Topology of compact solvmanifolds

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   The oscillator group
$M = G/\Gamma$: solvmanifold

$G$: real simply connected solvable Lie group

$\Gamma$: lattice (discrete cocompact subgroup)

**Aims:** find lattices, compute de Rham cohomology, study existence of symplectic structures, hard Lefschetz property, formality

We will explain:

- the difference with nilmanifolds
- known results on the computation of Rham cohomology in special cases (completely solvable, Mostow condition)
- a method to compute the de Rham cohomology in general (following results by Guan and Witte)
- applications: Nakamura manifold, almost abelian solvable Lie groups, hyperelliptic surface, three families of lattices on the oscillator group
Nilpotent and solvable Lie groups

$G$ is $k$-step nilpotent $\iff$ the descending chain of normal subgroups

$$G_0 = G \supset G_1 = [G, G] \supset \cdots \supset G_{i+1} = [G_i, G] \supset \cdots$$

degenerates, i.e. $G_i = \{e\} \ \forall i \geq k$, (e is the identity element).

$G$ is $k$-step solvable $\iff$ the derived series of normal subgroups

$$G^{(0)} = G \supset G^{(1)} = [G, G] \supset \cdots \supset G^{(i+1)} = [G^{(i)}, G^{(i)}] \supset \cdots$$

degenerates.

In particular a solvable Lie group is completely solvable if every eigenvalue $\lambda$ of every operator $\text{Ad}_g, \ g \in G$, is real.

Note that a nilpotent Lie group is completely solvable.
Nilmanifolds

$G$ simply connected nilpotent Lie group

Recall: $\exp : \mathfrak{g} \rightarrow G$ is a diffeomorphism

For nilpotent Lie groups there is a simple criterion for the existence of lattices:

**Theorem (Malčev)**

$G$ simply connected nilpotent Lie group

$\exists \Gamma$ lattice on $G$ $\iff$ the Lie algebra $\mathfrak{g}$ of $G$ has a basis such that the structure constants in this basis are rational $\iff \exists \mathfrak{g}_Q$ such that $\mathfrak{g} = \mathfrak{g}_Q \otimes \mathbb{R}$

If $\mathfrak{g} = \mathfrak{g}_Q \otimes \mathbb{R}$, one also says that $\mathfrak{g}$ has a rational structure

A nilmanifold is the a quotient $M = G/\Gamma$, where $G$ is a real simply connected nilpotent Lie group and $\Gamma$ is a lattice.
Solvmanifolds

There is no simple criterion for the existence of a lattice in a connected and simply-connected solvable Lie group $G$.

Here are some necessary criteria.

**Proposition (Milnor)**

If $G$ admits a lattice then it is unimodular. \[ \operatorname{tr} \operatorname{ad}_X = 0, \quad \forall X \in g \]

**The Mostow bundle**

Let $G/\Gamma$ be a solvmanifold that is not a nilmanifold. $\mathcal{N} = \text{nilradical}$ of $G$ = largest connected nilpotent normal subgroup of $G$. Then $\Gamma_{\mathcal{N}} := \Gamma \cap \mathcal{N}$ is a lattice in $\mathcal{N}$, $\Gamma \mathcal{N} = \mathcal{N} \Gamma$ is closed in $G$ and $G/(\mathcal{N}\Gamma) =: \mathbb{T}^k$ is a torus.

\[ \Rightarrow \text{ we have the fibration:} \]

\[ N/\Gamma_{\mathcal{N}} = (\mathcal{N}\Gamma)/\Gamma \hookrightarrow G/\Gamma \twoheadrightarrow G/(\mathcal{N}\Gamma) = \mathbb{T}^k \]

Much of the rich structure of solvmanifolds is encoded in this bundle. The nilradical has an important rôle in the study of solvmanifolds.
Solvmanifolds

$G$ connected and simply connected solvable Lie group

$$\implies G \overset{\text{diffeo}}{\cong} \mathbb{R}^n$$

(*BUT* $\exp : \mathfrak{g} \to G$ is not necessarily injective or surjective. *)

$\implies$ solvmanifolds $G/\Gamma$ are aspherical and $\pi_1(G/\Gamma) \cong \Gamma$.

The fundamental group plays an important rôle:

**Diffeomorphism Theorem**

$G_1/\Gamma_1$ and $G_2/\Gamma_2$ solvmanifolds and

$\varphi : \Gamma_1 \to \Gamma_2$ isomorphism.

$\implies \exists$ diffeomorphism $\Phi : G_1 \to G_2$ such that

(i) $\Phi|_{\Gamma_1} = \varphi$,

(ii) $\forall \gamma \in \Gamma_1 \forall \rho \in G_1 \quad \Phi(\rho \gamma) = \Phi(\rho)\varphi(\gamma)$.

**Corollary**

*Two solvmanifolds with isomorphic $\pi_1$ are diffeomorphic.*
G/Γ solvmanifold (nilmanifold)

General question: can one compute Dolbeault cohomology of $M$ by invariant forms, i.e., using the Chevalley-Eilenberg complex:

$$\ldots \rightarrow \Lambda^{k-1} g^* \xrightarrow{d} \Lambda^k g^* \xrightarrow{d} \Lambda^{k+1} g^* \rightarrow \ldots$$

$$d\alpha(x_1, \ldots, x_{k+1}) = \sum_{i<j} (-1)^{i+j} \alpha([x_i, x_j], x_1, \ldots, \hat{x_i}, \ldots, \hat{x_j}, \ldots, x_{k+1})$$

So, when is $H^*_{dR}(G/\Gamma) \cong H^*(g)$?
de Rham Cohomology of nilmanifolds

\( G/\Gamma \) nilmanifold

**Theorem (Nomizu)**

\[ H^*_{dR}(G/\Gamma) \cong H^*(g). \]

\( \wedge^* g \) is a minimal model of \( G/\Gamma \) (in the sense of Sullivan).

