Dynamical Cohomology and Topological Rigidity

Alejandro Kocsard

Universidade Federal Fluminense
Brasil

Encuentro de Geometría Diferencial
Rosario 2012

Cohomological Equation

$$Xu = \phi \qquad \text{(CE)}$$

Cohomological Equation

$$Xu = \phi \qquad \text{(CE)}$$

- M smooth (compact) manifold
- X smooth vector field
- $\phi\colon M \to \mathbb{R}$ given
- $u\colon M \to \mathbb{R}$ unknown

Cohomological Equation

$$Xu = \phi \qquad \text{(CE)}$$

- M smooth (compact) manifold
- X smooth vector field
- $\phi\colon M \to \mathbb{R}$ given
- $u\colon M \to \mathbb{R}$ unknown

Induced flow

$\Phi_X : M \times \mathbb{R} \to M$ defined by

$$\partial_t\big|_{t=0}\Phi_X^t(p) = X(p), \quad \forall p \in M$$

Cohomological Equation

$$Xu = \phi \qquad \text{(CE)}$$

- M smooth (compact) manifold
- X smooth vector field
- $\phi \colon M \to \mathbb{R}$ given
- $u \colon M \to \mathbb{R}$ unknown

Induced flow

$\Phi_X \colon M \times \mathbb{R} \to M$ defined by

$$\partial_t \big|_{t=0} \Phi_X^t(p) = X(p), \quad \forall p \in M$$

Line flows are characteristic curves!

Existence of solutions

Definition

For $r \geqslant 1$, define the space of C^r-**coboundaries**

$$B(X, C^r) := \{Xu : u \in C^r(M, \mathbb{R})\}$$

Existence of solutions

Definition

For $r \geqslant 1$, define the space of C^r-**coboundaries**

$$B(X, C^r) := \{Xu : u \in C^r(M, \mathbb{R})\}$$

Main Problem

How does the topology of M affect the $B(X, C^\infty)$?

Existence of solutions

Definition

For $r \geqslant 1$, define the space of C^r-**coboundaries**

$$B(X, C^r) := \{Xu : u \in C^r(M, \mathbb{R})\}$$

Main Problem

How does the topology of M affect the $B(X, C^\infty)$?

Example (uninteresting)

$M = \mathbb{R}^d$, $X \equiv (1, 0, \ldots, 0)$. Then $B(X, C^\infty) = C^\infty(\mathbb{R}^d)$

Existence of solutions

Definition

For $r \geqslant 1$, define the space of C^r-**coboundaries**

$$B(X, C^r) := \{Xu : u \in C^r(M, \mathbb{R})\}$$

Main Problem

How does the topology of M affect the $B(X, C^\infty)$?

Example (uninteresting)

$M = \mathbb{R}^d$, $X \equiv (1, 0, \dots, 0)$. Then $B(X, C^\infty) = C^\infty(\mathbb{R}^d)$

First evidence

Compactness of M breaks surjectivity, e.g. $1 \notin B(X, C^0)$

Invariant measures

Definition

- μ Borel probability measure is X-**invariant** iff

$$\int_M \psi \circ \Phi_X^t \, \mathrm{d}\mu = \int_M \psi \, \mathrm{d}\mu, \quad \forall \psi, \ \forall t$$

- Krylov & Bogolioubov: There exists an X-invariant proba measure

Invariant measures

Definition

- μ Borel probability measure is X-**invariant** iff

$$\int_M \psi \circ \Phi_X^t \,\mathrm{d}\mu = \int_M \psi \,\mathrm{d}\mu, \quad \forall \psi, \ \forall t$$

- Krylov & Bogolioubov: There exists an X-invariant proba measure

Proposition [(CE) vs. invariant measures]

$$\overline{B(X, C^\infty)}^{C^0} = \bigcap_{\mu \in \mathfrak{M}(X)} \ker \mu$$

Invariant measures

Definition

- μ Borel probability measure is X-**invariant** iff

$$\int_M \psi \circ \Phi_X^t \, \mathrm{d}\mu = \int_M \psi \, \mathrm{d}\mu, \quad \forall \psi, \ \forall t$$

- Krylov & Bogolioubov: There exists an X-invariant proba measure

Proposition [(CE) vs. invariant measures]

$$\overline{B(X, C^\infty)}^{C^0} = \bigcap_{\mu \in \mathfrak{M}(X)} \ker \mu$$

Measures are not enough!

