2. Uniqueness of Solutions

The example (1.8), with the solutions given in (1.9), shows that something more than the continuity of \( f \) in \( (E) \) is required in order to guarantee that a solution passing through a given point be unique. A simple condition which permits one to imply uniqueness is the Lipschitz condition. Suppose \( f \) is defined in a domain \( D \) of the \( (t, x) \) plane. If there exists a constant \( k > 0 \) such that for every \( (t, x_1) \) and \( (t, x_2) \) in \( D \)

\[
|f(t, x_1) - f(t, x_2)| \leq k|x_1 - x_2|
\]

then \( f \) is said to satisfy a Lipschitz condition (with respect to \( x \)) in \( D \), and this fact will be denoted by \( f \in \text{Lip} \) in \( D \). The constant \( k \) is called the Lipschitz constant. If, in addition \( f \in C) \) in \( D \), one writes \( f \in (C, \text{Lip}) \) in \( D \). If \( f \in \text{Lip} \) in \( D \), then \( f \) is uniformly continuous in \( x \) for each fixed \( t \), although nothing is implied concerning the continuity of \( f \) with respect to \( t \). If \( D \) is convex (that is, \( D \) contains the line segment connecting any two points in \( D \)), then an application of the mean-value theorem of differential calculus shows that the existence and boundedness of \( f_\varepsilon \) (\( = \delta f/\delta x \)) in \( D \) are sufficient for \( f \in \text{Lip} \) in \( D \).

Before proceeding to the uniqueness proof, an important inequality will be deduced. In the following, \( D \) is a domain in the \( (t, x) \) plane.

Theorem 2.1. Suppose \( f \in (C, \text{Lip}) \) in \( D \), with Lipschitz constant \( k \). Let \( \varphi_1, \varphi_2 \) be \( \varepsilon \) and \( \varepsilon \)-approximate solutions of \( (E) \) of class \( C^1 \) on some interval \((a, b)\), satisfying for some \( \tau, a < \tau < b \),

\[
|\varphi_1(t) - \varphi_2(t)| \leq \delta
\]  

(2.1)

where \( \delta \) is a nonnegative constant. If \( \varepsilon = \varepsilon_1 + \varepsilon_2 \), then for all \( t \in (a, b) \),

\[
|\varphi_1(t) - \varphi_2(t)| \leq \delta e^{kt-a} + \frac{\varepsilon}{k}(e^{kt-b} - 1)
\]  

(2.2)

Theorem 2.1 is of practical as well as theoretical interest since in computational procedures it is always approximate solutions of a differential equation that are found.

Proof of Theorem 2.1. Consider the case where \( \tau \leq t < b \); a corresponding proof holds for \( a < t \leq \tau \). Since \( \varphi_1, \varphi_2 \) are \( \varepsilon_1 \) and \( \varepsilon_2 \)-approximate solutions of \( (E) \),

\[
|\varphi_1(s) - f(s, \varphi_1(s))| \leq \varepsilon_1 \quad (i = 1, 2)
\]  

(2.3)

at all but a finite number of points on \( \tau \leq s < b \).

Integrating from \( \tau \) to \( t \), where \( \tau \leq t < b \), (2.3) yields

\[
|\varphi(t) - \varphi(\tau) - \int_{\tau}^{t} f(s, \varphi(s)) \, ds | \leq \varepsilon_i(t - \tau) \quad (i = 1, 2)
\]  

(2.4)
Using the fact that \(|\alpha - \beta| \leq |\alpha| + |\beta|\), the above gives
\[
\left| (\varphi_1(t) - \varphi_2(t)) - (\varphi_1(\tau) - \varphi_2(\tau)) - \int_{\tau}^{t} [f(s,\varphi_1(s)) - f(s,\varphi_2(s))] \, ds \right| \leq \epsilon(t - \tau)
\]
Let \(r\) be the function defined on \([r, b]\) by \(r(t) = |\varphi_1(t) - \varphi_2(t)|\). Then the preceding inequality gives
\[
r(t) \leq r(\tau) + \int_{\tau}^{t} |f(s,\varphi_1(s)) - f(s,\varphi_2(s))| \, ds + \epsilon(t - \tau)
\]
and using the fact that \(f \in \text{Lip} in D\), one gets
\[
r(t) \leq r(\tau) + k \int_{\tau}^{t} r(s) \, ds + \epsilon(t - \tau) \quad (2.4)
\]
Define the function \(R\) by
\[
R(t) = \int_{\tau}^{t} r(s) \, ds \quad (r \leq t < b)
\]
In terms of \(R\), (2.4) is
\[
R'(t) - kR(t) \leq \delta + \epsilon(t - \tau)
\]
since by (2.1) \(r(\tau) \leq \delta\). Multiply both sides of this inequality by \(e^{-k(t-\tau)}\) and integrate the resulting expression from \(\tau\) to \(t\), obtaining
\[
e^{-k(t-\tau)}R(t) \leq \frac{\delta}{k} (1 - e^{-k(t-\tau)}) - \frac{\epsilon}{k^2} e^{-k(t-\tau)}(1 + k(t - \tau)) + \frac{\epsilon}{k^2}
\]
or
\[
R(t) \leq \frac{\delta}{k} (e^{k(t-\tau)} - 1) - \frac{\epsilon}{k^2} (1 + k(t - \tau)) + \frac{\epsilon}{k^2} e^{k(t-\tau)} \quad (2.5)
\]
Combining (2.5) with (2.4), there results finally
\[
r(t) \leq \delta e^{k(t-\tau)} + \frac{\epsilon}{k^2} (e^{k(t-\tau)} - 1)
\]
which is the desired result on \([r, b]\).