\( \wedge^* g \) is formal \( \iff \) \( G \) is abelian and \( G/\Gamma \) is a torus.

If a nilmanifold is Kählerian, then it is a torus.
[Benson & Gordon, Hasegawa]
Idea of the proof of Nomizu’s Theorem

Suppose $G$ is $k$-step nilpotent. $\mathfrak{g}_\ell$ Lie algebra of $G_\ell$.

Let $H :=$ simply-connected Lie subgroup of $G$ with Lie algebra $\mathfrak{h} = \mathfrak{g}_{k-1} \implies H$ central and $H \cong \mathbb{R}^n$.

We have the fibration:

$$
\mathbb{T}^n = H/H \cap \Gamma \hookrightarrow M = G/\Gamma \xrightarrow{\pi} \overline{M} = G/H\Gamma
$$

$E_{\ast}^{p,q} :=$ Leray-Serre spectral sequence associated with the fibration:

$$
E_2^{p,q} = H_{dR}^p(\overline{M}, H_{dR}^q(\mathbb{T}^n)) \cong H_{dR}^p(\overline{M}) \otimes \bigwedge^q \mathbb{R}^n,
$$

$$
E_\infty^{p,q} \Rightarrow H_{dR}^{p+q}(M).
$$
Main idea: construct a second spectral sequence
\( \tilde{E}^{p,q} \) = Leray-Serre spectral sequence for the complex of
\( G \)-invariant forms \( \bigwedge^* g^* \)
\( \bigwedge^* g^* \) subcomplex of \( \bigwedge^* M \implies \tilde{E}^{p,q} \subseteq E^{p,q}_* \&
\( \tilde{E}^{p,q}_2 = H^p(g/h) \otimes \bigwedge^q \mathbb{R}^n, \)
\( \tilde{E}^{p,q}_\infty \Rightarrow H^{p+q}(g). \)
\( \dim \overline{M} < \dim M, \) induction on \( \dim \)
\( \implies E_2 = \tilde{E}_2 \& E_\infty = \tilde{E}_\infty. \)
i.e., \( H^\ell_{dR}(M) \cong H^\ell(g) \) for any \( \ell. \)
de Rham Cohomology of solvmanifolds

\( G/\Gamma \) solvmanifold

\( \text{Ad}_G(G) = \{ e^{\text{ad}X} \mid X \in \mathfrak{g} \} \) solvable and \( \text{Aut}(G) \cong \text{Aut}(\mathfrak{g}) \).

\( \mathcal{A}(\text{Ad}_G(G)) \) and \( \mathcal{A}(\text{Ad}_G(\Gamma)) \): real algebraic closures of \( \text{Ad}_G(G) \) and \( \text{Ad}_G(\Gamma) \) (respectively)

**Theorem (Borel density theorem)**

*Let \( \Gamma \) be a lattice of a simply connected solvable Lie group \( G \), \( \implies \exists \) a maximal compact torus \( \mathbb{T}_{\text{cpt}} \subset \mathcal{A}(\text{Ad}_G(G)) \), such that*

\[ \mathcal{A}(\text{Ad}_G(G)) = \mathbb{T}_{\text{cpt}} \mathcal{A}(\text{Ad}_G(\Gamma)). \]
When is the de Rham cohomology of a solvmanifold given by the Chevalley-Eilenberg complex?

There are 2 important cases:

1. **Hattori**: If $G$ is completely solvable, i.e., if the linear map $\text{ad}_X : g \to g$ has only real eigenvalues

2. **Mostow condition**: If $\mathcal{A}(\text{Ad}_G(G)) = \mathcal{A}(\text{Ad}_G(\Gamma))$

**Remarks**

- (1) is a particular case of (2).
  Indeed if all eigenvalues of $\text{Ad}$ are real, then $\mathcal{A}(\text{Ad}_G(G))$ has no nontrivial conn. compact subgroups $\implies \mathcal{A}(\text{Ad}_G(G)) = \mathcal{A}(\text{Ad}_G(\Gamma))$

- Recall that a nilpotent Lie group is completely solvable $\implies$ (1) and (2) generalize Nomizu’s Theorem.

- We will see that one can have the isomorphism $H^*(g) \cong H^*_{dR}(G/\Gamma)$ even if $\mathcal{A}(\text{Ad}_G(\Gamma)) \neq \mathcal{A}(\text{Ad}_G(G))$
  (Example on hyperelliptic surface)
Idea of the proof of (2)

We prove that if the Mostow condition holds, we still have a fibration of $M = G/\Gamma$ over a smaller dimensional solvmanifold with a torus as fibre. Then one can proceed as in the proof of Nomizu’s Theorem.

$$G(k) = [G(k-1), G(k-1)] \& \Gamma(k) = [\Gamma(k-1), \Gamma(k-1)]: \text{derived series}$$

Remark that $G(k)$ is nilpotent for any $k \geq 1$

Mostow condition + a gen. result on lattices in nilpotent Lie groups

$$\implies G(k)/\Gamma(k) \text{ is compact for any } k.$$  

$$\implies G(k)/\Gamma \cap G(k) \text{ is compact for any } k.$$  

Let $r$ be the last non-zero term in the derived series of $G$.

Namely $G(r+1) = (e)$ and $G(r) =: A \neq (e)$.