Distributions

- $\mathcal{D}'(M) := (C^\infty(M))^\star$ topological dual space
- $\mathcal{D}'(X) := \{\mu \in \mathcal{D}'(M) : \langle \mu, \psi \circ \Phi_X^t \rangle = \langle \mu, \psi \rangle, \ \forall \psi, \ \forall t\}$

Measures are not enough!

Distributions

- $\mathcal{D}'(M) := (C^\infty(M))^\star$ topological dual space
- $\mathcal{D}'(X) := \{\mu \in \mathcal{D}'(M) : \langle \mu, \psi \circ \Phi_X^t \rangle = \langle \mu, \psi \rangle, \ \forall \psi, \ \forall t\}$

Proposition [(CE) vs. invariant distributions]

$$\overline{B(X, C^\infty)}^{C^\infty} = \bigcap_{\mu \in \mathcal{D}'(X)} \ker \mu$$

An (important) example

- $X_\alpha \equiv (\alpha_1, \ldots, \alpha_d) \in \mathbb{R}^d$ constant vector field on $\mathbb{T}^d = \mathbb{R}^d / \mathbb{Z}^d$

An (important) example

- $X_\alpha \equiv (\alpha_1, \ldots, \alpha_d) \in \mathbb{R}^d$ constant vector field on $\mathbb{T}^d = \mathbb{R}^d / \mathbb{Z}^d$
- Irreducibility: $(\alpha_1, \ldots, \alpha_d) \in \mathbb{R}^d$ is **non-resonant** iff $\forall (n_1, \ldots, n_d) \in \mathbb{Z}^d$

$$\sum_{j=1}^{d} n_j \alpha_j = 0 \implies n_j = 0, \ \forall j$$

An (important) example

- $X_\alpha \equiv (\alpha_1, \ldots, \alpha_d) \in \mathbb{R}^d$ constant vector field on $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$
- Irreducibility: $(\alpha_1, \ldots, \alpha_d) \in \mathbb{R}^d$ is **non-resonant** iff $\forall (n_1, \ldots, n_d) \in \mathbb{Z}^d$

$$\sum_{j=1}^{d} n_j \alpha_j = 0 \implies n_j = 0, \ \forall j$$

Proposition

If X_α is irreducible, then:

1. $\dim \mathcal{D}'(X_\alpha) = 1$, i.e. X_α is **DUE**
2. $X_\alpha \colon C^\infty(\mathbb{T}^d) \to C^\infty(\mathbb{T}^d)$ is **closed** iff $(\alpha_1, \ldots, \alpha_d)$ Diophantine

An (important) example

- $X_\alpha \equiv (\alpha_1, \ldots, \alpha_d) \in \mathbb{R}^d$ constant vector field on $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$
- Irreducibility: $(\alpha_1, \ldots, \alpha_d) \in \mathbb{R}^d$ is **non-resonant** iff $\forall (n_1, \ldots, n_d) \in \mathbb{Z}^d$

$$\sum_{j=1}^{d} n_j \alpha_j = 0 \implies n_j = 0, \ \forall j$$

Proposition

If X_α is irreducible, then:

1. $\dim \mathcal{D}'(X_\alpha) = 1$, i.e. X_α is **DUE**
2. $X_\alpha \colon C^\infty(\mathbb{T}^d) \to C^\infty(\mathbb{T}^d)$ is **closed** iff $(\alpha_1, \ldots, \alpha_d)$ Diophantine

(1) and (2) key properties for **KAM theory**

Cohomological Rigidity

Conjecture (Herman, Katok, late 1970's)

If $X \in \mathscr{X}(M)$ is **cohomologically rigid**, i.e. DUE and closed, then

- $M = \mathbb{T}^d$
- $X = X_\alpha$, with $(\alpha_1, \ldots, \alpha_d)$ Diophantine

Cohomological Rigidity

Conjecture (Herman, Katok, late 1970's)

If $X \in \mathscr{X}(M)$ is **cohomologically rigid**, i.e. DUE and closed, then

- $M = \mathbb{T}^d$
- $X = X_\alpha$, with $(\alpha_1, \ldots, \alpha_d)$ Diophantine

Theorem (F. & J. Rodríguez-Hertz, 2005)

$X \in \mathscr{X}(M)$ cohomologically rigid, then M fibers over $\mathbb{T}^{\beta_1(M)}$.

