

Capítulo 7:
**Ecuaciones de Maxwell y Ondas
Electromagnéticas**



Hasta ahora:

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0} \quad \text{Ley de Gauss}$$

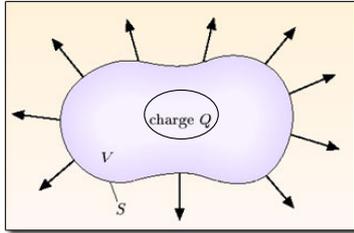
$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \quad \text{Ley de Faraday-Henry}$$

$$\nabla \cdot \vec{B} = 0 \quad \text{Ley de Gauss para el magnetismo}$$

$$\nabla \times \vec{B} = \mu_0 \vec{J} \quad \text{Ley de Ampere}$$

Veremos que la Ley de Ampere presenta problemas

Principio de conservación de la carga



$$-\frac{\partial}{\partial t} \int_V \rho dV = \int_S \mathbf{J} \cdot d\mathbf{S} = I = -\frac{\partial Q}{\partial t}$$

Usando el teorema de la divergencia:

$$\nabla \cdot \vec{J} = -\frac{\partial \rho}{\partial t}$$

Ecuación de continuidad
de la carga

Ley de Ampere-Maxwell



James Clerk Maxwell (1831-1879)

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{j} \quad \text{Ley de Ampere}$$

$$\underbrace{\vec{\nabla} \cdot (\vec{\nabla} \times \vec{B})}_{=0} = \mu_0 \vec{\nabla} \cdot \vec{j} \quad \Longrightarrow \quad \nabla \cdot \vec{j} = 0$$

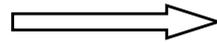
Maxwell propuso un termino adicional:

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{j} + \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t}$$

$$\vec{\nabla} \cdot (\vec{\nabla} \times \vec{B}) = \mu_0 \vec{\nabla} \cdot \vec{j} + \mu_0 \epsilon_0 \frac{\partial}{\partial t} (\vec{\nabla} \cdot \vec{E}) = 0$$

Usando la Ley de Gauss

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0}$$



$$\nabla \cdot \vec{j} = -\frac{\partial \rho}{\partial t}$$

Ley de Ampere-Maxwell en forma diferencial

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{j} + \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t}$$

Ley de Ampere-Maxwell en forma integral

$$\oint_C \vec{B} \cdot d\vec{s} = \mu_0 \int_S \vec{j} \cdot \vec{n} \cdot dS + \mu_0 \epsilon_0 \frac{d}{dt} \int_S \vec{E} \cdot \vec{n} \cdot dS$$

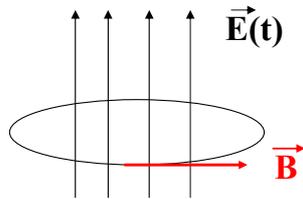
Los campos magnéticos son producidos por corrientes eléctricas y por campos eléctricos variables.

En ausencia de corrientes:

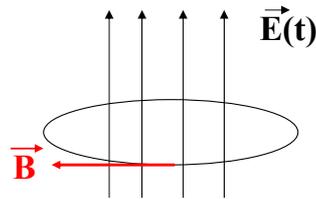
$$\oint \vec{B} \cdot d\vec{s} = \mu_0 \epsilon_0 \frac{d\Phi_E}{dt}$$

Comentar similitudes y diferencias con la Ley de Faraday

$$\oint \vec{E} \cdot d\vec{s} = -\frac{d\Phi_B}{dt}$$

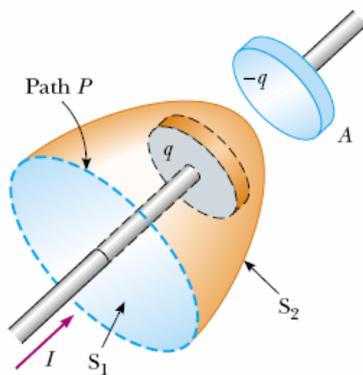


E aumenta



E disminuye

Ejemplo de la necesidad del nuevo término



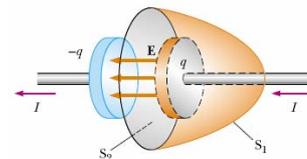
Usando Ley de Ampere:

$$S_1: \quad \oint \vec{B} \cdot d\vec{s} = \mu_0 I$$

$$S_2: \quad \oint \vec{B} \cdot d\vec{s} = 0$$

Esta contradicción la resuelve el nuevo término:

$I \neq 0 \rightarrow$ Capacitor se esta cargando $\rightarrow q(t) \rightarrow E(t)$

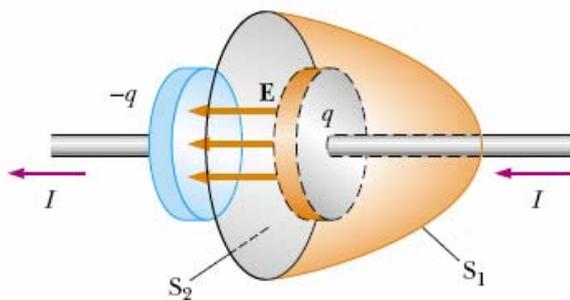


Por razones históricas, se denomina

$$I_d \equiv \epsilon_0 \frac{d\Phi_E}{dt}$$

$$\oint \mathbf{B} \cdot d\mathbf{s} = \mu_0(I + I_d) = \mu_0 I + \mu_0 \epsilon_0 \frac{d\Phi_E}{dt}$$

corriente de desplazamiento



S_1 :

$$\oint \mathbf{B} \cdot d\mathbf{s} = \mu_0 I$$

S_2 :

$$\oint \mathbf{B} \cdot d\mathbf{s} = \mu_0 I_d$$

$$\Phi_E = \int \mathbf{E} \cdot d\mathbf{A}$$

$$\Phi_E = EA = \frac{q}{\epsilon_0}$$

$$I_d = \epsilon_0 \frac{d\Phi_E}{dt} = \frac{dq}{dt}$$

Ecuaciones de Maxwell

Ley	Forma diferencial	Forma integral
Gauss	$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0}$	$\oint_S \vec{E} \cdot d\vec{s} = \frac{q}{\epsilon_0}$
Gauss para B	$\vec{\nabla} \cdot \vec{B} = 0$	$\oint_S \vec{B} \cdot d\vec{s} = 0$
Faraday-Henry	$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$	$\oint_C \vec{E} \cdot d\vec{l} = -\frac{d}{dt} \int_S \vec{B} \cdot d\vec{s}$
Ampere-Maxwell	$\vec{\nabla} \times \vec{B} = \mu_0 \vec{j} + \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t}$	$\oint_C \vec{B} \cdot d\vec{l} = \mu_0 \int_S \vec{j} \cdot \vec{n} \cdot d\mathbf{s} + \mu_0 \epsilon_0 \frac{d}{dt} \int_S \vec{E} \cdot \vec{n} \cdot d\mathbf{s}$

Fuerza de Lorentz $\mathbf{F} = q\mathbf{E} + q\mathbf{v} \times \mathbf{B}$

Aquí tenemos el electromagnetismo!!!!