$A$ is abelian $\implies A/A \cap \Gamma := \mathbb{T}^m$ is a compact torus.

Thus, $\overline{M} := G/A\Gamma$ is a compact solvmanifold with dimension smaller than $M := G/\Gamma$ and $\mathbb{T}^m \hookrightarrow M \xrightarrow{\pi} \overline{M}$ is a fibration.
If $\mathcal{A}(\text{Ad}_G(\Gamma)) \neq \mathcal{A}(\text{Ad}_G(\Gamma))$ it is more difficult to compute the de Rham cohomology.

We explain a method deriving from results of Guan and Witte

**Main Theorem [ – , A. Fino]**

Let $M = G/\Gamma$ be a compact solvmanifold and let $\mathbb{T}_{cpt}$ be a compact torus such that

$$\mathbb{T}_{cpt} \mathcal{A}(\text{Ad}_G(\Gamma)) = \mathcal{A}(\text{Ad}_G(G)).$$

Then there exists a subgroup $\tilde{\Gamma}$ of finite index in $\Gamma$ and a simply connected normal subgroup $\tilde{G}$ of $\mathbb{T}_{cpt} \ltimes G$ such that

$$\mathcal{A}(\text{Ad}_{\tilde{G}}(\tilde{\Gamma})) = \mathcal{A}(\text{Ad}_{\tilde{G}}(\tilde{G})).$$

$\implies \tilde{G}/\tilde{\Gamma}$ is diffeomorphic to $G/\tilde{\Gamma}$ and $H^*_dR(G/\tilde{\Gamma}) \cong H^*(\tilde{g}).$

Observe that $H^*_dR(G/\Gamma) \cong H^*_dR(G/\tilde{\Gamma})^{\Gamma/\tilde{\Gamma}}$

(the invariants by the action of the finite group $\Gamma/\tilde{\Gamma}$).
**Proof of the Main Theorem**

It is not restrictive to suppose that $\mathcal{A}(\text{Ad}_G(\Gamma))$ is connected. Otherwise we pass from $\Gamma$ to a finite index subgroup $\tilde{\Gamma}$ [equivalently $M = G/\Gamma \cong G/\tilde{\Gamma}$ finite-sheeted covering of $M$]

Let $T_{cpt}$ be a maximal compact torus of $\mathcal{A}(\text{Ad}_G G)$ which contains a maximal compact torus $\overline{S}_{cpt}$ of $\mathcal{A}(\text{Ad}_G(\tilde{\Gamma}))$.

Let $S_{cpt}$ be a subtorus of $T_{cpt}$ complementary to $\overline{S}_{cpt}$ so that $T_{cpt} = S_{cpt} \times \overline{S}_{cpt}$.

Let $\sigma$ be the composition of the homomorphisms:

$$\sigma : G \xrightarrow{\text{Ad}} \mathcal{A}(\text{Ad}_G G) \xrightarrow{\text{proj}} T_{cpt} \xrightarrow{\text{proj}} S_{cpt} \xrightarrow{x \mapsto x^{-1}} S_{cpt}.$$ 

*The point now is to get rid of $S_{cpt}$*
-nilshadow map:
\[ \Delta : G \to \mathbb{S}_{cpt} \ltimes G, \quad g \mapsto (\sigma(g), g), \]

[not a homomorphism (unless \( \mathbb{S}_{cpt} = \{0\} \) \( \implies \) \( \sigma = 0 \)]

One has:
\[ \Delta(ab) = \Delta(\sigma(b^{-1})a\sigma(b)) \Delta(b), \quad \forall a, b \in G \]

and \( \Delta(\gamma g) = \gamma \Delta(g) \), for every \( \gamma \in \tilde{\Gamma}, g \in G \).

\( \Delta \) is a diffeomorphism onto its image \( \implies \) \( \Delta(G) \) is simply connected.

The product in \( \Delta(G) \) is given by:
\[ \Delta(a)\Delta(b) = (\sigma(a), a) (\sigma(b), b) = (\sigma(a)\sigma(b), \sigma(b^{-1})a\sigma(b) b), \]

for any \( a, b \in G \).

By construction, \( \mathcal{A}(\text{Ad}_G(\tilde{\Gamma})) \) projects trivially on \( \mathbb{S}_{cpt} \) and \( \sigma(\tilde{\Gamma}) = \{e\} \). \( \implies \)
\[ \tilde{\Gamma} = \Delta(\tilde{\Gamma}) \subset \Delta(G). \]
Let $\tilde{G} = \Delta(G)$.

[Witte]: $\mathbb{S}_{cpt}$ is a maximal compact subgroup of $\mathcal{A}(\text{Ad} \tilde{G} (\tilde{G}))$ and $\mathbb{S}_{cpt} \subset \mathcal{A}(\text{Ad} \tilde{G} (\tilde{G})) \implies \mathcal{A}(\text{Ad} \tilde{G} (\tilde{G})) = \mathcal{A}(\text{Ad} \tilde{G} (\tilde{G}))$

Diffeomorphism Theorem $G/\tilde{\Gamma}$ is diffeomorphic to $\tilde{G}/\tilde{\Gamma}$.

Mostow condition holds $\implies H^*(G/\tilde{\Gamma}) \cong H^*(\tilde{g})$.

By the diffeomorphism $\Delta : G \rightarrow \tilde{G}$, $\Delta^{-1}$ induces a finite sheeted covering map $\Delta^* : \tilde{G}/\tilde{\Gamma} \rightarrow G/\Gamma$.

