and it is important to realize that the conclusions drawn in the previous section do not hold for all possible variants and parameterizations of the indirect approach. In this section we will study two other variants where first the closed-loop system is identified and then the open-loop system is estimated by solving an over-determined system of equations, relating the open-loop parameters to the closed-loop parameters, in the least-squares sense using the knowledge of the controller (cf. the discussion in Section 5.2). We will then see that it is possible to get the same level of accuracy with the indirect method as with the direct, even with finite model orders. This was first shown in [25] and in this section we will review and extend the results there. See also [15] and [27].

8.2.1 *Indirect Identification Using ARMAX Models*

Let us now outline the idea. Suppose that the regulator is linear and of the form (4), i.e.

$$u(t) = r(t) - K(q)y(t) \tag{133}$$

Factorize K as

$$K(q) = \frac{X(q)}{Y(q)} \tag{134}$$

where the polynomials X and Y are coprime. Let the closed-loop and open-loop model structures be

$$A_c(q)y(t) = B_c(q)r(t) + C_c(q)e(t) \tag{135}$$

and

$$A(q)y(t) = B(q)u(t) + C(q)e(t) \tag{136}$$

respectively. From

$$G_c(q) = \frac{G(q)}{1+G(q)K(q)} \text{ and } H_c(q) = \frac{H(q)}{1+G(q)K(q)} \tag{137}$$

(cf. (9)) it follows that we may solve for A, B and C in

$$\begin{cases} A_c(q) = A(q)Y(q) + B(q)X(q) \\ B_c(q) = B(q)Y(q) \\ C_c(q) = C(q)Y(q) \end{cases} \tag{138}$$

Let η and θ denote the closed-loop and open-loop parameter vectors, respectively. Then, for given η, (138) is a system of linear equations in θ from which θ can be estimated. If the Markov estimate is used, i.e. if (138) is solved in the least squares sense, using the estimated covariances of $\hat{\eta}_N$ as weights, Corollary 9 below states that this indirect method is consistent under reasonable assumptions and gives the same accuracy as the direct method with an ARMAX model structure.

Corollary 9 *Consider the situation in Theorem A.4 and let the model structure $\mathcal{M}$ be given by* (135). *Assume that the model structure $\mathcal{M}$ is identifiable at η_0 and that the data set $Z^\infty = \{y(1), r(1), \dots, y(\infty), r(\infty)\}$ is informative enough with respect to $\mathcal{M}$. Assume also that $\mathcal{S} \in \mathcal{M}$ holds for the model structure* (136). *Then, if the open-loop parameters θ are estimated from* (138) *using the Markov estimate, we have that*

$$\sqrt{N}(\hat{\theta}_N - \theta_0) \in AsN(0, P_\theta^{AX}) \tag{139a}$$

where

$$P_\theta^{AX} = \lambda_0 \left[\bar{E}\psi(t,\theta_0)\psi^T(t,\theta_0)\right]^{-1} \tag{139b}$$

$$\psi(t,\theta) = -\frac{d}{d\theta}\frac{A(q)}{C(q)}\left(y(t) - \frac{B(q)}{A(q)}u(t)\right) \tag{139c}$$

The proof is given in Appendix B.5.

Through this theorem we have established yet another link between the direct and indirect approaches. As will become more apparent below, the fact that the dynamics *model* and the noise *model* share the same poles is crucial for obtaining the same accuracy as with the direct approach.

8.2.2 Indirect Identification Using Box-Jenkins Models

Suppose now that we in the first step in the indirect method use the following model structure with an independently parameterized noise model:

$$\begin{aligned} y(t) &= G_c(q,\rho_\eta)r(t) + H_c(q,\beta_\eta)e(t) \\ &= \frac{B_c(q)}{F_c(q)}r(t) + \frac{C_c(q)}{D_c(q)}e(t) \end{aligned} \tag{140}$$

This is also known as a Box-Jenkins model structure. Let the open-loop model be

$$\begin{aligned} y(t) &= G(q,\rho)u(t) + H(q,\beta)e(t) \\ &= \frac{B(q)}{F(q)}r(t) + \frac{C(q)}{D(q)}e(t) \end{aligned} \tag{141}$$

To find the indirect estimate of G_0 we may solve for B and F in (cf. (138))

$$\begin{cases} B_c(q) = B(q)Y(q) \\ F_c(q) = F(q)Y(q) + B(q)X(q) \end{cases} \tag{142}$$

The final estimate will also this time be consistent (under mild conditions) but not of optimal accuracy. Instead we get

Corollary 10 *Consider the situation in Theorem A.4 and let the model structure $\mathcal{M}$ be given by* (140). *Assume that the model structure $\mathcal{M}$ is identifiable at η_0 and that the data set $Z^\infty = \{y(1), r(1), \ldots, y(\infty), r(\infty)\}$ is informative enough with respect to $\mathcal{M}$. Assume also that $G_0 \in \mathcal{G}$ holds for the model structure* (141). *Then, if the open-loop parameters θ are estimated from* (142) *using the Markov estimate, we have that*

$$\sqrt{N}(\hat{\rho}_N - \rho_0) \in AsN(0, P_\rho^{BJ}) \tag{143a}$$

where

$$P_\rho^{BJ} = \lambda_0 \left[\frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{\Phi_u^r}{|H_0|^2} G'_\rho G'^{*}_\rho d\omega \right]^{-1} \tag{143b}$$

The proof is given in Appendix B.6.

Thus

$$P_\rho^{\mathrm{BJ}} = P_\rho^{\mathrm{I}} \; (\geq P_\rho^{\mathrm{D}}) \tag{144}$$

and once again we conclude that with an independently parameterized noise model the indirect method gives worse accuracy than the direct method. The difference is quantified by the term Δ (cf. (122)) which is missing in (143).