Física Clásica

Maxwell's equations

I. $\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}$ (Flux of \mathbf{E} through a closed surface) = (Charge inside)/ ϵ_0

II. $\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$ (Line integral of \mathbf{E} around a loop) = $-\frac{d}{dt}$ (Flux of \mathbf{B} through the loop)

III. $\nabla \cdot \mathbf{B} = 0$ (Flux of \mathbf{B} through a closed surface) = 0

IV. $c^2 \nabla \times \mathbf{B} = \frac{\mathbf{j}}{\epsilon_0} + \frac{\partial \mathbf{E}}{\partial t}$ c^2 (Integral of \mathbf{B} around a loop) = (Current through the loop)/ ϵ_0
 $+\frac{\partial}{\partial t}$ (Flux of \mathbf{E} through the loop)

Conservation of charge

$\nabla \cdot \mathbf{j} = -\frac{\partial \rho}{\partial t}$ (Flux of current through a closed surface) = $-\frac{\partial}{\partial t}$ (Charge inside)

Force law

$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$

Law of motion

$\frac{d}{dt}(\mathbf{p}) = \mathbf{F}$, where $\mathbf{p} = \frac{m\mathbf{v}}{\sqrt{1 - v^2/c^2}}$ (Newton's law, with Einstein's modification)

Gravitation

$\mathbf{F} = -G \frac{m_1 m_2}{r^2} \mathbf{e}_r$

Gran descubrimiento!!! ➔ Ondas electromagnéticas

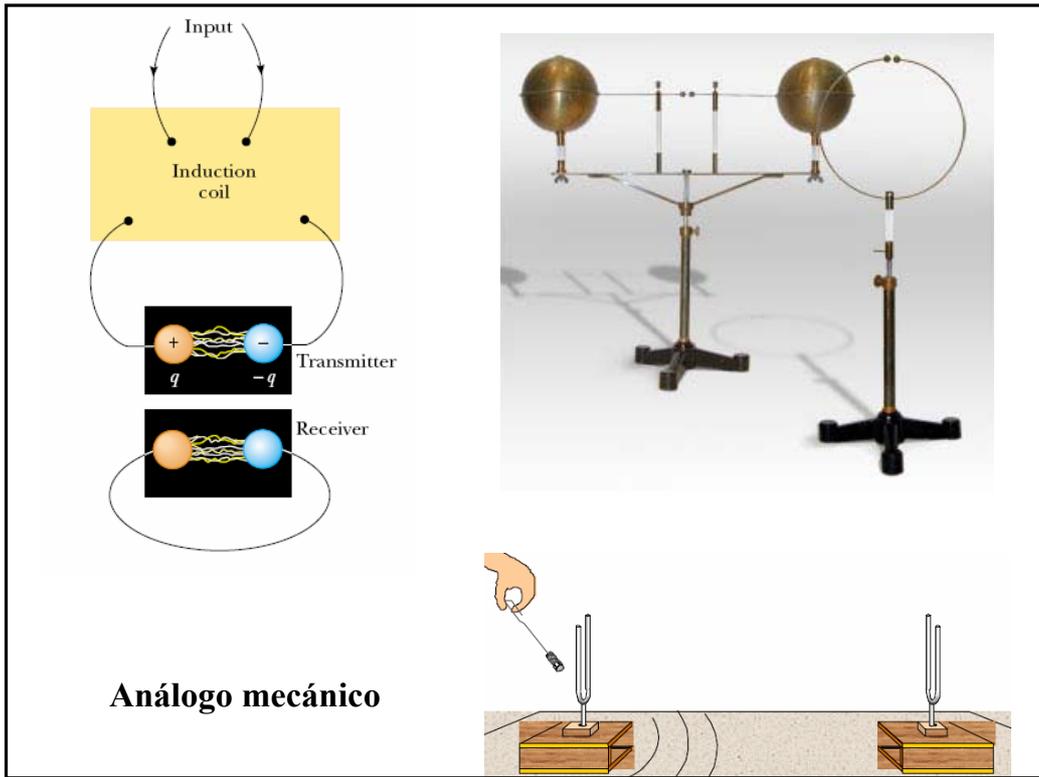
$\mathbf{E}(t) \rightarrow \mathbf{B}(t) \rightarrow \mathbf{E}(t) \rightarrow \mathbf{B}(t) \rightarrow \mathbf{E}(t) \rightarrow \mathbf{B}(t)$

Maxwell predijo en forma teórica la existencia de ondas de E y B (ondas electromagnéticas)



generó y detectó ondas electromagnéticas

Heinrich Hertz (1857-1894)



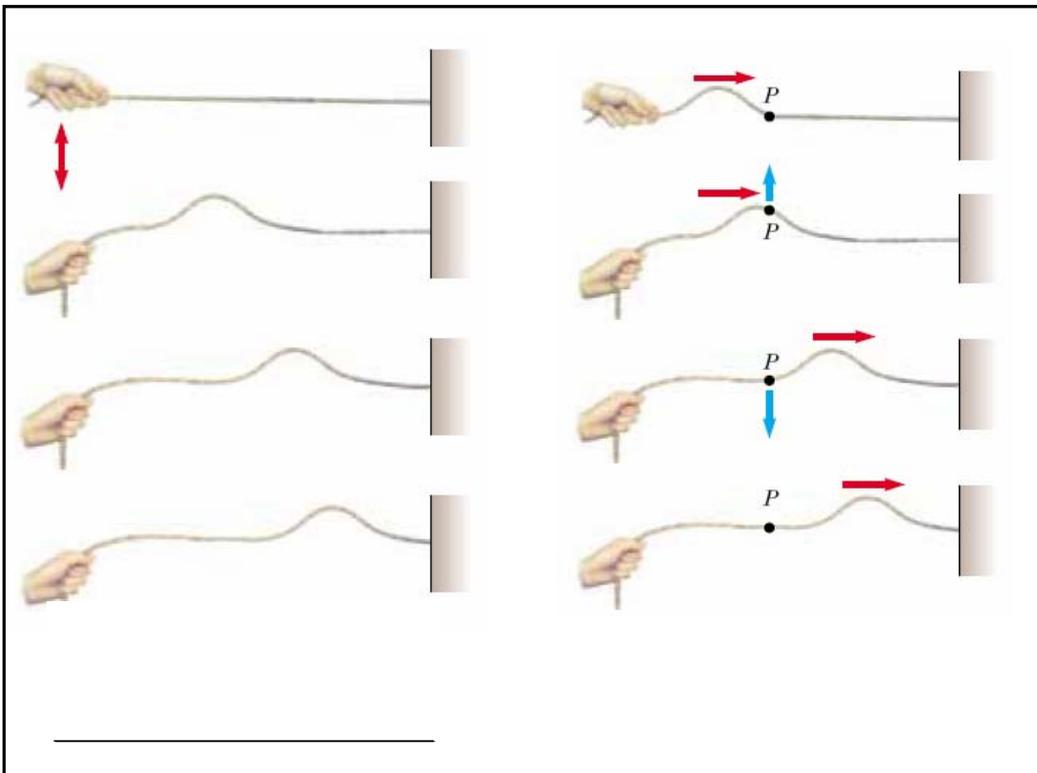
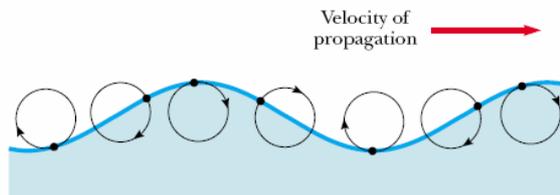
Movimiento Ondulatorio