**Corollary**

The Lie algebra $\tilde{g}$ of $\tilde{G}$ can be identified by

$$\tilde{g} = \{(X_5, X) \mid X \in g\}$$

with Lie bracket:

$$[(X_5, X), (Y_5, Y)] = (0, [X, Y] - \text{ad}(X_5)(Y) + \text{ad}(Y_5)(X)).$$

where $X_5$ the image $\sigma_*(X)$, for $X \in g$
Applications

Now we obtain some applications of the Main Theorem, by computing explicitly the Lie group $\tilde{G}$.

Example (Nakamura manifold – description)

Consider the simply connected complex solvable Lie group $G$:

$$G = \left\{ \begin{pmatrix} e^z & 0 & 0 & w_1 \\ 0 & e^{-z} & 0 & w_2 \\ 0 & 0 & 1 & z \\ 0 & 0 & 0 & 1 \end{pmatrix}, \ w_1, w_2, z \in \mathbb{C} \right\}.$$

$G \cong \mathbb{C} \ltimes \varphi \mathbb{C}^2$, where

$$\varphi(z) = \begin{pmatrix} e^z & 0 \\ 0 & e^{-z} \end{pmatrix}.$$

Let

$$L_{1,2\pi} = \mathbb{Z}[t_0, 2\pi i] = \{ t_0 k + 2\pi h i, \ h, k \in \mathbb{Z} \},$$

$$L_2 = \left\{ P \begin{pmatrix} \mu \\ \alpha \end{pmatrix}, \mu, \alpha \in \mathbb{Z}[i] \right\}.$$

Then, by Yamada $\Gamma = L_{1,2\pi} \ltimes \varphi L_2$ is a lattice of $G$.

$G/\Gamma$: Nakamura manifold
Example (Nakamura manifold – computation of cohomology)

$G$ has trivial center $\implies \text{Ad}_G(G) \cong G \cong \mathbb{R}^2 \ltimes \mathbb{R}^4$. Moreover,

\[
\mathcal{A} (\text{Ad}_G G) = (\mathbb{R}^\# \times S^1) \ltimes \mathbb{R}^4, \\
\mathcal{A} (\text{Ad}_G \Gamma) = \mathbb{R}^\# \times \mathbb{R}^4,
\]

where the split torus $\mathbb{R}^\#$ corresponds to the action of $e^{\frac{1}{2}(z+\bar{z})}$ and the compact torus $S^1$ to the one of $e^{\frac{1}{2}(z-\bar{z})}$.

$\implies \mathcal{A} (\text{Ad}_G G) = S^1 \mathcal{A} (\text{Ad}_G \Gamma)$ and $\mathcal{A} (\text{Ad}_G \Gamma)$ is connected.

Main Theorem $\implies \exists$ a simply connected normal subgroup $\tilde{G} = \Delta(G)$ of $S^1 \ltimes G$.

The new Lie group $\tilde{G}$ is obtained by killing the action of $e^{\frac{1}{2}(z-\bar{z})}$:

\[
\tilde{G} \cong \begin{cases} 
  e^{\frac{1}{2}(z+\bar{z})} & 0 & 0 & w_1 \\
  0 & e^{-\frac{1}{2}(z+\bar{z})} & 0 & w_2 \\
  0 & 0 & 1 & z \\
  0 & 0 & 0 & 1 
\end{cases}, \ w_1, w_2, z \in \mathbb{C}
\]

$G/\Gamma \text{ diffeo } \tilde{G}/\Gamma$ was already shown by Yamada. $\implies$

$H^*_d(G/\Gamma) \cong H^*(\tilde{g})$, ($\tilde{g}$ Lie algebra of $\tilde{G}$) and $H^*_d(G/\Gamma) \not\cong H^*(g)$.
Example (A three dimensional example)

\( G = \mathbb{R} \ltimes \mathbb{R}^2 \) with structure equations
\[
\left\{
\begin{array}{l}
    de^1 = 0, \\
    de^2 = 2\pi e^1 \wedge e^3 \\
    de^3 = -2\pi e^1 \wedge e^2
\end{array}
\right.
\]
is non-completely solvable and admits a lattice \( \Gamma = \mathbb{Z} \ltimes \mathbb{Z}^2 \).

Indeed,
\[
\mathbb{R} \ltimes \mathbb{R}^2 = \left\{ \begin{pmatrix}
    \cos(2\pi t) & \sin(2\pi t) & 0 & x \\
    -\sin(2\pi t) & \cos(2\pi t) & 0 & y \\
    0 & 0 & 1 & t \\
    0 & 0 & 0 & 1
\end{pmatrix} \right\}
\]
and \( \Gamma \) is generated by 1 in \( \mathbb{R} \) and the standard lattice \( \mathbb{Z}^2 \).

\( \mathcal{A}(\text{Ad}_G(G)) = S^1 \ltimes \mathbb{R}^2 \) and \( \mathcal{A}(\text{Ad}_G(\Gamma)) = \mathbb{R}^2 \)

Main Theorem

\( \tilde{G} \cong \mathbb{R}^3 \subset S^1 \ltimes G \).

Indeed, it is well known that \( G/\Gamma \) is diffeomorphic to a torus.
The previous example $\mathbb{R} \ltimes \mathbb{R}^2$ is an *almost abelian* Lie group:

A Lie algebra $\mathfrak{g}$ is called almost abelian if it has an abelian ideal of codimension 1, i.e., $\mathfrak{g} \cong \mathbb{R} \ltimes \mathfrak{b}$, where $\mathfrak{b} \cong \mathbb{R}^n$ is an abelian ideal of $\mathfrak{g}$.

In this case the Mostow bundle is a torus bundle over $S^1$.