8.3 Variance Results for the Projection Method

In this section some results for the efficiency of the projection method will be presented. We will consider the model

$$\hat{y}(t|\theta) = H_*^{-1}(q) G(q, \theta) \hat{u}(t) + (1 - H_*^{-1}(q)) y(t) \tag{145}$$

Note that the true system (3) can be rewritten as

$$y(t) = G_0(q) \hat{u}(t) + w(t) \tag{146}$$

where

$$w(t) = v(t) + G_0(q)(u(t) - \hat{u}(t)) \tag{147}$$

The key feature of the projection method is now that $\hat{u}$ will be uncorrelated with w. Estimating the model (145) is therefore an open-loop problem and all standard open-loop results hold. As an example we have

Corollary 11 *Consider the estimate $\hat{\theta}_N$ determined by* (25)-(27). *Assume that the model structure* (145) *is uniformly stable and that Assumptions 1 and*

2 hold. Suppose that $\hat{u}$ and w given by (147) *are uncorrelated. Suppose also that for a unique value θ^* interior to $D_{\mathcal{M}}$ we have*

$$\hat{\theta}_N \to \theta^*, \quad w.\ p.\ 1 \text{ as } N \to \infty \tag{148}$$

$$R = \bar{V}''(\theta^*) > 0 \tag{149}$$

$$\sqrt{N} E\left[\frac{1}{N}\sum_{t=1}^{N}[\psi(t,\theta^*)\varepsilon(t,\theta^*) - \bar{E}\psi(t,\theta^*)\varepsilon(t,\theta^*)]\right] \to 0, \quad as\ N \to \infty \tag{150}$$

Then

$$\sqrt{N}(\hat{\theta}_N - \theta^*) \in AsN(0, P_\theta) \tag{151a}$$

$$P_\theta = R^{-1}QR^{-1} \tag{151b}$$

$$R = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{\Phi_{\hat{u}}}{|H_*|^2} G'_\rho G'^*_\rho d\omega \tag{151c}$$

$$Q = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{\Phi_w \Phi_{\hat{u}}}{|H_*|^4} G'_\rho G'^*_\rho d\omega \tag{151d}$$

Remarks:

(1) Suppose that $\Phi_w = \lambda|\bar{H}_0|^2$ for some constant λ and some monic filter $\bar{H}_0$. For all H_*,

$$P_\theta \geq \lambda R^{-1} \triangleq P_\theta^{opt} \tag{152}$$

Equality holds if $H_* = \bar{H}_0$.

(2) The result (151) is equivalent to the open-loop result, except that Φ_u is replaced by $\Phi_{\hat{u}}$ and Φ_v by Φ_w. Typically

$$\Phi_{\hat{u}} < \Phi_u \text{ and } \Phi_w > \Phi_v \tag{153}$$

so that the signal to noise ratio is worse than for open-loop identification ($\Phi_{\hat{u}}/\Phi_w < \Phi_u/\Phi_v$), hence the variance will be larger.

9 Summarizing Discussion

We have attempted to give a status report on identification of closed-loop systems. Several methods have been studied and we have shown that most of the common methods can be viewed as special parameterizations of the general prediction error method. This allows simultaneous analysis of the statistical properties of the methods using the general prediction error theory. The main results can be summarized as follows.

The direct method

- gives consistency and optimal accuracy, regardless of the feedback, if the noise and dynamic models contain a true description of the system.
- requires a correct noise model (possibly estimated) to avoid bias in the dynamics transfer function for closed-loop data (Corollary 4). For this reason, user chosen weightings for custom-made bias distribution cannot be used.

The indirect method

- requires perfect knowledge of the regulator but can be applied with fixed noise models and still guarantee consistency (Corollary 4).

The projection method

- allows the true system to be approximated in an arbitrary, user chosen frequency domain norm regardless of the feedback (Corollary 4).
- typically gives worse accuracy than the direct method (Corollary 11).

Further comments:

(1) The asymptotic variance expressions for increasing model orders are the same for all methods (Corollary 6).
(2) For the finite model order case, the direct method meets the Cramer-Rao bound and is thus optimal. The indirect method generally gives worse accuracy. The improved accuracy for the direct method can be traced to the fact that a constrained noise model (that includes the true noise description) allows some leverage from the noise part of the input to estimate the dynamics (Corollary 7 and Lemma 8).

The optimal statistical properties, the simplicity, and the general applicability of the direct method implies that this should be seen as the first choice of methods for closed-loop identification. When this method fails no other method will succeed. In the literature several other methods, e.g., the indirect method, have been advocated mainly due to their ability to shape the bias distribution by use of fixed noise models (prefilters). However, it is important to realize that the total error in the resulting estimate will be due to *both* bias and variance errors and since these methods typically give sub-optimal accuracy, this "advantage" may be questionable.

If a low order model is sought that should approximate the system dynamics in a pre-specified frequency norm there are two ways to proceed:

- First estimate a higher order model, with small bias, using the direct approach and then reduce this model to lower order using the proper frequency weighting.
- Use the projection method. This has the advantage that a higher order model need not be estimated, but has the drawback that it may have worse

accuracy.

Acknowledgement

The authors wish to thank all friends and colleagues, including the the anonymous reviewers, that have given helpful comments and suggestions for improvements of the manuscript during the course of this work.

A Theoretical Results for the Prediction Error Method

A.1 Complement to Section 4.2

Here we will prove (33). Let us formalize the result in a theorem:

Theorem A.1 *Let $\hat{\theta}_N$ be defined by* (25) *and* (27) *where $L(q,\theta) = I$. Suppose that the model structure* (22) *is uniformly stable and that Assumptions 1 and 2 hold. Suppose also that all signals involved are quasistationary. Then*

$$\hat{\theta}_N \to D_c = \arg\min_{\theta \in D_{\mathcal{M}}} \int_{-\pi}^{\pi} \operatorname{tr}\Big[\Lambda^{-1} H_\theta^{-1} \begin{bmatrix} (G_0 - G_\theta)^* \\ (H_0 - H_\theta)^* \end{bmatrix}^* \begin{bmatrix} \Phi_u & \Phi_{ue} \\ \Phi_{eu} & \Lambda_0 \end{bmatrix} \cdot \begin{bmatrix} (G_0 - G_\theta)^* \\ (H_0 - H_\theta)^* \end{bmatrix} H_\theta^{-*}\Big] d\omega \quad w.\ p.\ 1\ as\ N \to \infty \tag{A.1}$$

PROOF. From (32) we have that $\hat{\theta}_N$ will tend to the minimizing argument of $\bar{V}(\theta)$, w. p. 1 as $N \to \infty$. Using Parseval's relationship we get

$$\bar{V}(\theta) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \operatorname{tr}\left[\Lambda^{-1}\Phi_\varepsilon\right] d\omega \tag{A.2}$$

Here Φ_ε is the spectrum of the prediction errors $\varepsilon(t,\theta) = H^{-1}(q,\theta)(y(t) - G(q,\theta)u(t))$. Write H_θ as short for $H(e^{i\omega},\theta)$, etc. Then, using (3) we get

$$\varepsilon = H_\theta^{-1}(y - G_\theta u) = H_\theta^{-1}\Big[(G_0 - G_\theta)u + (H_0 - H_\theta)e\Big] + e \tag{A.3}$$

The last term is independent of the rest since $(G_0(q) - G(q,\theta))u(t)$ depends only on $e(s)$ for $s < t$ (by assumption) and $H_0(q) - H(q,\theta)$ contains a delay (since both H_0 and H are monic). The spectrum of ε thus becomes

$$\Phi_\varepsilon = H_\theta^{-1} \begin{bmatrix} (G_0 - G_\theta)^* \\ (H_0 - H_\theta)^* \end{bmatrix}^* \begin{bmatrix} \Phi_u & \Phi_{ue} \\ \Phi_{eu} & \Lambda_0 \end{bmatrix} \begin{bmatrix} (G_0 - G_\theta)^* \\ (H_0 - H_\theta)^* \end{bmatrix} H_\theta^{-*} + \Lambda_0 \tag{A.4}$$

The result now follows since the last term is independent of θ. □

A.2 Complement to Section 4.3

In this appendix we will give the exact statements of the results (39)-(41), including the necessary technical assumptions. The main results are given in Theorem A.2 and Corollary A.3 below. These are used in Corollary 6, Section 7.