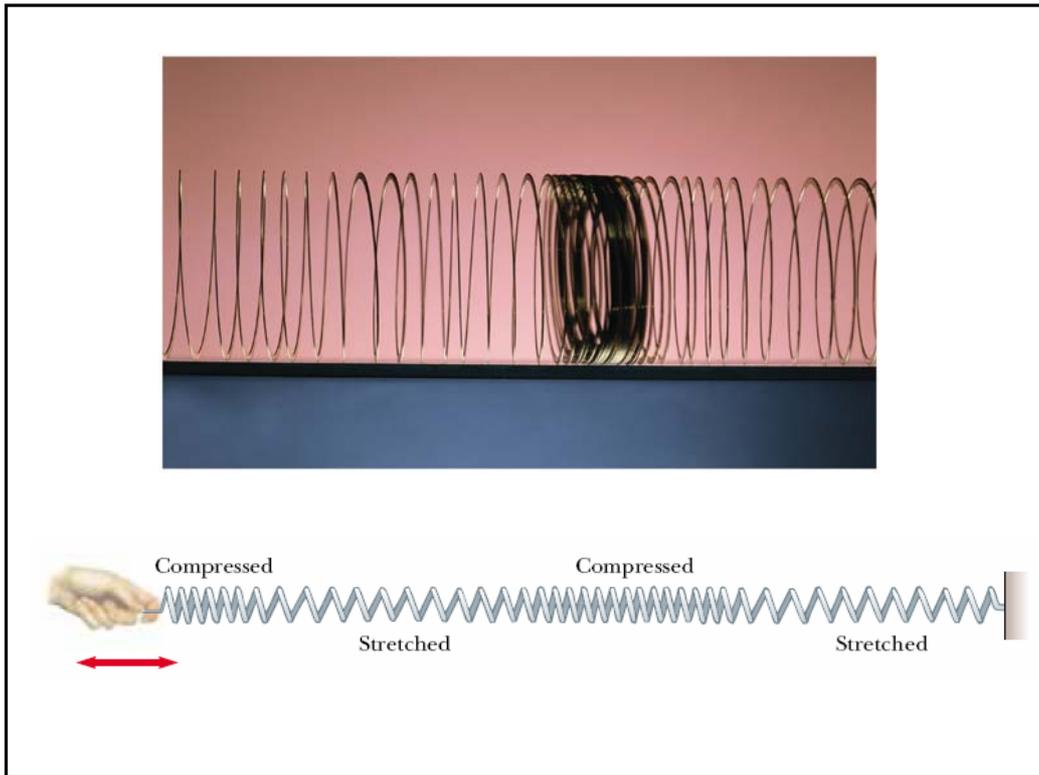




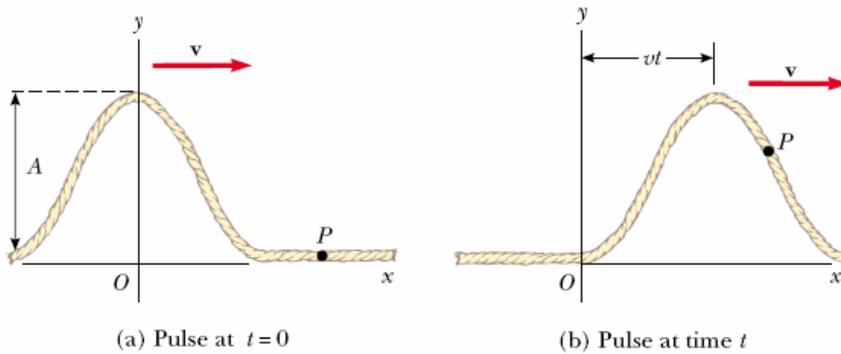
Características comunes

- 1- estado inicial de equilibrio
- 2- perturbación en un punto del espacio
- 3- propagación de la perturbación con velocidad v
- 4- oscilación del medio perturbado alrededor de la posición de equilibrio





Descripción matemática de la propagación



$$y(x, t) = f(x - vt)$$

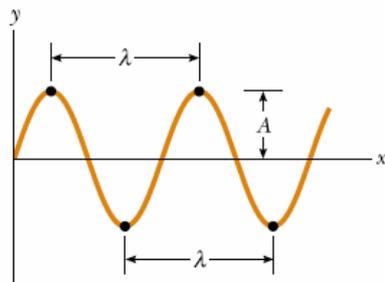
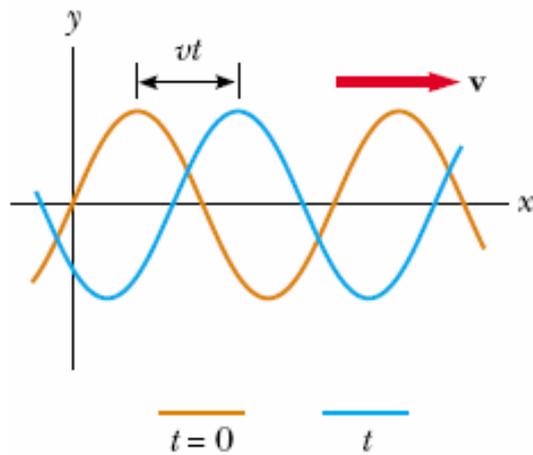
$$y(x, t) = f(x + vt)$$

Describe una situación física que viaje o se propaga

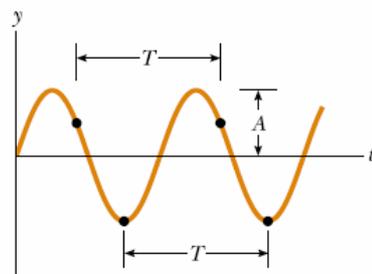
Caso especialmente interesante:

$$y(x, t) = A \sin \left[\frac{2\pi}{\lambda} (x - vt) \right]$$

onda sinusoidal o armónica



longitud de onda



periodo

$$k \equiv \frac{2\pi}{\lambda}$$

nº de onda

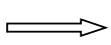
$$f \equiv \frac{1}{T}$$

frecuencia

$$\omega \equiv \frac{2\pi}{T}$$

frecuencia angular

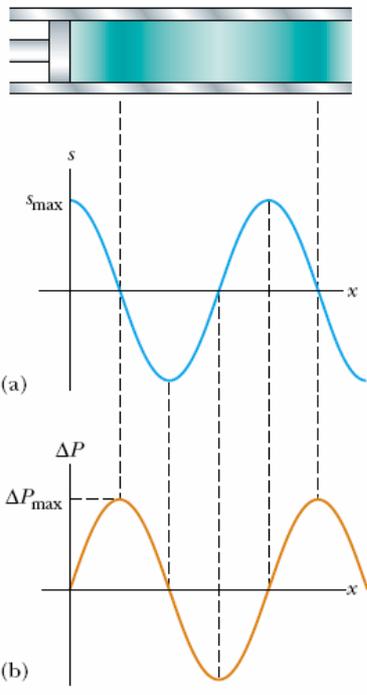
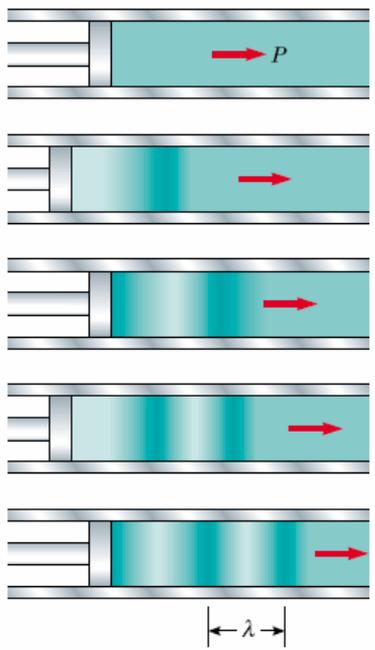
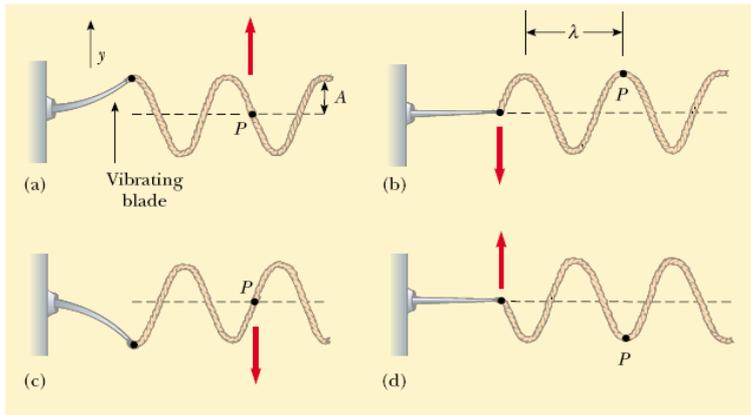
$$y(x, t) = A \sin \left[\frac{2\pi}{\lambda} (x - vt) \right]$$



$$y = A \sin(kx - \omega t)$$

donde $v = \frac{\omega}{k}$ $v = \frac{\lambda}{T}$

$$v = \lambda f$$



$$\Delta P = \Delta P_{\max} \sin(kx - \omega t)$$

Ecuación diferencial del movimiento oscilatorio

supongamos $\psi(x \pm vt)$ llamamos $x' = x \pm vt \longrightarrow$ $\frac{\partial x'}{\partial x} = 1$
 $\frac{\partial x'}{\partial t} = \pm v$

$$\frac{\partial \psi(x')}{\partial x} = \frac{\partial \psi}{\partial x'} \frac{\partial x'}{\partial x} = \frac{\partial \psi}{\partial x'} \longrightarrow \frac{\partial^2 \psi}{\partial x^2} = \frac{\partial}{\partial x} \left(\frac{\partial \psi}{\partial x'} \right) = \frac{\partial^2 \psi}{\partial x'^2} \frac{\partial x'}{\partial x} = \frac{\partial^2 \psi}{\partial x'^2}$$