The action $\varphi$ of $\mathbb{R}$ on $\mathbb{R}^n$ is represented by a family of matrices $\varphi(t)$, which encode the monodromy or “twist” in the Mostow bundle.

A nice feature of almost abelian solvable groups is that there is a criterion on the existence of a lattice.

**Proposition (Bock)**

Let $G = \mathbb{R} \ltimes_\varphi \mathbb{R}^n$ be almost abelian solvable Lie group. Then $G$ admits a lattice if and only if there exists a $t_0 \neq 0$ for which $\varphi(t_0)$ can be conjugated to an integer matrix.
The Lie algebra $\mathfrak{g}$ of $G$ has form

$$\mathbb{R} \ltimes \text{ad}_{X_{n+1}} \mathbb{R}^n,$$

where we consider $\mathbb{R}^n$ generated by $\{X_1, \ldots, X_n\}$ and $\mathbb{R}$ by $X_{n+1}$, and $\varphi(t) = e^{t \text{ad}_{X_{n+1}}}$. Moreover, a lattice can be always represented as $\Gamma = \mathbb{Z} \ltimes \mathbb{Z}^n$.

For almost abelian solvmanifolds, Gorbatsevich found a criterion to decide whether the Mostow condition holds:

**Proposition (Gorbatsevich)**

The Mostow condition is satisfied if and only if $\pi i$ can not be written as linear combination in $\mathbb{Q}$ of the eigenvalues of $t_0 \text{ad}_{X_{n+1}}$, where $\Gamma$ is generated by $t_0$. 
• If $i\pi$ is not representable as a $\mathbb{Q}$-linear combination of the numbers $\lambda_k$, \[ H^*_{dR}(G/\Gamma) \cong H^*(\mathfrak{g}). \]

• Otherwise the only known result on cohomology [Bock]

\[ b_1(G/\Gamma) = n + 1 - \text{rank}(\varphi(1) - \text{id}). \]

By applying the Main Theorem one obtains a method to compute the de Rham cohomology of $G/\Gamma$.

[ – , M. Macrì] construct lattices on six dimensional not completely solvable almost abelian Lie groups, for which the Mostow condition does not hold. We compute

• cohomology (does not agree with the one of $\mathfrak{g}$)

• minimal model

• show that some of these solvmanifolds admit not invariant symplectic structures and we study formality and Lefschetz properties
### Example (6-dimensional indecomposable almost abelian solvmanifolds not satisfying the Mostow condition)

<table>
<thead>
<tr>
<th>$G$</th>
<th>$\tilde{t}$</th>
<th>$H^*(\mathfrak{g})$</th>
<th>$H^*(G/\Gamma_{\tilde{t}})$</th>
<th>F</th>
<th>IS</th>
<th>S</th>
<th>HL</th>
</tr>
</thead>
<tbody>
<tr>
<td>$G_{6.8}^{p=0}$</td>
<td>$\tilde{t} = 2\pi$</td>
<td>$b_1 = 1, b_2 = 1, b_3 = 2$</td>
<td>$b_1 = 3, b_2 = 3, b_3 = 2$</td>
<td>Yes</td>
<td>No</td>
<td>No</td>
<td>\</td>
</tr>
<tr>
<td></td>
<td>$\tilde{t} = \pi, \frac{\pi}{2}, \frac{\pi}{3}$</td>
<td></td>
<td>$b_1 = 1, b_2 = 1, b_3 = 2$</td>
<td>Yes</td>
<td>No</td>
<td>\</td>
<td>\</td>
</tr>
<tr>
<td>$G_{6.10}^{a=0}$</td>
<td>$\tilde{t} = 2\pi$</td>
<td>$b_1 = 2, b_2 = 3, b_3 = 4$</td>
<td>$b_1 = 4, b_2 = 7, b_3 = 8$</td>
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<td>Yes</td>
<td>Yes</td>
<td>No*</td>
</tr>
<tr>
<td></td>
<td>$\tilde{t} = \pi, \frac{\pi}{2}, \frac{\pi}{3}$</td>
<td></td>
<td>$b_1 = 2, b_2 = 3, b_3 = 4$</td>
<td>No</td>
<td>Yes</td>
<td>\</td>
<td>No*</td>
</tr>
<tr>
<td>$G_{6.11}^{p=0}$</td>
<td>$\tilde{t} = 2\pi$</td>
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<td>$b_1 = 3, b_2 = 4, b_3 = 4$</td>
<td>Yes</td>
<td>No</td>
<td>No</td>
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<tr>
<td></td>
<td>$\tilde{t} = \pi, \frac{\pi}{2}, \frac{\pi}{3}$</td>
<td></td>
<td>$b_1 = 1, b_2 = 1, b_3 = 1$</td>
<td>Yes</td>
<td>No</td>
<td>\</td>
<td>\</td>
</tr>
</tbody>
</table>

$\times$ = for both the invariant and not invariant symplectic structures considered.

$\ast$ = for the invariant symplectic structures.