First of all the vec-operator introduced in (36) is defined as ($A_{i,j}$ is the (i,j) element of the $(n \times m)$ matrix A)

$$\mathrm{vec}\Big[A\Big] = \Big[A_{1,1}, A_{2,1}, \ldots, A_{n,1}, A_{1,2}, A_{2,2}, \ldots, A_{n,m}\Big]^T \tag{A.5}$$

The model structure considered is

$$\varepsilon(t,\theta) = y(t) - \hat{y}(t|\theta) = H^{-1}(q,\theta)(y(t) - G(q,\theta)u(t)) \tag{A.6}$$

Let

$$\theta^*(n) = \arg\min_{\theta \in D_{\mathcal{M}}} \bar{E}\varepsilon^T(t,\theta)\varepsilon(t,\theta) \tag{A.7}$$

(If the minimum is not unique, let $\theta^*(n)$ denote *any* minimizing element.) Now define the estimate $\hat{\theta}_N(n,\delta)$ by

$$\hat{\theta}_N(n,\delta) = \arg\min_{\theta \in D_{\mathcal{M}}} V_N(\theta, n, \delta, Z^N) \tag{A.8}$$

$$V_N(\theta, n, \delta, Z^N) = \frac{1}{N}\sum_{t=1}^{N} \varepsilon^T(t,\theta)\varepsilon(t,\theta) + \delta\|\theta - \theta^*(n)\|^2 \tag{A.9}$$

Here δ is a regularization parameter helping us to select a unique minimizing element in (A.8) in case $\delta = 0$ leads to non-unique minima. Let

$$T_n^*(e^{i\omega}) = T(e^{i\omega}, \theta^*(n)) \tag{A.10}$$

$$\hat{T}_N(e^{i\omega}, n, \delta) = T(e^{i\omega}, \hat{\theta}_N(n,\delta)) \tag{A.11}$$

$$T_0(e^{i\omega}) = \mathrm{vec}\Big[G_0(e^{i\omega}) \ H_0(e^{i\omega})\Big] \tag{A.12}$$

It will be assumed that

$$\lim_{n\to\infty} n^2 E\Big[(\varepsilon(t,\theta^*(n)) - e(t))^T(\varepsilon(t,\theta^*(n)) - e(t))\Big] = 0 \tag{A.13}$$

which implies that $T_n^*(e^{i\omega})$ tends to $T_0(e^{i\omega})$ as n tends to infinity. Define $Z(q,\theta)$ through (cf. (38))

$$\frac{\partial}{\partial\theta_k}T(q,\theta) = q^{-k+1}\frac{\partial}{\partial\theta_1}T(q,\theta) \triangleq q^{-k+1}Z(q,\theta) \tag{A.14}$$

Let the matrix function $Z(e^{i\omega},\theta)$ be denoted by $Z_0(e^{i\omega})$ when evaluated for $T_0(e^{i\omega})$. It will also be assumed that

$$Z_0(e^{i\omega})Z_0^T(e^{i\omega}) \tag{A.15}$$

is invertible and that

$$\lim_{N\to\infty}\frac{1}{\sqrt{N}}\sum_{t=1}^{N}E\Big[\frac{d}{d\theta}\varepsilon^T(t,\theta^*(n))\varepsilon(t,\theta^*(n))\Big] = 0 \quad (n \text{ fixed}) \tag{A.16}$$

Finally it will be assumed that

$$\bar{\sigma}(\Phi_u(\omega)) \leq C, \quad \underline{\sigma}(\Phi_u(\omega)) \geq \lambda > 0 \quad \forall\omega \tag{A.17}$$

$$\frac{1}{\sqrt{n(N)}}\sum_{\tau=-n(N)}^{n(N)}|\tau|\bar{\sigma}(R_u(\tau)) \to 0 \quad \text{as } N\to\infty \;\; n(N)\to\infty \tag{A.18}$$

Here

$$R_u(\tau) = \bar{E}u(t)u^T(t-\tau) \tag{A.19}$$

The main variance result is now

Theorem A.2 *Consider the estimate* $\hat{T}_N(e^{i\omega},n,\delta)$ *under the assumptions* (38) *and* (A.6)-(A.18). *Suppose also that Assumptions 1 and 2 hold. Then*

$$\sqrt{N}\,\mathrm{vec}\Big[\hat{T}_N(e^{i\omega},n,\delta) - T_n^*(e^{i\omega})\Big] \in AsN(0,P(\omega,n,\delta)) \quad \text{as } N\to\infty \text{ for fixed } n,\ \delta \tag{A.20}$$

where

$$\lim_{\delta\to 0}\lim_{n\to\infty}\frac{1}{n}P(\omega,n,\delta) = \begin{bmatrix}\Phi_u(\omega) & \Phi_{ue}(\omega)\\ \Phi_{eu}(\omega) & \Lambda_0\end{bmatrix}^{-T} \otimes \Phi_v(\omega) \tag{A.21}$$

(Here $\otimes$ denotes the Kronecker product.)

This was proven for the SISO case in [21]. The result for the MIMO case, cited here, was established in [34]. The expression (39) is an intuitive, but not formally correct, interpretation of (A.20), (A.21). In open loop we have

Corollary A.3 *Consider the same situation as in Theorem A.2 but assume that $H(q, \theta)$ is independent of θ. Assume that $\Phi_{ue} = 0$ and that* (A.13) *is relaxed to*

$$n^2 \| G_n^*(e^{i\omega}) - G_0(e^{i\omega}) \| \to 0 \quad \text{as } n \to \infty \tag{A.22}$$

Then

$$\sqrt{N}\,\mathrm{vec}\Big[\hat{G}_N(e^{i\omega}, n, \delta) - G_n^*(e^{i\omega})\Big] \in AsN(0, P(\omega, n, \delta)) \quad \text{as } N \to \infty \text{ for fixed } n,\ \delta \tag{A.23}$$

where

$$\lim_{\delta \to 0} \lim_{n \to \infty} \frac{1}{n} P(\omega, n, \delta) = \Phi_u(\omega)^{-T} \otimes \Phi_v(\omega) \tag{A.24}$$