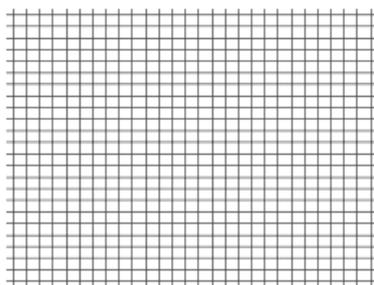
$$\frac{\partial \psi}{\partial t} = \frac{\partial \psi}{\partial x'} \frac{\partial x'}{\partial t} = \pm v \frac{\partial \psi}{\partial x'} \longrightarrow \frac{\partial^2 \psi}{\partial t^2} = \frac{\partial}{\partial t} \left(\pm v \frac{\partial \psi}{\partial x'} \right) = \pm v \frac{\partial^2 \psi}{\partial x'^2} \frac{\partial x'}{\partial t} = v^2 \frac{\partial^2 \psi}{\partial x'^2}$$

$$\frac{\partial^2 \psi}{\partial x'^2} = \frac{\partial^2 \psi}{\partial x^2} = \frac{1}{v^2} \frac{\partial^2 \psi}{\partial t^2} \quad \text{ecuación de onda}$$

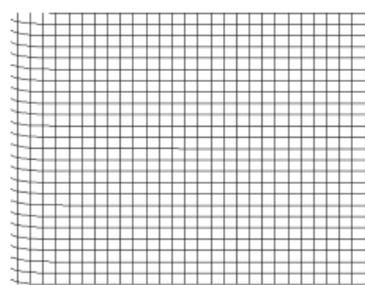
Si $\psi_1(x, t)$ y $\psi_2(x, t)$ son solución \longrightarrow $\psi_1(x, t) \pm \psi_2(x, t)$
es solución

dirección de propagación de la onda

perturbación



onda longitudinal

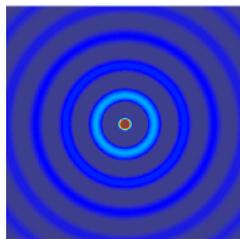


onda transversal

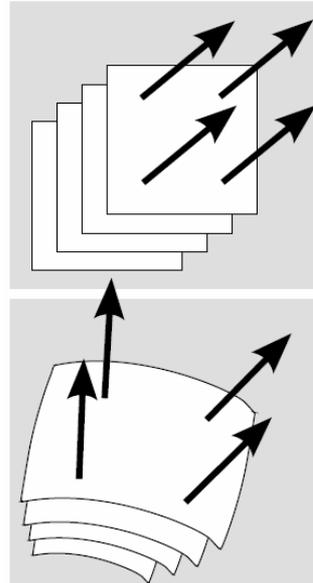
Frente de onda: lugar del espacio donde la perturbación toma el mismo valor en un dado instante de tiempo.



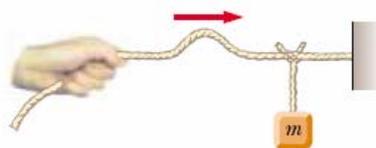
onda plana



onda esférica



Que se propaga en un movimiento ondulatorio?



(a)



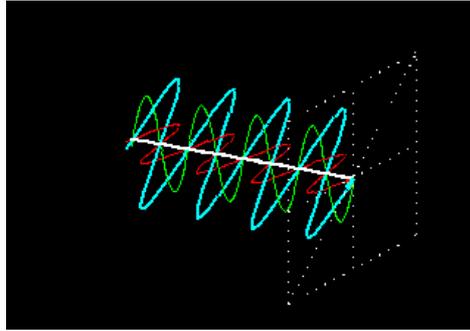
(b)

Respuesta general:

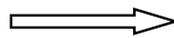
una condición física generada en algún lugar y que se transmite a otra regiones

En un mov. ondulatorio se propaga momento y energía

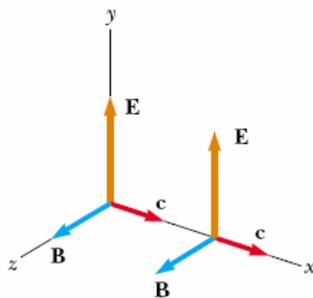
Ondas Electromagnéticas Planas



Ecuaciones de Maxwell en el vacío ($j=0, \rho=0$)



Ondas EM



Suponemos:

$$\vec{E} = (0, E, 0)$$

$$\vec{B} = (0, 0, B)$$

$$1- \nabla \cdot \vec{E} = 0 \longrightarrow \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} = 0$$

$$\frac{\partial E}{\partial y} = 0 \longrightarrow E \neq E(y)$$

$$2- \nabla \cdot \vec{B} = 0 \longrightarrow \frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z} = 0$$

$$\frac{\partial B}{\partial z} = 0 \longrightarrow \mathbf{B} \neq B(z)$$

$$3- \nabla \times \vec{E} = \frac{\partial \vec{B}}{\partial t} \longrightarrow \nabla \times \vec{E} = \begin{vmatrix} \vec{\mu}_x & \vec{\mu}_y & \vec{\mu}_z \\ \partial_x & \partial_y & \partial_z \\ 0 & E & 0 \end{vmatrix}$$

$$\left(\frac{-\partial E}{\partial z}, 0, \frac{\partial E}{\partial x} \right) = \left(0, 0, \frac{-\partial B}{\partial t} \right)$$

$$\mathbf{E} \neq E(z) \quad \frac{\partial E}{\partial x} = -\frac{\partial B}{\partial t}$$

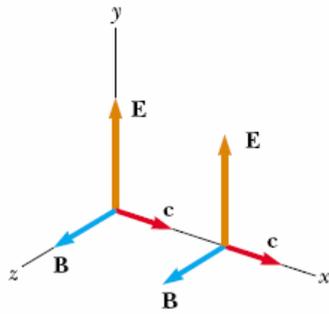
$$4- \nabla \times \vec{B} = \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t} \longrightarrow \nabla \times \vec{B} = \begin{vmatrix} \vec{\mu}_x & \vec{\mu}_y & \vec{\mu}_z \\ \partial_x & \partial_y & \partial_z \\ 0 & 0 & B \end{vmatrix}$$

$$\left(\frac{\partial B}{\partial y}, \frac{-\partial B}{\partial x}, 0 \right) = \mu_0 \epsilon_0 \left(0, \frac{\partial E}{\partial t}, 0 \right)$$

$$\mathbf{B} \neq B(y) \quad -\frac{\partial B}{\partial x} = \mu_0 \epsilon_0 \frac{\partial E}{\partial t}$$

$$\mathbf{E} \neq E(y,z) \quad \longrightarrow \quad \mathbf{E} = E(x,t)$$

$$\mathbf{B} \neq B(y,z) \quad \longrightarrow \quad \mathbf{B} = B(x,t)$$



$$\frac{\partial E}{\partial x} = -\frac{\partial B}{\partial t}$$

$$\frac{\partial B}{\partial x} = -\mu_0 \epsilon_0 \frac{\partial E}{\partial t}$$

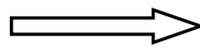
$$\frac{\partial^2 E}{\partial x^2} = -\frac{\partial}{\partial x} \left(\frac{\partial B}{\partial t} \right) = -\frac{\partial}{\partial t} \left(\frac{\partial B}{\partial x} \right) = -\frac{\partial}{\partial t} \left(-\mu_0 \epsilon_0 \frac{\partial E}{\partial t} \right)$$

$$\frac{\partial^2 E}{\partial x^2} = \mu_0 \epsilon_0 \frac{\partial^2 E}{\partial t^2}$$

$$\frac{\partial^2 B}{\partial x^2} = \mu_0 \epsilon_0 \frac{\partial^2 B}{\partial t^2}$$

$$\frac{\partial^2 E}{\partial x^2} = \mu_0 \epsilon_0 \frac{\partial^2 E}{\partial t^2}$$

Velocidad de la onda



$$c = \frac{1}{\sqrt{\mu_0 \epsilon_0}}$$

$$\frac{\partial^2 B}{\partial x^2} = \mu_0 \epsilon_0 \frac{\partial^2 B}{\partial t^2}$$

Pero: $\mu_0 = 4\pi \times 10^{-7} \text{ T} \cdot \text{m/A}$ $\epsilon_0 = 8.85419 \times 10^{-12} \text{ C}^2/\text{N} \cdot \text{m}^2$

$$c = 2.99792 \times 10^8 \text{ m/s.}$$

Velocidad de la luz en el vacio !!!!!