**F:** formality

**IS:** existence of invariant symplectic structures

**S:** existence of symplectic structures

*(induced by ones on the modified Lie alg)*

**HL:** Hard Lefschetz property
### Example (6-dimensional decomposable almost abelian solvmanifolds not satisfying the Mostow condition)

<table>
<thead>
<tr>
<th>$G$</th>
<th>$\Gamma_R$</th>
<th>$H^*(g)$</th>
<th>$H^*(G/\Gamma_R)$</th>
<th>F</th>
<th>IS</th>
<th>S</th>
<th>HL</th>
</tr>
</thead>
<tbody>
<tr>
<td>$G_{5.14}^0 \times \mathbb{R}$</td>
<td>$\bar{t} = 2\pi$</td>
<td>$b_1 = 3, b_2 = 5, b_3 = 6$</td>
<td>$b_1 = 5, b_2 = 11, b_3 = 14$</td>
<td>No</td>
<td>Yes</td>
<td>Yes</td>
<td>No*</td>
</tr>
<tr>
<td></td>
<td>$\bar{t} = \pi, \frac{\pi}{2}, \frac{\pi}{3}$</td>
<td>$b_1 = 3, b_2 = 5, b_3 = 6$</td>
<td></td>
<td>No</td>
<td>Yes</td>
<td>\</td>
<td>No*</td>
</tr>
<tr>
<td>$G_{5.17}^{p-r} \times \mathbb{R}$</td>
<td>$\bar{t} = 2\pi r_2$</td>
<td>if $p \neq 0, r \neq \pm 1$</td>
<td>$p \neq 0: b_1 = 6, b_2 = 15, b_3 = 20$</td>
<td>Yes</td>
<td>Yes</td>
<td>Yes</td>
<td>Yes*</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$b_1 = 2, b_2 = 1, b_3 = 0$</td>
<td>$p = 0: b_1 = 2, b_2 = 5, b_3 = 8$</td>
<td>Yes</td>
<td>Yes</td>
<td>\</td>
<td>Yes*</td>
</tr>
<tr>
<td></td>
<td>$\bar{t} = \pi, r$ even</td>
<td>if $p = 0, r \neq \pm 1$</td>
<td>$p \neq 0: b_1 = 2, b_2 = 1, b_3 = 0$</td>
<td>Yes</td>
<td>Yes</td>
<td>\</td>
<td>Yes*</td>
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<tr>
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<td>$b_1 = 2, b_2 = 5, b_3 = 4$</td>
<td>$p = 0: b_1 = 2, b_2 = 7, b_3 = 8$</td>
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<td>Yes*</td>
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<tr>
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<td>Yes</td>
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<tr>
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<td>$b_1 = 2, b_2 = 5, b_3 = 8$</td>
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<td>Yes</td>
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<td>Yes*</td>
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<tr>
<td></td>
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<tr>
<td>$G_{5.18}^0 \times \mathbb{R}$</td>
<td>$\bar{t} = 2\pi$</td>
<td>$b_1 = 2, b_2 = 3, b_3 = 4$</td>
<td>$b_1 = 4, b_2 = 9, b_3 = 13$</td>
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<td>Yes</td>
<td>No*</td>
</tr>
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<td>$\bar{t} = \pi, \frac{\pi}{2}, \frac{\pi}{3}$</td>
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<td>No</td>
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<td>\</td>
<td>No*</td>
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<tr>
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<td>$\bar{t} = \pi, \frac{\pi}{2}, \frac{\pi}{3}$</td>
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<td>Yes</td>
<td>\</td>
<td>No*</td>
</tr>
<tr>
<td>$G_{3.5}^0 \times \mathbb{R}^3$</td>
<td>$\bar{t} = 2\pi$</td>
<td>$b_1 = 4, b_2 = 7, b_3 = 8$</td>
<td>$b_1 = 6, b_2 = 15, b_3 = 20$</td>
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<td>Yes</td>
<td>Yes</td>
<td>Yes*</td>
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<td></td>
<td>$\bar{t} = \pi, \frac{\pi}{2}, \frac{\pi}{3}$</td>
<td>$b_1 = 4, b_2 = 7, b_3 = 8$</td>
<td></td>
<td>Yes</td>
<td>Yes</td>
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<td>Yes*</td>
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</table>

* = for both the invariant and the not invariant symplectic structures considered.
## Example (6-dimensional decomposable almost abelian solvmanifolds not satisfying the Mostow condition)

<table>
<thead>
<tr>
<th>$G$</th>
<th>$\Gamma_{\ell}$</th>
<th>$H^*(g)$</th>
<th>$H^*(G/\Gamma_{\ell})$</th>
<th>F</th>
<th>IS</th>
<th>S</th>
<th>HL</th>
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<tbody>
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<td>Yes</td>
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<tr>
<td></td>
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<td>$b_1 = 3, b_2 = 5, b_3 = 6$</td>
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<td>No*</td>
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<tr>
<td></td>
<td>$\ell = 2\pi r_2$</td>
<td>if $p \neq 0, r \neq \pm 1$</td>
<td>$p \neq 0: b_1 = 6, b_2 = 15, b_3 = 20$</td>
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<td>Yes</td>
<td>Yes</td>
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</tr>
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<td></td>
<td></td>
<td>$b_1 = 2, b_2 = 1, b_3 = 0$</td>
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<td>Yes</td>
<td>Yes</td>
<td>\</td>
<td>Yes*</td>
</tr>
<tr>
<td></td>
<td>$\ell = \pi$, $r$ even</td>
<td>if $p = 0, r \neq \pm 1$</td>
<td>$p \neq 0: b_1 = 2, b_2 = 1, b_3 = 0$</td>
<td>Yes</td>
<td>Yes</td>
<td>\</td>
<td>Yes*</td>
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<tr>
<td></td>
<td></td>
<td>$b_1 = 2, b_2 = 3, b_3 = 4$</td>
<td>$p = 0: b_1 = 2, b_2 = 7, b_3 = 8$</td>
<td>Yes</td>
<td>Yes</td>
<td>\</td>
<td>Yes*</td>
</tr>
<tr>
<td></td>
<td>$\ell = \pi, r \equiv 1, 3$</td>
<td>if $p = 0, r \equiv \pm 1$</td>
<td>$p \neq 0: b_1 = 2, b_2 = 3, b_3 = 4$</td>
<td>Yes</td>
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<td>Yes*</td>
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<td>Yes*</td>
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<td></td>
<td>$\ell = \pi, r \equiv 4 2$</td>
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<td>$p = 0: b_1 = 2, b_2 = 3, b_3 = 4$</td>
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<td>Yes</td>
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<td>Yes*</td>
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<td>$b_1 = 4, b_2 = 9, b_3 = 13$</td>
<td>No</td>
<td>Yes</td>
<td>Yes</td>
<td>No*</td>
</tr>
<tr>
<td></td>
<td>$\ell = \pi$, $\ell = \pi, \frac{\pi}{2}, \frac{\pi}{3}$</td>
<td></td>
<td></td>
<td>No</td>
<td>Yes</td>
<td>\</td>
<td>No*</td>
</tr>
<tr>
<td>$G_{3.5}^0 \times \mathbb{R}^3$</td>
<td>$\ell = 2\pi$</td>
<td>$b_1 = 4, b_2 = 7, b_3 = 8$</td>
<td>$b_1 = 6, b_2 = 15, b_3 = 20$</td>
<td>Yes</td>
<td>Yes</td>
<td>Yes</td>
<td>Yes*</td>
</tr>
<tr>
<td></td>
<td>$\ell = \pi, \frac{\pi}{2}, \frac{\pi}{3}$</td>
<td>$b_1 = 2, b_2 = 3, b_3 = 4$</td>
<td>$b_1 = 4, b_2 = 7, b_3 = 8$</td>
<td>Yes</td>
<td>Yes</td>
<td>\</td>
<td>Yes*</td>
</tr>
</tbody>
</table>