A.3 Complement to Section 4.4

We will now turn to the question of asymptotic distribution of the parameter vector estimates. For notational convenience the discussion will be limited to the SISO case. Let λ_0 denote the variance of the noise e in (3) in the SISO case. The following result is a variant of Theorem 9.1 in [22]:

Theorem A.4 *Consider the estimate $\hat{\theta}_N$ determined by* (25)-(27) *with $L(q, \theta) = 1$. Assume that the model structure is linear and uniformly stable and that Assumptions 1 and 2 hold. Suppose that the regulator is given by* (4) *and that for a unique value θ^* interior to $D_{\mathcal{M}}$ we have*

$$\hat{\theta}_N \to \theta^*, \quad \text{w. p. 1 as } N \to \infty \tag{A.25}$$

$$R = \bar{V}''(\theta^*) > 0 \tag{A.26}$$

$$\sqrt{N} E\Big[\frac{1}{N} \sum_{t=1}^{N} [\psi(t, \theta^*)\varepsilon(t, \theta^*) - \bar{E}\psi(t, \theta^*)\varepsilon(t, \theta^*)]\Big] \to 0, \quad \text{as } N \to \infty \tag{A.27}$$

Then

$$\sqrt{N}(\hat{\theta}_N - \theta^*) \in AsN(0, P_\theta) \tag{A.28a}$$

$$P_\theta = R^{-1} Q R^{-1} \tag{A.28b}$$

$$Q = \lim_{N \to \infty} N \cdot E\Big\{[V_N'(\theta^*, Z^N)][V_N'(\theta^*, Z^N)]^T\Big\} \tag{A.28c}$$

Remark: Frequency domain expressions for R and Q can easily be derived using Parseval's relationship (cf. [22], Section 9.4).

B Additional Proofs

B.1 Proof of Corollary 4

From Theorem A.1 we know that

$$D_c = \arg\min_{\theta \in D_{\mathcal{M}}} \int_{-\pi}^{\pi} \operatorname{tr}\left[\Lambda^{-1}\Phi_\varepsilon\right] d\omega \tag{B.1}$$

where Φ_ε is the spectrum of the prediction errors ε. Consider first the direct approach. The spectrum of the prediction errors is given by (A.4). Note that (cf. (35))

$$\begin{bmatrix} \Phi_u & \Phi_{ue} \\ \Phi_{eu} & \Lambda_0 \end{bmatrix} = \begin{bmatrix} I & 0 \\ \Phi_{eu}\Phi_u^{-1} & I \end{bmatrix} \begin{bmatrix} \Phi_u & 0 \\ 0 & (\Lambda_0 - \Phi_{eu}\Phi_u^{-1}\Phi_{ue}) \end{bmatrix} \begin{bmatrix} I & \Phi_u^{-1}\Phi_{ue} \\ 0 & I \end{bmatrix} \tag{B.2}$$

Thus with B_θ as in (97) we get

$$\Phi_\varepsilon = H_\theta^{-1} \begin{bmatrix} (G_0 - G_\theta + B_\theta)^* \\ (H_0 - H_\theta)^* \end{bmatrix}^* \begin{bmatrix} \Phi_u & 0 \\ 0 & (\Lambda_0 - \Phi_{eu}\Phi_u^{-1}\Phi_{ue}) \end{bmatrix} \begin{bmatrix} (G_0 - G_\theta + B_\theta)^* \\ (H_0 - H_\theta)^* \end{bmatrix} H_\theta^{-*} + \Lambda_0 \tag{B.3}$$

Equation (96) now follows since the last term is independent of θ. For the indirect approach we get, using (20),

$$\varepsilon = H_*^{-1}(y - G_{c,\theta}r) = H_*^{-1}\left[(G_{c,0} - G_{c,\theta})r + S_0 v + G_{c,0}d\right] \tag{B.4}$$

Thus

$$\begin{aligned}\Phi_\varepsilon = H_*^{-1}\Big[&(G_{c,0} - G_{c,\theta})\Phi_r(G_{c,0} - G_{c,\theta})^* + (G_{c,0} - G_{c,\theta})\Phi_{rd}G_{c,0}^* + \\ &+ G_{c,0}\Phi_{dr}(G_{c,0} - G_{c,\theta})^* + \theta\text{-indep. terms}\Big](H_*)^{-*}\end{aligned} \tag{B.5}$$

or

$$\begin{aligned}\Phi_\varepsilon = H_*^{-1}\Big[&(G_{c,0} - G_{c,\theta} + G_{c,0}\Phi_{dr}\Phi_r^{-1})\Phi_r(G_{c,0} - G_{c,\theta} + G_{c,0}\Phi_{dr}\Phi_r^{-1})^* + \\ &+ \theta\text{-indep. terms}\Big](H_*)^{-*}\end{aligned} \tag{B.6}$$

With D as in (100), we thus have

$$\Phi_\varepsilon = H_*^{-1}\left[(G_{c,0}\cdot D - G_{c,\theta})\Phi_r(G_{c,0}\cdot D - G_{c,\theta})^* + \theta\text{-indep. terms}\right](H_*)^{-*} \tag{B.7}$$

from which (99) follows. The proofs of (102) and (104) are analogous. □

B.2 Proof of Corollary 6

For the direct method, note that the result in Theorem A.2 applies also to the closed-loop case. The covariance of $\operatorname{vec}\Big[\hat{G}_N(e^{i\omega}, n)\Big]$ is given by the top left block in P given by (A.21). Note that

$$\begin{bmatrix} \Phi_u & \Phi_{ue} \\ \Phi_{eu} & \Lambda_0 \end{bmatrix}^{-1} = \begin{bmatrix} (\Phi_u - \Phi_{ue}\Lambda_0^{-1}\Phi_{eu})^{-1} & \times \\ \times & \times \end{bmatrix} \quad (\times = \text{don't care}) \tag{B.8}$$

From (16) and (19) we get

$$\Phi_u - \Phi_{ue}\Lambda_0^{-1}\Phi_{eu} = S_0^i \Phi_r (S_0^i)^* = \Phi_u^r \tag{B.9}$$

which proves the result for the direct approach.