Cohomological Rigidity

Conjecture (Herman, Katok, late 1970's)

If $X \in \mathscr{X}(M)$ is **cohomologically rigid**, i.e. DUE and closed, then

- $M = \mathbb{T}^d$
- $X = X_\alpha$, with $(\alpha_1, \ldots, \alpha_d)$ Diophantine

Theorem (F. & J. Rodríguez-Hertz, 2005)

$X \in \mathscr{X}(M)$ cohomologically rigid, then M fibers over $\mathbb{T}^{\beta_1(M)}$.

Theorem (Forni - K____ - Matsumoto, 2007)

Herman-Katok conjecture is true for $d \leqslant 3$.

DUE vector fields - Rigidity

Folkloric Idea

The topology of M “generates” X-invariant distributions

DUE vector fields - Rigidity

Folkloric Idea

The topology of M "generates" X-invariant distributions

Example

$X \in \mathfrak{X}(M)$ DUE $\implies \chi(M) = 0$

DUE vector fields - Rigidity

Folkloric Idea

The topology of M "generates" X-invariant distributions

Example

$X \in \mathfrak{X}(M)$ DUE $\implies \chi(M) = 0$

Conjecture (Forni, 2007)

$X \in \mathfrak{X}(M)$ is DUE, then $M = \mathbb{T}^d$

DUE vector fields - Rigidity

Folkloric Idea

The topology of M "generates" X-invariant distributions

Example

$X \in \mathfrak{X}(M)$ DUE $\implies \chi(M) = 0$

Conjecture (Forni, 2007)

$X \in \mathfrak{X}(M)$ is DUE, then $M = \mathbb{T}^d$

Main Theorem (Avila, Fayad, K____, 2011)

Let $P := G/H$ be homogeneous space of following kind:

1. G **nilpotent** simply connected Lie group, $H < G$ uniform lattice;
2. G **compact** Lie group, $H < G$ any closed subgroup.

Then $M := \mathbb{T}^2 \times P$ admits DUE vector fields.

Proof Main Theorem

Proof Main Theorem

Main difficulty: (Flaminio, Forni, F. Rodríguez-Hertz)

There is no new homogeneous DUE vector field

Proof Main Theorem

Main difficulty: (Flaminio, Forni, F. Rodríguez-Hertz)
There is no new homogeneous DUE vector field

- $\Phi_X : (\mathbb{T}^2 \times P) \times \mathbb{R} \to \mathbb{T}^2 \times P$ as **suspension of a diffeomorphism** $f : \mathbb{T}^1 \times P \to \mathbb{T}^1 \times P$

Proof Main Theorem

Main difficulty: (Flaminio, Forni, F. Rodríguez-Hertz)
There is no new homogeneous DUE vector field

- $\Phi_X : (\mathbb{T}^2 \times P) \times \mathbb{R} \to \mathbb{T}^2 \times P$ as **suspension of a diffeomorphism** $f : \mathbb{T}^1 \times P \to \mathbb{T}^1 \times P$
- **Homogeneous skew-products:** $\alpha \in \mathbb{T}$, $\gamma \in C^\infty(\mathbb{T}, G)$, define

$$T_{\alpha,\gamma} : \mathbb{T} \times G/H \ni (t, gH) \mapsto (t + \alpha, \gamma(t)gH)$$

Proof Main Theorem

Main difficulty: (Flaminio, Forni, F. Rodríguez-Hertz)
There is no new homogeneous DUE vector field

- $\Phi_X : (\mathbb{T}^2 \times P) \times \mathbb{R} \to \mathbb{T}^2 \times P$ as **suspension of a diffeomorphism** $f : \mathbb{T}^1 \times P \to \mathbb{T}^1 \times P$
- **Homogeneous skew-products:** $\alpha \in \mathbb{T}$, $\gamma \in C^\infty(\mathbb{T}, G)$, define

$$T_{\alpha,\gamma} : \mathbb{T} \times G/H \ni (t, gH) \mapsto (t + \alpha, \gamma(t) gH)$$

- **Anosov-Katok space:**

$$\overline{\mathcal{O}} := \overline{\left\{ T_{0,\gamma} \circ T_{\alpha,e_G} \circ T_{0,\gamma}^{-1} : \alpha \in \mathbb{T},\ \gamma \in C^\infty(\mathbb{T}, G) \right\}}^{C^\infty}$$