$\times$ = for both the invariant and the not invariant symplectic structures considered.

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### Topology of compact solvmanifolds

**Sergio Console**

**Aims**

- Nilpotent and solvable
  - Nilmanifolds
  - Solvmanifolds

**de Rham Cohomology**

- Nilmanifolds
- Solvmanifolds

**Main Theorem**

- Proof of the Main Theorem

**Applications**

- Nakamura manifold
- Almost abelian
- 6 dim almost abelian
  - Kähler and symplectic structures on solvmanifolds
  - Hyperelliptic surface
  - The oscillator group
Example (6-dimensional decomposable almost abelian solvmanifolds not satisfying the Mostow condition)

\[ \exists \text{ examples where the cohomology depends strongly on the lattice:} \]

\[ H^*_{dR}(G/\Gamma_\pi) \not\cong H^*_{dR}(G/\Gamma_{2\pi}) \not\cong H^*(g), \quad G = G^0_{5.18} \times \mathbb{R}. \]
Kähler structures on solvmanifolds

**Benson-Gordon conjecture:** a compact solvmanifold has a Kähler structure if and only if it is a complex torus

**Hasegawa (2006):**

*A solvmanifold carries a Kähler metric if and only if it is covered by a finite quotient of a complex torus, which has the structure of a complex torus bundle over a complex torus.*

An example is provided by the hyperelliptic surface

in particular,

*a compact solvmanifold of completely solvable type has a Kähler structure if and only if it is a complex torus*
Half-flat symplectic structures on solvmanifolds

Six dimensional almost abelian solvmanifolds were consider in string backgrounds where the internal compactification manifold is a solvmanifold (see e.g. [Andriot, Goi, Minasian and Petrini]).

They are related to solutions of the supersymmetry (SUSY) equations. By [Fino-Ugarte], solution of the SUSY equations IIA possess a symplectic half-flat structure, whereas solutions of the SUSY equations IIB admit a half-flat structure.

An $SU(3)$ structure on a six-dimensional manifold $M$ (i.e., an $SU(3)$ reduction of the frame bundle of $M$) $\leadsto$

- a non-degenerate 2-form $\Omega$,
- an almost-complex structure $J$,
- a complex volume form $\Psi$.

The $SU(3)$ structure is called half-flat if $\Omega \wedge \Omega$ and the real part of $\Psi$ are closed [Chiossi-Salamon].

If in addition $\Omega$ is closed, the half-flat structure is called symplectic.
**Proposition ( – , M. Macrì)**

We have the following behavior concerning half flatness of (invariant) symplectic structures for the above solvmanifolds:

- $G^{a=0}_{6.10}/\Gamma_{2\pi}$ and $G^0_{5.14} \times \mathbb{R}/\Gamma_{2\pi}$ admit (not) invariant symplectic forms which are not half flat.

- $G^{p,-p,r}_{5.17} \times \mathbb{R}/\Gamma_{2\pi r_2} \ (r = \frac{r_1}{r_2} \in \mathbb{Q})$ admits an invariant symplectic form which is half flat only for $p \geq 0$ and $r = 1$ and it admits a not invariant symplectic form which is half flat.

- $G^0_{5.18} \times \mathbb{R}/\Gamma_{2\pi}$ and $G^0_{3.5} \times \mathbb{R}^3/\Gamma_{2\pi}$ admit (not) invariant symplectic forms which are half flat.
Example (Hyperelliptic surface)

\[ G = \mathbb{R} \ltimes \varphi (\mathbb{C} \times \mathbb{R}), \text{ with } \varphi : \mathbb{R} \to \text{Aut}(\mathbb{C} \times \mathbb{R}) \text{ defined by} \]

\[ \varphi(t)(z, s) = (e^{i\eta t} z, s), \quad \text{where } \eta = \pi, \frac{2}{3}\pi, \frac{1}{2}\pi \text{ or } \frac{1}{3}\pi \]

Hasegawa: \( G \) has 7 isomorphism classes of lattices \( \Gamma = \mathbb{Z} \ltimes \varphi \mathbb{Z}^3 \), where \( \varphi : \mathbb{Z} \to \text{Aut}(\mathbb{Z}^3) \) has matrix \( \varphi(1) \) with eigenvalues \( 1, e^{i\eta}, e^{-i\eta} \).