Let us now turn to the proof for the indirect approach. The model is given by (45). According to Corollary A.3 the random variable

$$\sqrt{N}\operatorname{vec}\Big[\hat{G}_{c,N}(e^{i\omega}, n, \delta) - G_{c,n}^*(e^{i\omega})\Big] \tag{B.10}$$

will have an asymptotic normal distribution with covariance matrix $\bar{P}(\omega, n, \delta)$ satisfying

$$\lim_{\delta \to 0} \lim_{n \to \infty} \frac{1}{n} \bar{P}(\omega, n, \delta) = (\Phi_r(\omega))^{-T} \otimes \Phi_{v_c}(\omega) \tag{B.11}$$

where $\Phi_{v_c} = S_0 \Phi_v S_0^*$. This result holds regardless of the noise model H_* used in (45). From the Taylor expansion

$$\begin{aligned} \operatorname{vec}\Big[\hat{G}_N(e^{i\omega}, n, \delta) - G_n^*(e^{i\omega})\Big] = \\ = \left(\frac{d \operatorname{vec}[G]}{d \operatorname{vec}[G_c]}(e^{i\omega}, \theta^*(n))\right)^T \left\{\operatorname{vec}\Big[\hat{G}_{c,N}(e^{i\omega}, n, \delta) - G_{c,n}^*(e^{i\omega})\Big]\right\} \\ + O(\|\operatorname{vec}\Big[\hat{G}_{c,N}(e^{i\omega}, n, \delta) - G_{c,n}^*(e^{i\omega})\Big]\|) \end{aligned} \tag{B.12}$$

we see that the covariance matrix of

$$\sqrt{N}\operatorname{vec}\Big[\hat{G}_N(e^{i\omega}, n, \delta) - G_n^*(e^{i\omega})\Big] \tag{B.13}$$

will be

$$P(\omega, n, \delta) = \left(\frac{d \operatorname{vec}[G]}{d \operatorname{vec}[G_c]}(e^{i\omega}, \theta^*(n))\right)^{-T} \bar{P}(\omega, n, \delta) \left(\frac{d \operatorname{vec}[G]}{d \operatorname{vec}[G_c]}(e^{-i\omega}, \theta^*(n))\right)^{-1} \tag{B.14}$$

Note that

$$G_c = SG = (I + GK)^{-1}G = G - GKG + GKGKG - \dots \tag{B.15}$$

Repeated use of the formula [14]

$$\frac{d\,\mathrm{vec}[AXB]}{d\,\mathrm{vec}[X]} = (B \otimes A^T) \tag{B.16}$$

and the chain rule now gives

$$\begin{aligned}\frac{d\,\mathrm{vec}[G_c]}{d\,\mathrm{vec}[G]} = I - (KG \otimes I) - (I \otimes K^T G^T) + (KG \otimes K^T G^T) + \\ + (KGKG \otimes I) + (I \otimes K^T G^T K^T G^T) - \dots\end{aligned} \tag{B.17}$$

As can be readily verified, with $S^i = (I + KG)^{-1}$ and $S = (I + GK)^{-1}$, we thus have that

$$\frac{d\,\mathrm{vec}[G_c]}{d\,\mathrm{vec}[G]} = \left(S^i \otimes S^T\right) = \left((S^i)^T \otimes S\right)^T \tag{B.18}$$

Consequently

$$\lim_{n\to\infty} \frac{1}{n} P(\omega, n, \delta) = \left((S_0^i)^T \otimes S_0\right)^{-1} \left((\Phi_r)^{-T} \otimes \Phi_{v_c}\right) \left((S_0^i)^T \otimes S_0\right)^{-*} \tag{B.19}$$

$$= \left((S_0^i)^{-T} \otimes S_0^{-1}\right) \left((\Phi_r)^{-T} \otimes \Phi_{v_c}\right) \left(((S_0^i)^T)^{-*} \otimes S_0^{-*}\right) \tag{B.20}$$

$$= \left((S_0^i)^{-T}(\Phi_r)^{-T}((S_0^i)^*)^{-T}\right) \otimes \left(S_0^{-1}\Phi_{v_c}S_0^{-*}\right) \tag{B.21}$$

$$= (\Phi_u^r)^{-T} \otimes \Phi_v \tag{B.22}$$

which concludes the proof for the indirect approach.

For the joint input-output approach the model is (in case an output error model is used):

$$\begin{bmatrix} \hat{y}(t,\theta) \\ \hat{u}(t,\theta) \end{bmatrix} = \begin{bmatrix} G_c(q,\theta) \\ S^i(q,\theta) \end{bmatrix} r(t) \tag{B.23}$$

The open-loop estimate is computed as $\hat{G}_N = \hat{G}_{c,N}(\hat{S}_N^i)^{-1}$. Write Q as short for $\left[G_c^T \ (S^i)^T\right]^T$. According to Corollary A.3 the random variable

$$\sqrt{N}\,\mathrm{vec}\Big[\hat{Q}_N(e^{i\omega}, n, \delta) - Q_n^*(e^{i\omega})\Big] \tag{B.24}$$

will have an asymptotic normal distribution with covariance matrix $\bar{P}(\omega, n, \delta)$

satisfying

$$\lim_{\delta\to 0}\lim_{n\to\infty}\frac{1}{n}\bar{P}(\omega,n,\delta)=(\Phi_r(\omega))^{-T}\otimes\begin{bmatrix} I \\ -K(e^{i\omega})\end{bmatrix}\Phi_{v_c}(\omega)\begin{bmatrix} I \\ -K(e^{i\omega})\end{bmatrix}^* \tag{B.25}$$

where $\Phi_{v_c}=S_0\Phi_v S_0^*$. From the Taylor expansion

$$\begin{aligned}\operatorname{vec}\Big[\hat{G}_N(e^{i\omega},n,\delta)-G_n^*(e^{i\omega})\Big] &= \\ &=\left(\frac{d\operatorname{vec}[G]}{d\operatorname{vec}\big[Q\big]}(e^{i\omega},\theta^*(n))\right)^T\Big\{\operatorname{vec}\Big[\hat{Q}_N(e^{i\omega},n,\delta)-Q_n^*(e^{i\omega})\Big]\Big\} \\ &\qquad + O(\|\operatorname{vec}\Big[\hat{Q}_N(e^{i\omega},n,\delta)-Q_n^*(e^{i\omega})\Big]\|)\end{aligned} \tag{B.26}$$

we see that the covariance matrix of

$$\sqrt{N}\operatorname{vec}\Big[\hat{G}_N(e^{i\omega},n,\delta)-G_n^*(e^{i\omega})\Big] \tag{B.27}$$

will be

$$P(\omega,n,\delta)=\left(\frac{d\operatorname{vec}[G]}{d\operatorname{vec}[Q]}(e^{i\omega},\theta^*(n))\right)^{-T}\bar{P}(\omega,n,\delta)\left(\frac{d\operatorname{vec}[G]}{d\operatorname{vec}[Q]}(e^{-i\omega},\theta^*(n))\right)^{-1} \tag{B.28}$$