Caso particular: onda armónica

$$E = E_{\max} \cos(kx - \omega t)$$

$$B = B_{\max} \cos(kx - \omega t)$$

$$\frac{\omega}{k} = \frac{2\pi f}{2\pi/\lambda} = \lambda f = c$$

$$\lambda = \frac{c}{f} = \frac{3.00 \times 10^8 \text{ m/s}}{f}$$

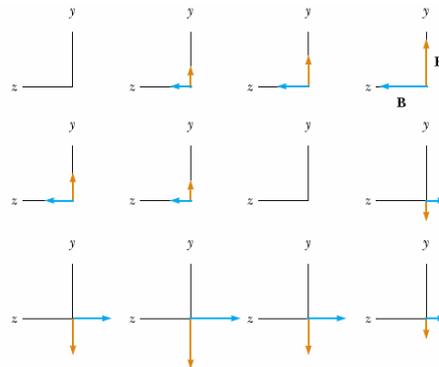
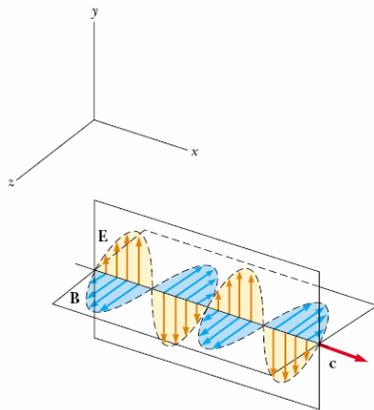
$$\frac{\partial E}{\partial x} = -kE_{\max} \sin(kx - \omega t)$$

$$\frac{\partial B}{\partial t} = \omega B_{\max} \sin(kx - \omega t)$$

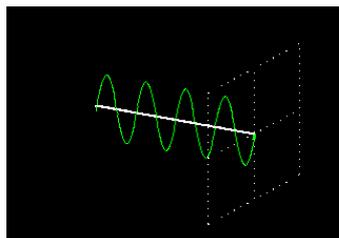
$$\Rightarrow kE_{\max} = \omega B_{\max}$$

$$\frac{E_{\max}}{B_{\max}} = \frac{\omega}{k} = c$$

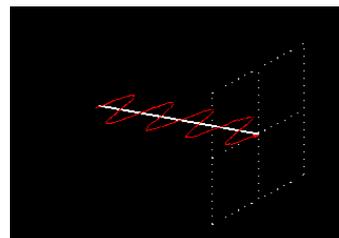
$$\frac{E_{\max}}{B_{\max}} = \frac{E}{B} = c$$