Proof Main Theorem

Main difficulty: (Flaminio, Forni, F. Rodríguez-Hertz)
There is no new homogeneous DUE vector field

- $\Phi_X : (\mathbb{T}^2 \times P) \times \mathbb{R} \to \mathbb{T}^2 \times P$ as **suspension of a diffeomorphism** $f : \mathbb{T}^1 \times P \to \mathbb{T}^1 \times P$
- **Homogeneous skew-products:** $\alpha \in \mathbb{T}$, $\gamma \in C^\infty(\mathbb{T}, G)$, define

$$T_{\alpha,\gamma} : \mathbb{T} \times G/H \ni (t, gH) \mapsto (t + \alpha, \gamma(t)gH)$$

- **Anosov-Katok space:**

$$\overline{\mathcal{O}} := \overline{\left\{ T_{0,\gamma} \circ T_{\alpha,e_G} \circ T_{0,\gamma}^{-1} : \alpha \in \mathbb{T},\ \gamma \in C^\infty(\mathbb{T}, G) \right\}}^{C^\infty}$$

- Theorem: DUE diffeos are generic in $\overline{\mathcal{O}}$.

Main "Dynamical" Lemma

$\exists$ **"finite dim filtration"** (E_k) of subspaces of $C^\infty_\mu(\mathbb{T} \times G/H)\}$ satisfying: $\forall n \in \mathbb{N}$ and $\forall q_0 \in \mathbb{N}$, $\exists q \in \mathbb{N}$ and $\exists \gamma \colon \mathbb{T} \to G$ such that:

1. γ is $1/q_0$-periodic,
2. $\forall p \in \mathbb{N}$, coprime with q,

$$E_n \subset B\left(T_{0,\gamma} \circ T_{p/q,e_G} \circ T_{0,\gamma}^{-1}, C^\infty\right),$$

Main "Dynamical" Lemma

$\exists$ **"finite dim filtration"** (E_k) of subspaces of $C^\infty_\mu(\mathbb{T} \times G/H)\}$ satisfying:
$\forall n \in \mathbb{N}$ and $\forall q_0 \in \mathbb{N}$, $\exists q \in \mathbb{N}$ and $\exists \gamma \colon \mathbb{T} \to G$ such that:

1. γ is $1/q_0$-periodic,
2. $\forall p \in \mathbb{N}$, coprime with q,

$$E_n \subset B\left(T_{0,\gamma} \circ T_{p/q,e_G} \circ T_{0,\gamma}^{-1}, C^\infty\right),$$

i.e.

$$\sum_{j=0}^{q-1} \phi \circ \left(T_{0,\gamma} \circ T_{p/q,e_G} \circ T_{0,\gamma}^{-1}\right)^j \equiv 0, \quad \forall \phi \in E_n$$

Nilmanifold case: pseudo-polynomials

- Any C^∞-function $\phi\colon \mathbb{T} \times (G/H) \to \mathbb{R}$ can be written

$$\phi(t, gH) = \sum_{k\in\mathbb{Z}} \hat{\phi}_k^{(0)}(gH) e^{2\pi i k t}$$

Nilmanifold case: pseudo-polynomials

- Any C^∞-function $\phi\colon \mathbb{T} \times (G/H) \to \mathbb{R}$ can be written

$$\phi(t, gH) = \sum_{k\in\mathbb{Z}} \hat{\phi}_k^{(0)}(gH)e^{2\pi ikt}$$

- $\hat{\phi}_0^{(0)} \in C^\infty(G/H, \mathbb{C})$ can be lifted to its universal cover $\hat{\phi}_0^{(0)}\colon \mathfrak{g} \to \mathbb{C}$

Nilmanifold case: pseudo-polynomials

- Any C^∞-function $\phi\colon \mathbb{T}\times(G/H)\to\mathbb{R}$ can be written

$$\phi(t,gH)=\sum_{k\in\mathbb{Z}}\hat{\phi}_k^{(0)}(gH)e^{2\pi ikt}$$

- $\hat{\phi}_0^{(0)}\in C^\infty(G/H,\mathbb{C})$ can be lifted to its universal cover $\hat{\phi}_0^{(0)}\colon\mathfrak{g}\to\mathbb{C}$
- Mal'čev: $\exists$ basis $\{v_1,\dots,v_d\}$ of $\mathfrak{g}$ s.t. $\operatorname{span}\{v_1,\dots,v_i\}$ is ideal and