By use of the formula (B.16) and the chain rule it can be shown that

$$\frac{d\operatorname{vec}[G]}{d\operatorname{vec}[Q]}=\Big((S^i)^{-1}\otimes\Big[I\ 0\Big]^T\Big)+\Big((S^i)^{-1}\otimes\Big[0\ \ -G_c(S^i)^{-1}\Big]^T\Big) \tag{B.29}$$

$$=(S^i)^{-1}\otimes\Big[I\ \ -G_c(S^i)^{-1}\Big]^T \tag{B.30}$$

Note that $G_c(S^i)^{-1}\to G_0$ as $n\to\infty$. Thus

$$\begin{aligned}\lim_{n\to\infty}\frac{1}{n}P(\omega,n,\delta) &= \Big((S_0^i)^{-T}\otimes\Big[I\ \ -G_0\Big]\Big)\cdot \\ &\quad\cdot\left((\Phi_r)^{-T}\otimes\begin{bmatrix} I \\ -K\end{bmatrix}\Phi_{v_c}\begin{bmatrix} I \\ -K\end{bmatrix}^*\right)\Big((S_0^i)^{-T}\otimes\Big[I\ \ -G_0\Big]\Big)^*\end{aligned} \tag{B.31}$$

$$=\Big((S_0^i)^{-T}(\Phi_r)^{-T}((S_0^i)^*)^{-T}\Big)\otimes\Big(S_0^{-1}\Phi_{v_c}S_0^{-*}\Big) \tag{B.32}$$

$$=(\Phi_u^r)^{-T}\otimes\Phi_v \tag{B.33}$$

which concludes the proof for the joint input-output approach and the also the proof of Corollary 6. □

B.3 Proof of Corollary 7

Consider first the direct approach. Under the conditions in the corollary we have that $\hat{\theta}_N \to \theta_0$ w. p. 1 as $N \to \infty$. See, e.g., Theorem 8.3 in [22]. It follows from Theorem A.4 that

$$\sqrt{N}(\hat{\theta}_N - \theta_0) \in AsN(0, P_\theta) \tag{B.34}$$

where

$$P_\theta = \lambda_0 \left[\bar{E}\psi(t,\theta_0)\psi^T(t,\theta_0)\right]^{-1} \tag{B.35}$$

$$\triangleq \lambda_0 R_\theta^{-1} \tag{B.36}$$

Introduce $\chi_0 = [u\ e]^T$. The spectrum of χ_0 is

$$\Phi_{\chi_0} = \begin{bmatrix} \Phi_u & \Phi_{ue} \\ \Phi_{ue}^* & \lambda_0 \end{bmatrix} \tag{B.37}$$

Since $\Phi_{ue} = -KS_0H_0\lambda_0$, we may also write this as

$$\Phi_{\chi_0} = \Phi_{\chi_0}^r + \Phi_{\chi_0}^e \tag{B.38}$$

where

$$\Phi_{\chi_0}^r = \begin{bmatrix} \Phi_u^r & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad \Phi_{\chi_0}^e = \lambda_0 \begin{bmatrix} KS_0H_0 \\ -1 \end{bmatrix} \begin{bmatrix} KS_0H_0 \\ -1 \end{bmatrix}^* \tag{B.39}$$

Using Parseval's relation R_θ can be written as (see, e.g., [22])

$$R_\theta = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1}{|H_0|^2} T_\theta' \Phi_{\chi_0} T_\theta'^* d\omega \tag{B.40}$$

where $T = [G\ H]$ and where $T_\theta' = \frac{d}{d\theta}T|_{\theta=\theta_0}$. From (B.38) it follows that

$$T_\theta'\Phi_{\chi_0}T_\theta'^* = T_\theta'\Phi_{\chi_0}^r T_\theta'^* + T_\theta'\Phi_{\chi_0}^e T_\theta'^* \tag{B.41}$$

We may thus write

$$R_\theta = R_\theta^r + R_\theta^e \tag{B.42}$$

where, due to the chosen parameterization,

$$R_\theta^r = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1}{|H_0|^2} T_\theta' \Phi_{\chi_0}^e T_\theta'^* d\omega = \begin{bmatrix} R_\rho^r & 0 \\ 0 & 0 \end{bmatrix} \tag{B.43}$$

and

$$R_\theta^e = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1}{|H_0|^2} T_\theta' \Phi_{\chi_0}^r T_\theta'^* d\omega = \begin{bmatrix} R_\rho^e & R_{\rho\beta}^e \\ R_{\beta\rho}^e & R_\beta^e \end{bmatrix} \tag{B.44}$$

Straight-forward calculations will also show that

$$R_\rho^r = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{\Phi_u^r}{|H_0|^2} G_\rho' G_\rho'^* d\omega \tag{B.45}$$

and

$$R_\theta^e = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1}{|H_0|^2} T_\theta' \Phi_{\chi_0}^e T_\theta'^* d\omega = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{\lambda_0}{|H_0|^2} \begin{bmatrix} K S_0 H_0 G_\rho' \\ -H_\beta' \end{bmatrix} \begin{bmatrix} K S_0 H_0 G_\rho' \\ -H_\beta' \end{bmatrix}^* d\omega \tag{B.46}$$

P_ρ, the covariance matrix for $\hat{\rho}_N$ can be found as the top left block of P_θ. Combining (B.35), (B.36), (B.43), and (B.44) we thus have that

$$P_\rho = \lambda_0 (R_\rho^r + \Delta)^{-1} \tag{B.47}$$

$$\Delta = R_\rho^e - R_{\rho\beta}^e (R_\beta^e)^{-1} R_{\beta\rho}^e \tag{B.48}$$

where R_ρ^r is given by (B.45) and expressions for R_ρ^e, $R_{\rho\beta}^e$, and R_β^e can be extracted from (B.46). This finishes the proof of (122). Let us now turn to the proof of (124). The covariance matrix P_θ is given by (B.35). Applying Parseval's formula to $R_\theta = \bar{E}\psi(t,\theta)\psi^T(t,\theta)$ gives

$$R_\theta = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1}{|H_{c,0}|^2} \begin{bmatrix} G_{c,\theta}' & H_{c,\theta}' \end{bmatrix} \begin{bmatrix} \Phi_r & 0 \\ 0 & \lambda_0 \end{bmatrix} \begin{bmatrix} G_{c,\theta}' & H_{c,\theta}' \end{bmatrix}^* d\omega \tag{B.49}$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1}{|H_{c,0}|^2} \begin{bmatrix} \Phi_r G_{c,\rho}' G_{c,\rho}'^* & 0 \\ 0 & \lambda_0 H_{c,\beta}' H_{c,\beta}'^* \end{bmatrix} d\omega \tag{B.50}$$