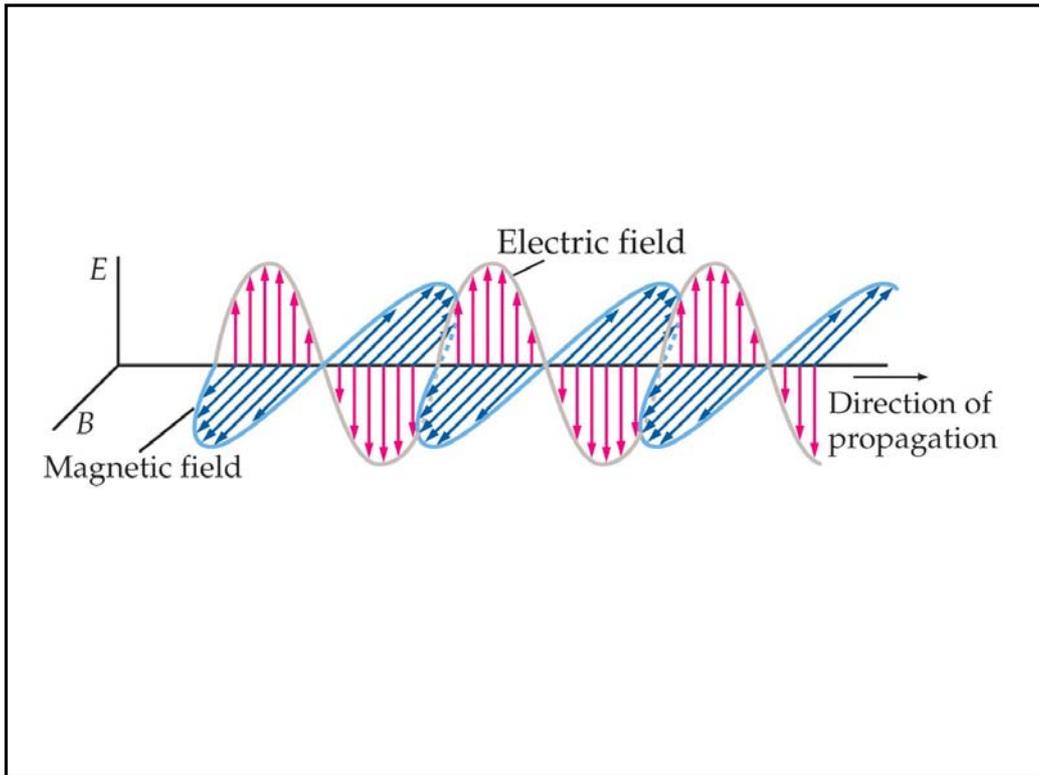


E

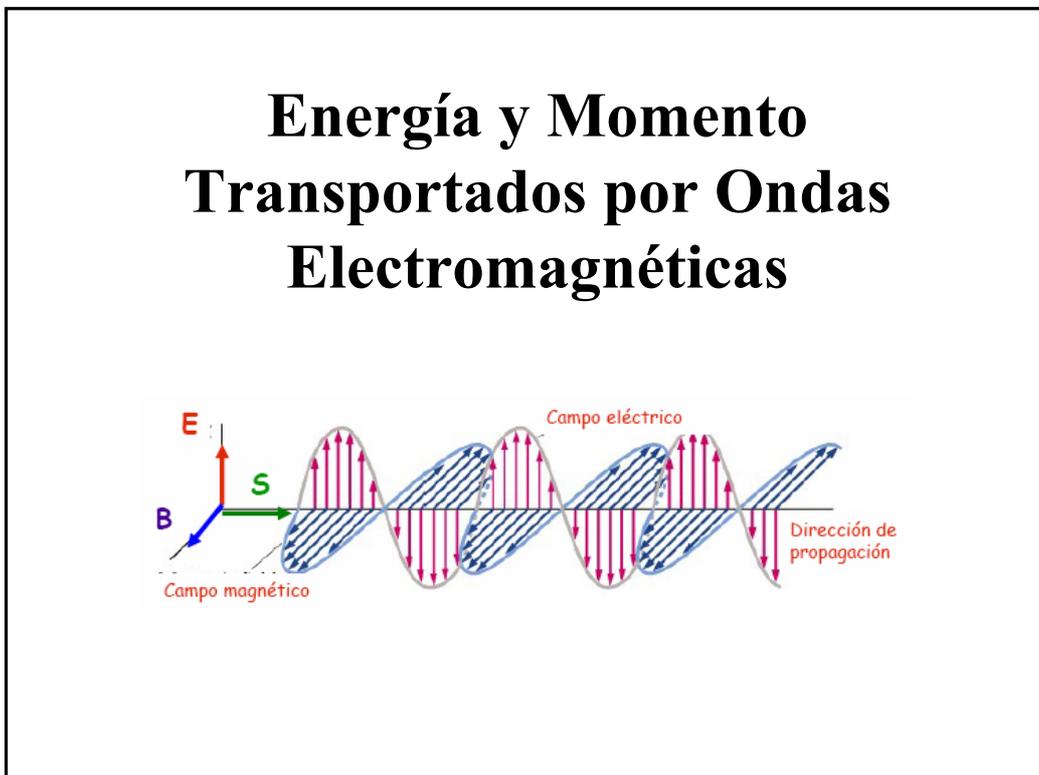


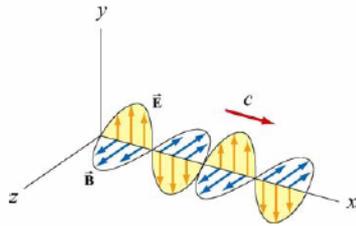
B





Energía y Momento Transportados por Ondas Electromagnéticas





$$u_E = \frac{1}{2} \epsilon_0 E^2$$

densidad de energía eléctrica

$$u_B = \frac{B^2}{2\mu_0}$$

densidad de energía magnética

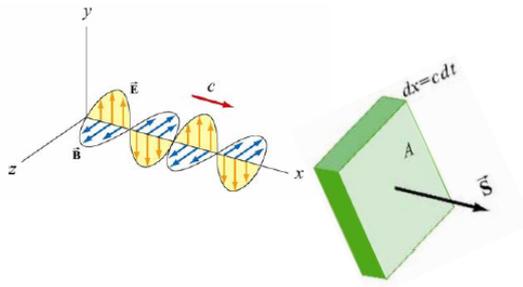
pero $B = \bar{E}/c$ donde $c = 1/\sqrt{\epsilon_0\mu_0}$,

$$u_B = \frac{(E/c)^2}{2\mu_0} = \frac{\epsilon_0\mu_0}{2\mu_0} E^2 = \frac{1}{2} \epsilon_0 E^2$$

$$u_B = u_E = \frac{1}{2} \epsilon_0 E^2 = \frac{B^2}{2\mu_0}$$

$$u = u_E + u_B = \epsilon_0 E^2 = \frac{B^2}{\mu_0}$$

densidad de energía de la onda EM



$$dU = u A dx = (u_E + u_B) A dx = \frac{1}{2} \left(\epsilon_0 E^2 + \frac{B^2}{\mu_0} \right) A dx$$

$$S = \frac{dU}{A dt} = \frac{c}{2} \left(\epsilon_0 E^2 + \frac{B^2}{\mu_0} \right)$$

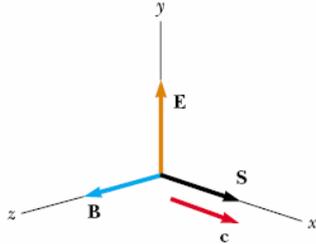
energía por unidad de área y unidad de tiempo

$$[S] = \text{W/m}^2$$

$$S = \frac{c}{2} \left(\epsilon_0 E^2 + \frac{B^2}{\mu_0} \right) = \frac{c B^2}{\mu_0} = c \epsilon_0 E^2 = \frac{EB}{\mu_0}$$

Se define el vector de Poynting:

$$\vec{S} \equiv \frac{1}{\mu_0} \vec{E} \times \vec{B}$$



$$S = \frac{EB}{\mu_0}$$

$$S = \frac{E^2}{\mu_0 c} = \frac{c}{\mu_0} B^2$$

Se define la intensidad de una onda EM:

$$I = \langle S \rangle = \frac{E_0 B_0}{\mu_0} \langle \cos^2(kx - \omega t) \rangle = \frac{E_0 B_0}{2\mu_0} = \frac{E_0^2}{2c\mu_0} = \frac{cB_0^2}{2\mu_0}$$

donde hemos utilizado: $\langle \cos^2(kx - \omega t) \rangle = \frac{1}{2}$

En general, la energía que atraviesa una superficie por unidad de tiempo es:

$$\frac{dU}{dt} = \iint \vec{S} \cdot d\vec{A}$$

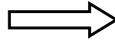
La densidad de energía promedio de la onda EM es:

$$\begin{aligned} \langle u \rangle &= \langle u_E + u_B \rangle = \epsilon_0 \langle E^2 \rangle = \frac{\epsilon_0}{2} E_0^2 \\ &= \frac{1}{\mu_0} \langle B^2 \rangle = \frac{B_0^2}{2\mu_0} \end{aligned}$$

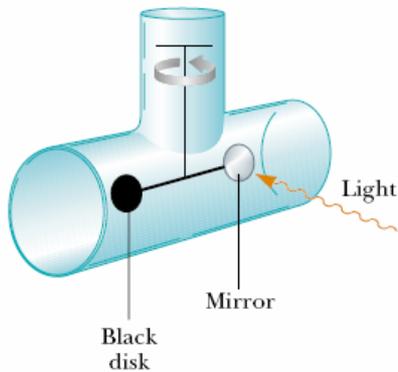
La intensidad y la densidad de energía promedio de la onda EM se vinculan de la siguiente manera:

$$I = \langle S \rangle = c \langle u \rangle$$

onda EM también transporta momento



ejerce una **presión de radiación** sobre una superficie (absorción o reflexión)



Asumimos incidencia normal. Maxwell mostró que si la onda es completamente reflejada por la superficie, la transferencia de momento está relacionada con la energía reflejada

$$\Delta p = \frac{2\Delta U}{c}$$

Si la onda es completamente absorbida

$$\Delta p = \frac{\Delta U}{c}$$

Para el caso de absorción, la presión de radiación promedio (fuerza por unidad de área) es dada por:

$$P = \frac{\langle F \rangle}{A} = \frac{1}{A} \left\langle \frac{dp}{dt} \right\rangle = \frac{1}{Ac} \left\langle \frac{dU}{dt} \right\rangle$$

$$\left\langle \frac{dU}{dt} \right\rangle = \langle S \rangle A = IA$$

$$P = \frac{I}{c}$$

Absorción

$$P = \frac{2I}{c}$$

Reflexión

Ej.: bombita de luz de 60 W a una distancia de 1 m

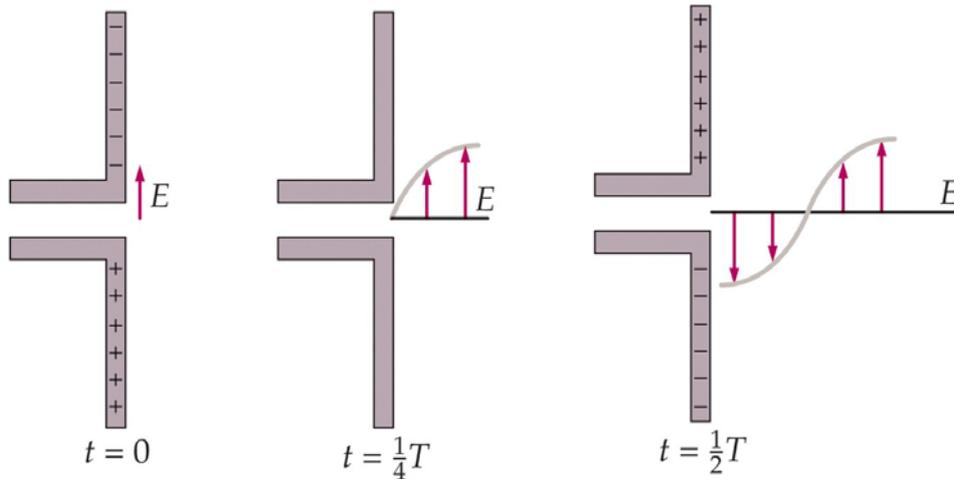
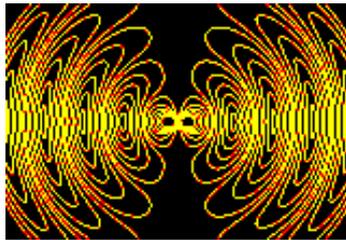
$$I(r) = \text{Pot} / 4 \pi r^2 = 4.77 \text{ W/m}^2$$