\( \varphi(1) \) has a pair of cx conj roots \( \Rightarrow \) \( A(\text{Ad}_G(G)) \neq A(\text{Ad}_G(\Gamma)) \).

In this case \( A(\text{Ad}_G(\Gamma)) \) is not connected, but

\( \Gamma \) contains as a finite index subgroup \( \tilde{\Gamma} \cong \mathbb{Z}^4 \)

\[ \Rightarrow \quad G/\Gamma \text{ is a finite covering of a torus} \]

Note: \( H^1_{dR}(G/\Gamma) \cong H^1(\mathfrak{g}) \) even if \( A(\text{Ad}_G(G)) \neq A(\text{Ad}_G(\Gamma)) \)

Indeed, \( G \) has structure equations:

\[
\begin{align*}
    de^1 &= e^2 \wedge e^4 \\
    de^2 &= -e^1 \wedge e^4 \\
    de^3 &= 0 \\
    de^4 &= 0
\end{align*}
\]

and \( H^1(\mathfrak{g}) = \text{span} < e^3, e^4 > \).
Example (Three families of lattices in the oscillator group \[ - , \ G. \ Ovando, \ M. \ Subils \] )

**Oscillator group:** \( G = \mathbb{R} \ltimes_{\alpha} H_3(\mathbb{R}) \)

\( H_3(\mathbb{R}) \) (real) three dimensional Heisenberg group

\[ \alpha : \mathbb{R} \rightarrow \text{Aut}(h_3), \quad t \mapsto \begin{pmatrix} \cos(t) & \sin(t) & 0 \\ -\sin(t) & \cos(t) & 0 \\ 0 & 0 & 1 \end{pmatrix}. \]

The oscillator group is an **almost nilpotent** solvable Lie group

If we regard \( H_3(\mathbb{R}) \) as \( \mathbb{R}^3 \) endowed with the operation

\[ (x, y, z) \cdot (x', y', z') = (x + x', y + y', z + z' + \frac{1}{2}(xy' - x'y)) \]

\( \Rightarrow \) \( H_3(\mathbb{R}) \) admists the co-compact subgroups \( \Gamma_k \subset H_3(\mathbb{R}) \) given by

\[ \Gamma_k = \mathbb{Z} \times \mathbb{Z} \times \frac{1}{2k} \mathbb{Z}. \]
Example (The oscillator group)

The lattice $\Gamma_k$ (for any $k$) is invariant under the subgroups generated by $\alpha(0) = \alpha(2\pi)$, $\alpha(\pi)$ and $\alpha(\frac{\pi}{2}) \implies$

we have three families of lattices in $G = \mathbb{R} \ltimes \alpha H_3(\mathbb{R})$:

\[ \Lambda_{k,0} = 2\pi \mathbb{Z} \ltimes \Gamma_k \subset G, \]
\[ \Lambda_{k,\pi} = \pi \mathbb{Z} \ltimes \Gamma_k \subset G, \]
\[ \Lambda_{k,\pi/2} = \frac{\pi}{2} \mathbb{Z} \ltimes \Gamma_k \subset G. \]

$\implies \Lambda_{k,0} \supset \Lambda_{k,\pi} \supset \Lambda_{k,\pi/2}$ ($\supset$: “contains as a normal subgroup”),

$\iff$ we have the solvmanifolds

\[ M_{k,0} = G/\Lambda_{k,0}, \]
\[ M_{k,\pi} = G/\Lambda_{k,\pi}, \]
\[ M_{k,\pi/2} = G/\Lambda_{k,\pi/2}. \]

All subgroups of the families $\Lambda_{k,i}$ are not pairwise isomorphic $\implies$
determine non-diffeomorphic solvmanifolds.
Example (The oscillator group)

The action of $\alpha(0)$ is trivial, so $\Lambda_{k,0} = 2\pi\mathbb{Z} \times \Gamma_k$

Diffeomorphism Theorem

\[ M_{k,0} = G/\Lambda_{k,0} \cong S^1 \times H_3(\mathbb{R})/\Gamma_k, \]

a *Kodaira–Thurston manifold.*

Moreover, for any fixed $k$, we have the finite coverings

\[ p_\pi : M_{k,0} \to M_{k,\pi}, \quad p_{\pi/2} : M_{k,0} \to M_{k,\pi/2}, \]

[ –, G. Ovando, M. Subiels ]

The Betti numbers $b_i$ of the solvmanifolds $M_{k,*}$ are given by

<table>
<thead>
<tr>
<th></th>
<th>$b_0$</th>
<th>$b_1$</th>
<th>$b_2$</th>
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</thead>
<tbody>
<tr>
<td>$M_{k,0}$</td>
<td>1</td>
<td>3</td>
<td>4</td>
</tr>
<tr>
<td>$M_{k,\pi}$</td>
<td>1</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>$M_{k,\pi/2}$</td>
<td>1</td>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>

(clearly $b_3 = b_1$ and $b_4 = b_0$, by Poincaré duality).

There are many symplectic structures on $M_{k,0}$ which are invariant by the group $\mathbb{R} \times H_3(\mathbb{R})$ but not under the oscillator group $G$. 