Hence

$$P_\rho = \lambda_0 \left[\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{\Phi_r}{|H_{c,0}|^2} G_{c,\rho}' G_{c,\rho}'^* d\omega \right]^{-1} \tag{B.51}$$

The theorem now follows after noting that $G_{c,\rho}' = S_0^2 G_\rho'$ and $H_{c,0} = S_0 H_0$. The proof of (126) is similar. $\square$

B.4 Proof of Lemma 8

Consider Δ defined by (122d). Note that Δ is the Schur complement of R^e_β in the matrix R^e_θ given by (B.44). Since $R^e_\theta \geq 0$ we thus have $\Delta \geq 0$. This proves the lower bound in (127). Next, define

$$\psi_e(t, \theta_0) = T'_\theta(q, \theta_0) \begin{bmatrix} K(q)S_0(q)H_0(q) \\ -1 \end{bmatrix} e(t) \tag{B.52}$$

Then we can alternatively write R^e_θ as

$$R^e_\theta = \bar{E}\psi_e(t, \theta_0)\psi_e^T(t, \theta_0) \tag{B.53}$$

Now introduce the notation

$$w(t) = H_0^{-1}(q)e(t) \tag{B.54}$$

$$\bar{G}'_\rho(q) = K(q)S_0(q)H_0(q)G'_\rho(q) \tag{B.55}$$

so that

$$\psi_e(t, \theta_0) = \begin{bmatrix} \bar{G}'_\rho(q)w(t) \\ -H'_\beta(q)w(t) \end{bmatrix} \tag{B.56}$$

Here the number of rows in $\bar{G}'_\rho$ and H'_β are consistent with the partitioning (B.43), (B.44). From well known least-squares projections (see, e.g., [2]), we now recognize Δ as *the error covariance matrix when estimating $\bar{G}'_\rho w$ from $H'_\beta w$.*

Suppose now that the order of the noise model tends to infinity. Without loss of generality we can assume that

$$H(q, \beta) = \lim_{M\to\infty} 1 + \sum_{k=1}^{M} h_k q^{-k} \tag{B.57}$$

Then knowing $H'_\beta w$ is equivalent to knowing all past w. Then $\bar{G}'_\rho w$ can be determined exactly from $H'_\beta w$, and $\Delta = 0$. This proves (128)

At the other extreme, a fixed (and correct) noise model will make $\Delta = R^e_\rho = \bar{E}\bar{G}'_\rho w(\bar{G}'_\rho w)^T$, which is the largest value Δ may have. This proves (129) and the upper bound in (127). $\square$

B.5 Proof of Corollary 9

Let

$$
\begin{aligned}
A(q) &= 1 + a_1 q^{-1} + \cdots + a_{n_a} q^{-n_a} \\
B(q) &= b_1 q^{-1} + \cdots + b_{n_b} q^{-n_b} \\
C(q) &= 1 + c_1 q^{-1} + \cdots + c_{n_c} q^{-n_c}
\end{aligned}
\tag{B.58}
$$

and similarly for the closed-loop polynomials. Define θ as

$$
\theta = \Big[a_1, \ \ldots, \ a_{n_a}, \ b_1, \ \ldots, \ b_{n_b}, c_1, \ \ldots, \ c_{n_c}\Big]^T \tag{B.59}
$$

and η equivalently. Furthermore let the regulator polynomials be

$$
\begin{aligned}
X(q) &= x_0 + x_1 q^{-1} + \cdots + x_{n_x} q^{-n_x} \\
Y(q) &= 1 + y_1 q^{-1} + \cdots + y_{n_y} q^{-n_y}
\end{aligned}
\tag{B.60}
$$

With these definitions the system of equations (138) can be written as

$$
\Gamma\theta = \bar{\eta} \tag{B.61}
$$

where

$$
\Gamma = \begin{bmatrix} \Gamma_Y & \Gamma_X & 0 \\ 0 & \Gamma_Y & 0 \\ 0 & 0 & \Gamma_Y \end{bmatrix}, \quad
\Gamma_Y = \begin{bmatrix} 1 & & \\ y_1 & \ddots & \\ \vdots & \ddots & 1 \\ & & y_1 \\ & & \vdots \end{bmatrix}, \quad
\Gamma_X = \begin{bmatrix} x_0 & & \\ x_1 & \ddots & \\ \vdots & \ddots & x_0 \\ & & x_1 \\ & & \vdots \end{bmatrix}
\tag{B.62}
$$

while

$$
\bar{\eta} = \Big[a_{c,1} - y_1, \ a_{c,2} - y_2, \ \ldots, \ b_{c,1}, \ b_{c,2}, \ \ldots, \ c_{c,1} - y_1, \ c_{c,2} - y_2, \ \ldots\Big]^T \tag{B.63}
$$

Now, if the closed-loop parameters η are estimated from a set of N data we have from Theorem A.4 that

$$
\sqrt{N}(\hat{\eta}_N - \eta_0) \in AsN(0, P_\eta) \tag{B.64}
$$

$$
P_\eta = \lambda_0 \Big[\bar{E}\psi_c(t,\eta_0)\psi_c^T(t,\eta_0)\Big]^{-1} \tag{B.65}
$$

$$
\psi_c(t,\eta) = -\frac{d}{d\eta}\frac{A_c(q)}{C_c(q)}\left(y(t) - \frac{B_c(q)}{A_c(q)} r(t)\right) \tag{B.66}
$$

Consider equation (B.61). The Markov estimate of θ is

$$\hat{\theta}_N = \Big[\Gamma^T(\mathrm{Cov}\,\hat{\hat{\eta}}_N)^{-1}\Gamma\Big]^{-1}\Gamma^T(\mathrm{Cov}\,\hat{\hat{\eta}}_N)^{-1}\hat{\hat{\eta}}_N \tag{B.67}$$

from which it follows that

$$\sqrt{N}(\hat{\theta}_N - \theta_0) \in AsN(0, P_\theta) \tag{B.68}$$

$$P_\theta = N\Big[\Gamma^T(\mathrm{Cov}\,\hat{\hat{\eta}}_N)^{-1}\Gamma\Big]^{-1} \tag{B.69}$$

We also have that

$$\mathrm{Cov}\,\hat{\hat{\eta}}_N = \frac{1}{N}P_\eta = \frac{\lambda_0}{N}\Big[\bar{E}\psi_c(t,\eta_0)\psi_c^T(t,\eta_0)\Big]^{-1} \tag{B.70}$$

Hence

$$P_\theta = \lambda_0\Big[\bar{E}\Gamma^T\psi_c(t,\eta_0)\psi_c^T(t,\eta_0)\Gamma\Big]^{-1} \tag{B.71}$$