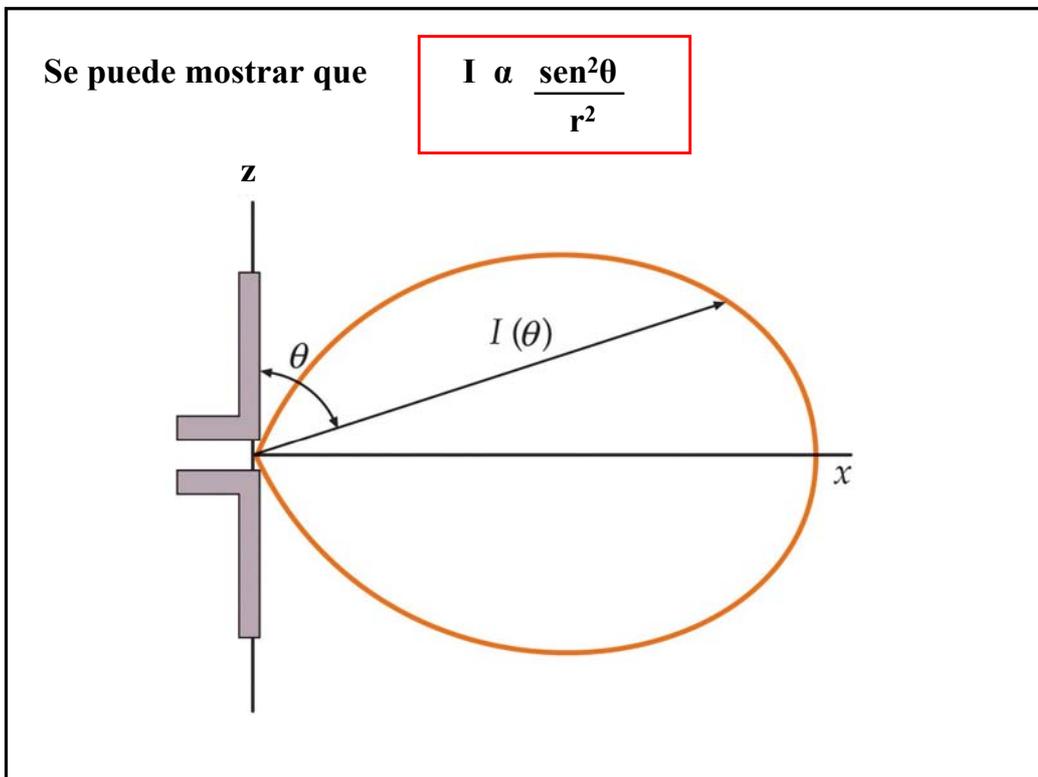
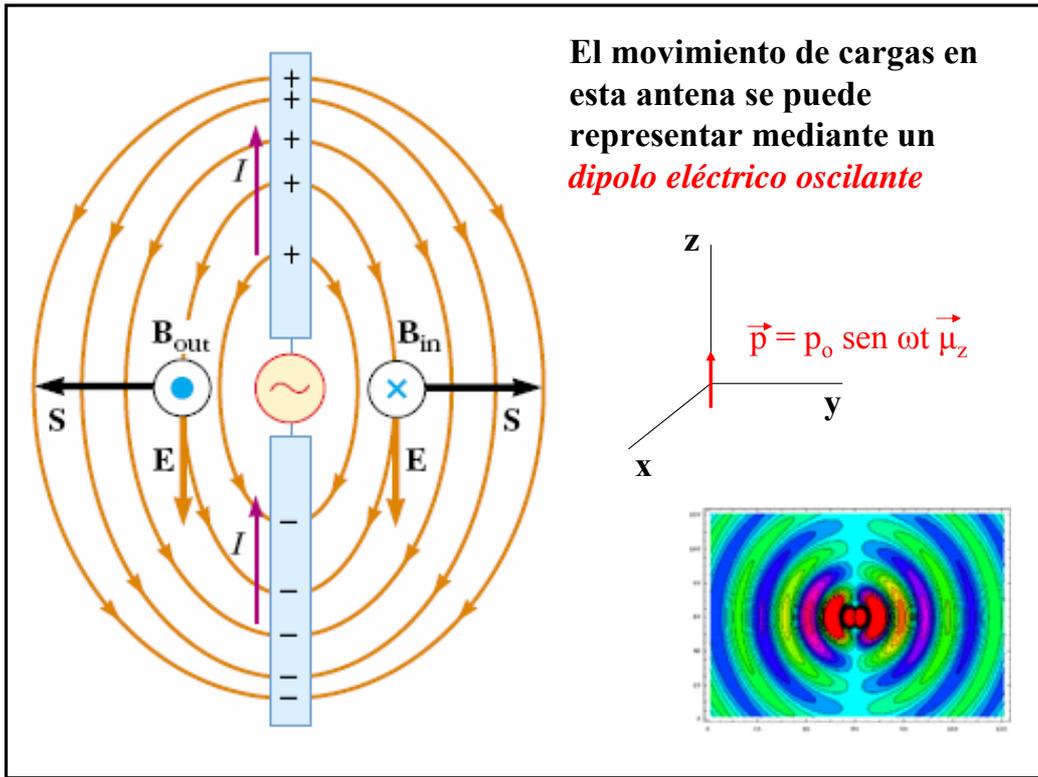
Fuerza sobre mi mano
(20 cm x 10 cm)

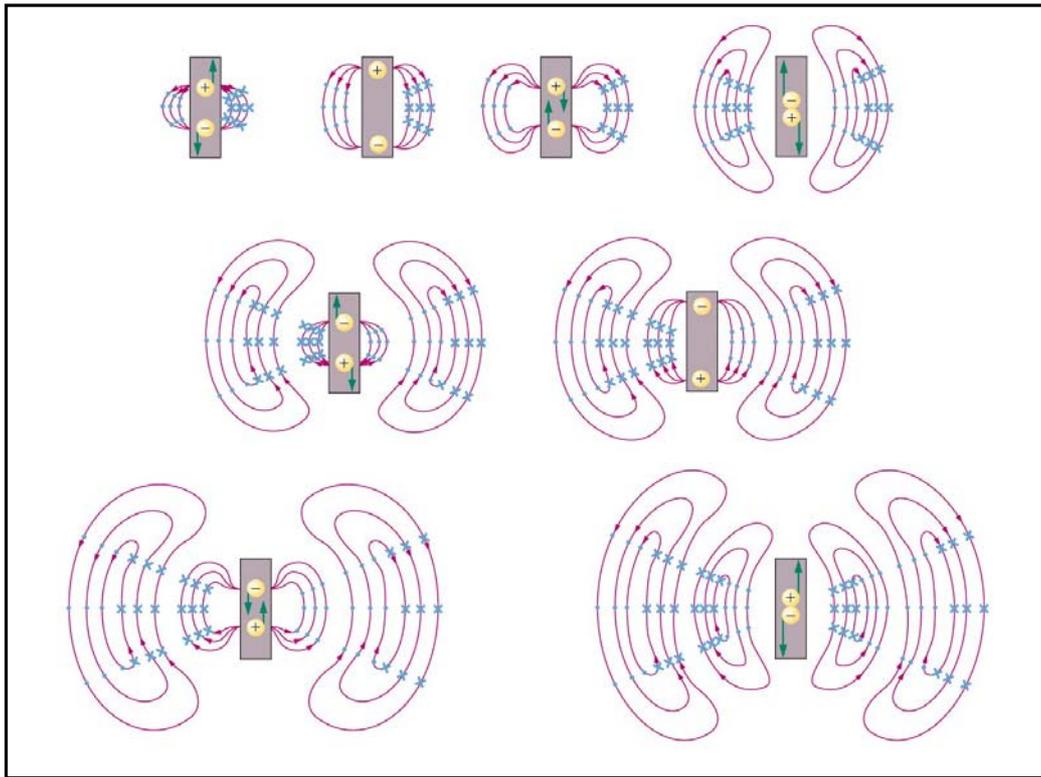
$$P = I / c = 15.8 \cdot 10^{-9} \text{ Pa}$$

$$F = 3 \cdot 10^{-8} \text{ N}$$

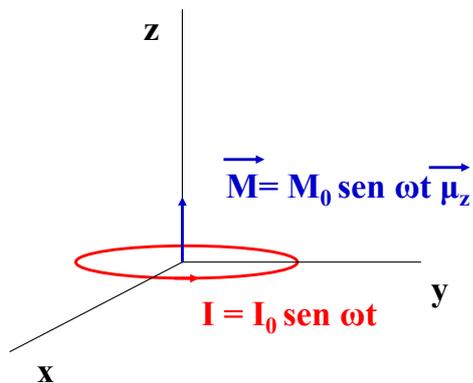
Producción y Recepción de Ondas Electromagnéticas