$$H=\exp(\mathbb{Z}v_1)\exp(\mathbb{Z}v_2)\dots\exp(\mathbb{Z}v_d).$$

Nilmanifold case: pseudo-polynomials

- Any C^∞-function $\phi\colon \mathbb{T} \times (G/H) \to \mathbb{R}$ can be written

$$\phi(t, gH) = \sum_{k\in\mathbb{Z}} \hat{\phi}_k^{(0)}(gH)e^{2\pi ikt}$$

- $\hat{\phi}_0^{(0)} \in C^\infty(G/H, \mathbb{C})$ can be lifted to its universal cover $\hat{\phi}_0^{(0)}\colon \mathfrak{g} \to \mathbb{C}$
- Mal'čev: $\exists$ basis $\{v_1, \ldots, v_d\}$ of $\mathfrak{g}$ s.t. $\mathrm{span}\{v_1, \ldots, v_i\}$ is ideal and

$$H = \exp(\mathbb{Z}v_1)\exp(\mathbb{Z}v_2)\ldots\exp(\mathbb{Z}v_d).$$

-

$$\phi_0^{(0)}(x_1, \ldots, x_d) = \sum_{k\in\mathbb{Z}} \phi_k^{(1)}(x_2, \ldots, x_d)e^{2\pi ikx_1}$$

Nilmanifold case: pseudo-polynomials

- Any C^∞-function $\phi\colon \mathbb{T} \times (G/H) \to \mathbb{R}$ can be written

$$\phi(t, gH) = \sum_{k\in\mathbb{Z}} \hat{\phi}_k^{(0)}(gH) e^{2\pi i k t}$$

- $\hat{\phi}_0^{(0)} \in C^\infty(G/H, \mathbb{C})$ can be lifted to its universal cover $\hat{\phi}_0^{(0)}\colon \mathfrak{g} \to \mathbb{C}$
- Mal'čev: $\exists$ basis $\{v_1, \ldots, v_d\}$ of $\mathfrak{g}$ s.t. $\operatorname{span}\{v_1, \ldots, v_i\}$ is ideal and

$$H = \exp(\mathbb{Z}v_1)\exp(\mathbb{Z}v_2)\ldots\exp(\mathbb{Z}v_d).$$

-

$$\phi_0^{(0)}(x_1, \ldots, x_d) = \sum_{k\in\mathbb{Z}} \phi_k^{(1)}(x_2, \ldots, x_d) e^{2\pi i k x_1}$$

$$\phi_0^{(1)}(x_2, \ldots, x_d) = \sum_{k\in\mathbb{Z}} \phi_k^{(2)}(x_3, \ldots, x_d) e^{2\pi i k x_2}$$

$$\ldots$$

Compact case

- $L^2_0(G/H,\mu) = \bigoplus V_n$ decomposition irreducible G-subrepresentations

Compact case

- $L^2_0(G/H,\mu) = \bigoplus V_n$ decomposition irreducible G-subrepresentations
- Peter-Weyl: Each $V_n \subset C^\infty(G/H)$ and $\dim V_n < \infty$

Compact case

- $L^2_0(G/H,\mu) = \bigoplus V_n$ decomposition irreducible G-subrepresentations
- Peter-Weyl: Each $V_n \subset C^\infty(G/H)$ and $\dim V_n < \infty$
- $E_n := \bigoplus_{k\leqslant n} V_n$ "finite dim filtration" of $C^\infty_0(G/H,\mu)$

Compact case

- $L_0^2(G/H,\mu) = \bigoplus V_n$ decomposition irreducible G-subrepresentations
- Peter-Weyl: Each $V_n \subset C^\infty(G/H)$ and $\dim V_n < \infty$
- $E_n := \bigoplus_{k \leqslant n} V_n$ "finite dim filtration" of $C_0^\infty(G/H,\mu)$

Theorem [E-equidistributed loops]

$\forall E \subset C_0^\infty(M,\mu)$ finite dimensional subspace, $\exists \gamma \in C^\infty(\mathbb{T}, M)$ and $\exists m \in \mathbb{N}$ s.t.

$$\sum_{j=1}^{m} \phi\left(\gamma\left(t + \frac{1}{m}\right)\right) = 0, \quad \forall t \in \mathbb{T},\ \forall \phi \in E$$

Gracias!