The gradient vector $\psi_c(t,\eta_0)$ is

$$\begin{aligned}\psi_c(t,\eta_0) = \frac{1}{C_{c,0}(q)}\Big[&-y(t-1),\ldots,-y(t-n_{a_c}),\\ &r(t-1),\ldots,r(t-n_{b_c}),e(t-1),\ldots,e(t-n_{c_c})\Big]^T\end{aligned} \tag{B.72}$$

and since

$$\frac{Y(q)}{C_{c,0}(q)} = \frac{1}{C_0(q)} \quad \text{and} \quad \frac{X(q)y(t)}{C_{c,0}(q)} - \frac{Y(q)r(t)}{C_{c,0}(q)} = -\frac{1}{C_0(q)}u(t) \tag{B.73}$$

we have

$$\begin{aligned}\Gamma^T\psi_c(t,\eta_0) = \frac{1}{C_0(q)}\Big[&-y(t-1),\ldots,-y(t-n_a),\\ &u(t-1),\ldots,u(t-n_b),e(t-1),\ldots,e(t-n_c)\Big]^T\end{aligned} \tag{B.74}$$

It follows that

$$\Gamma^T\psi_c(t,\eta_0) = -\frac{d}{d\theta}\frac{A(q)}{C(q)}\Big(y(t) - \frac{B(q)}{A(q)}u(t)\Big) \tag{B.75}$$

where the right-hand side should be evaluated at $\theta = \theta_0$. But the expression on the right is equal to the negative gradient of the open-loop prediction errors, i.e. $\psi(t,\theta)$, evaluated at $\theta = \theta_0$. Thus $\Gamma^T\psi_c(t,\eta_0) = \psi(t,\theta_0)$ and

$$P_\theta = \lambda_0\Big[\bar{E}\psi(t,\theta_0)\psi^T(t,\theta_0)\Big]^{-1} \tag{B.76}$$

which together with (B.68) proves the corollary. □

B.6 Proof of Corollary 10

Let

$$\begin{aligned} B(q) &= b_1 q^{-1} + \cdots + b_{n_b} q^{-n_b} \\ F(q) &= 1 + f_1 q^{-1} + \cdots + f_{n_f} q^{-n_f} \\ C(q) &= 1 + c_1 q^{-1} + \cdots + c_{n_c} q^{-n_c} \\ D(q) &= 1 + d_1 q^{-1} + \cdots + d_{n_d} q^{-n_d} \end{aligned} \tag{B.77}$$

and similarly for the closed-loop polynomials and let X, Y, Γ_X, and Γ_Y be as in the proof of Corollary 9. Then (142) can be written

$$\tilde{\Gamma}\rho = \bar{\rho}_\eta \tag{B.78}$$

where

$$\tilde{\Gamma} = \begin{bmatrix} \Gamma_Y & 0 \\ \Gamma_X & \Gamma_Y \end{bmatrix}, \quad \bar{\rho}_\eta = \Big[b_{c,1},\ b_{c,2},\ \ldots,\ f_{c,1} - y_1,\ f_{c,2} - y_2,\ \ldots \Big]^T \tag{B.79}$$

Using the same arguments as in the proof of Corollary 9 we get

$$\sqrt{N}(\hat{\rho}_N - \rho_0) \in AsN(0, P_\rho) \tag{B.80}$$

$$P_\rho = N\Big[\tilde{\Gamma}^T (\operatorname{Cov} \hat{\bar{\rho}}_{\eta,N})^{-1} \tilde{\Gamma}\Big]^{-1} \tag{B.81}$$

We also have that

$$\operatorname{Cov} \hat{\bar{\rho}}_{\eta,N} = \operatorname{Cov} \hat{\rho}_{\eta,N} = \frac{\lambda_0}{N} \Big[\bar{E}\psi_{c,\rho_\eta}(t,\eta_0)\psi^T_{c,\rho_\eta}(t,\eta_0) \Big]^{-1} \tag{B.82}$$

where ψ_{c,ρ_η} is the negative gradient of the closed-loop predictions errors,

$$\begin{aligned} \varepsilon_c(t,\eta) &= \frac{1}{H_c(q,\beta_\eta)} \Big(y(t) - G_c(q,\rho_\eta) r(t) \Big) && \text{(B.83)} \\ &= \frac{1}{H_c(q,\beta_\eta)} \Big(y(t) - \frac{B_c(q)}{F_c(q)} r(t) \Big) && \text{(B.84)} \end{aligned}$$

taken w.r.t. ρ_η, giving

$$\begin{aligned} \psi_{c,\rho_\eta}(t,\eta_0) = \frac{1}{H_{c,0}(q) F_{c,0}(q)} \Big[& r(t-1), \ldots, r(t - n_{b_c}), \\ & -\frac{B_{c,0}(q)}{F_{c,0}(q)} (r(t-1), \ldots, r(t - n_{f_c})) \Big] \end{aligned} \tag{B.85}$$

Returning to P_ρ we thus have that

$$P_\rho = \lambda_0 \Big[\bar{E}\tilde{\Gamma}^T \psi_c(t,\eta_0)\psi^T_c(t,\eta_0) \tilde{\Gamma} \Big]^{-1} \tag{B.86}$$

However, since

$$H_{c,0} = S_0 H_0, \quad \frac{F_{c,0}Y - B_{c,0}X}{F_{c,0}^2} = \frac{S_0^2}{F_0}, \quad \frac{B_{c,0}Y}{F_{c,0}^2} = \frac{S_0^2}{F_0} \cdot \frac{B_0}{F_0} \tag{B.87}$$

we get (after some calculations)

$$\begin{aligned} \tilde{\Gamma}^T \psi_{c,\rho_\eta}(t, \eta_0) = \frac{S_0(q)}{H_0(q)F_0(q)} \Big[& r(t-1), \ldots, r(t-n_b), \\ & -\frac{B_0(q)}{F_0(q)}(r(t-1), \ldots, r(t-n_f)) \Big]^T \end{aligned} \tag{B.88}$$

so

$$\tilde{\Gamma}^T \psi_{c,\rho_\eta}(t, \eta_0) = -\frac{S_0(q)}{H_0(q)} \cdot \frac{d}{d\rho_\theta} \Big(y(t) - \frac{B(q)}{F(q)} r(t) \Big) \tag{B.89}$$

$$= \frac{S_0(q)}{H_0(q)} \cdot \frac{d}{d\rho_\theta} G(q, \rho_\theta) r(t) \tag{B.90}$$

where the derivatives on the right should be evaluated at $\rho_\theta = \rho_{\theta,0}$. Using Parseval's formula we thus have that

$$P_\rho = \lambda_0 \left[\frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{\Phi_u^r}{|H_0|^2} G'_\rho G'^*_\rho d\omega \right]^{-1} \tag{B.91}$$

and the corollary follows. □

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