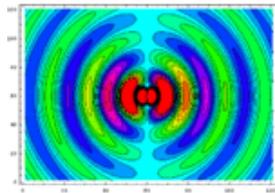




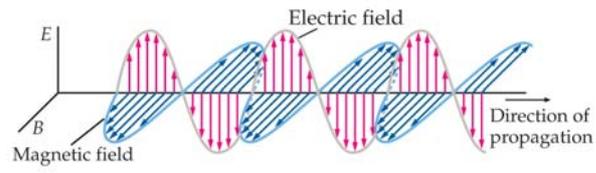
Dipolo magnético oscilante



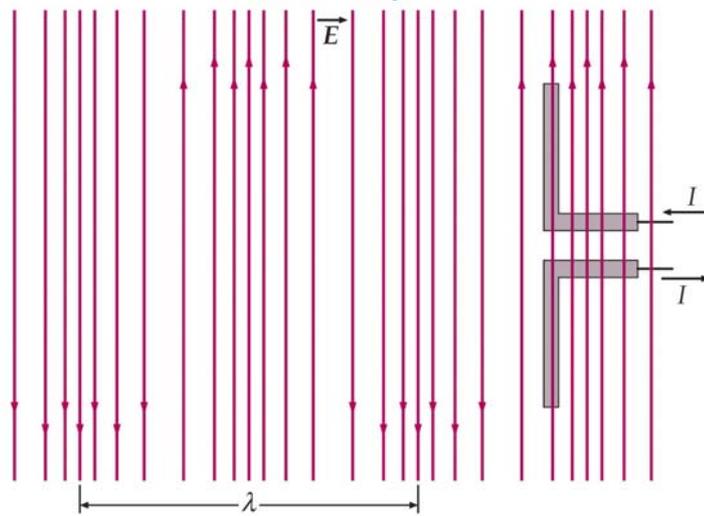
$$I \propto \frac{\sin^2 \theta}{r^2}$$



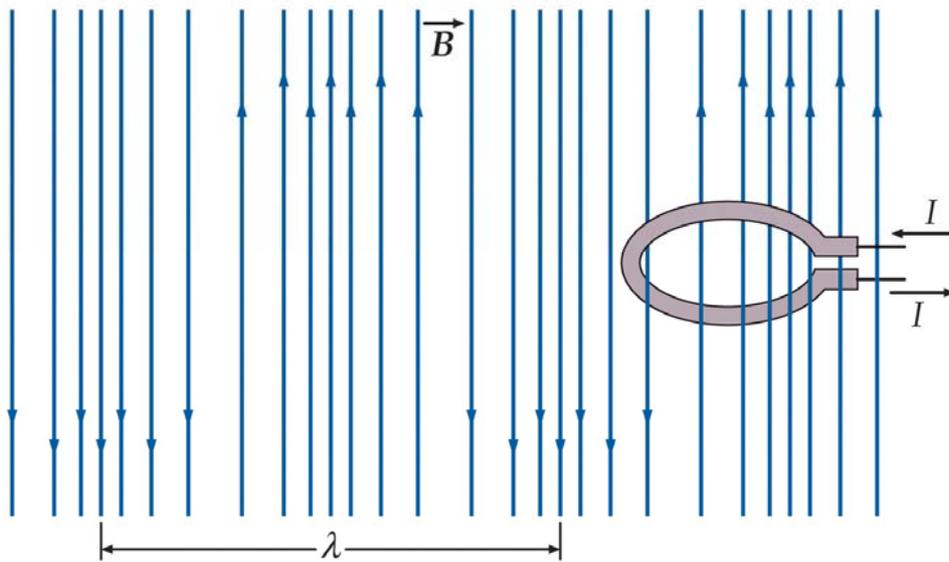
Recepción



Wave velocity

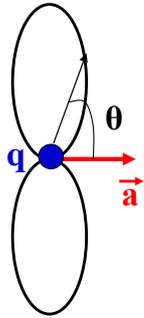


Wave velocity



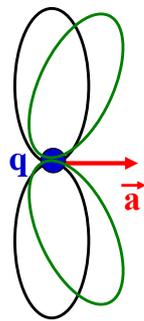
Radiación de una carga acelerada

Una carga acelerada irradia energía electromagnética

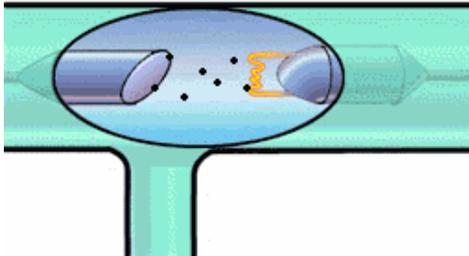


$$I = \frac{q^2 a^2 \sin^2\theta}{16 \pi^2 c^3 \epsilon_0 r^2}$$

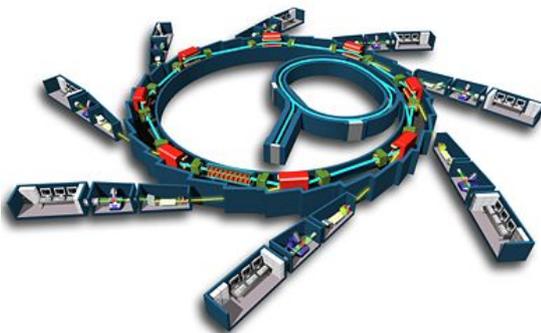
$v \ll c$



$v = 0.64 c$

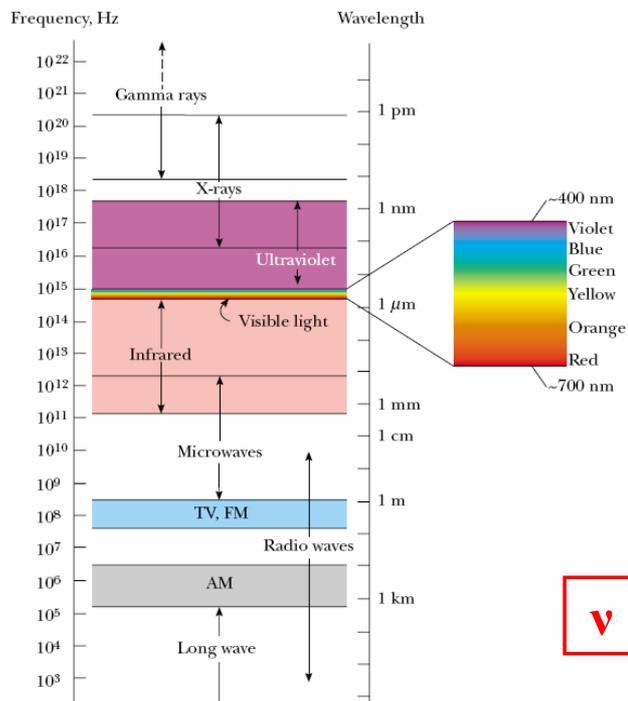
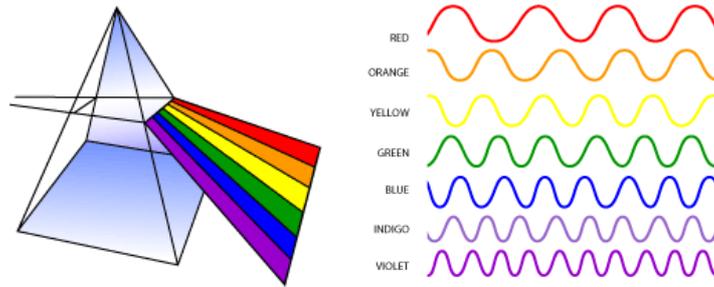


Radiación de frenado
Generación de Rayos X



Radiación Sincrotrónica

El Espectro Electromagnético



$$v \lambda = c$$

THE ELECTROMAGNETIC SPECTRUM