A particularly important case of Theorem 2.1 occurs when \(\varphi_1 = \varphi\) is an actual solution of (E). The theorem then shows that as \(\epsilon_1\) and \(\delta \to 0\) the approximate solution tends to the actual solution.

The inequality (2.2) is the best possible, in the sense that equality can be attained for nontrivial \(\varphi_1\) and \(\varphi_2\). For example, let \(k, \epsilon_1, \epsilon_2\) be any real constants, and let \(P_1: (0, \xi_1), P_2: (0, \xi_2)\) be two points in the \((t, x)\) plane. Let \(\varphi_1(0) = \xi_1\), and \(\varphi_2(0) = \xi_2\) and let \(\varphi_1\) and \(\varphi_2\) be solutions of the equations
\[
x' = kx - \epsilon_1 \quad x' = kx + \epsilon_2
\]
respectively, on \([0, 1]\). Then \(\varphi_1\) and \(\varphi_2\) are clearly \(\epsilon_1\)- and \(\epsilon_2\)-approximate solutions of
\[
x' = kx
\]
there. A simple calculation shows that for \(\varphi_1\) and \(\varphi_2\) the equality sign in (2.2) must hold if \(\xi_1 \geq \xi_1\).

Note that, roughly speaking, the inequality says that, if \(\delta\) and \(\epsilon\) are small, then so is \(|\varphi_1(t) - \varphi_2(t)|\). In fact, if \(\delta = \epsilon = 0\), then \(\varphi_1 = \varphi_2\) and there is at most one solution of (E) going through any given point \((r, \xi)\) in \(D\). This proves the following uniqueness result:

Theorem 2.2. Let \(f \in \text{C}(Lip) in D\), and \((r, \xi) \in D\). If \(\varphi_1\) and \(\varphi_2\) are any two solutions of (E) on \((a, b)\), \(a < r < b\), such that \(\varphi_1(r) = \varphi_2(r) = \xi\), then \(\varphi_1 = \varphi_2\).

Actually, in order to obtain uniqueness, it is not necessary to assume as strong a restriction on \(f\) as the Lipschitz condition. However, a more general discussion of the uniqueness problem will be deferred until Chap. 2, Sec. 2.

An existence proof can be based on the inequality (2.2) also.

Theorem 2.3. Suppose \(f \in \text{C}(Lip) on the rectangle \(R\): \(|t - r| \leq \alpha\) \quad |x - \xi| \leq b\) \((a, b > 0)\) and let
\[
M = \max |f(t, x)| \quad ((t, x) \in R)
\]
and
\[
\alpha = \min \left( \frac{\alpha}{M}, \frac{b}{M} \right)
\]
Then there exists a (unique) solution \(\varphi \in \text{C}^1\) of (E) on \(|t - r| \leq \alpha\) for which \(\varphi(r) = \xi\).

Proof. Let \(\{\epsilon_n\}\) be a monotone decreasing sequence of positive real numbers tending to zero as \(n \to \infty\). Choose for each \(\epsilon_n\) an \(\epsilon_n\)-approximate solution \(\varphi_n\). These functions satisfy the relation
\[
\varphi_n(t) = \xi + \int_{\tau}^{t} \left( f(s, \varphi_n(s)) + \Delta_n(s) \right) \, ds \quad (2.6)
\]
where \(\Delta_n(t) = \varphi'(t) - f(t, \varphi_n(t))\) at those points where \(\varphi_n\) exists, and \(\Delta_n(t) = 0\) otherwise. Now \(\Delta_n(t) \to 0\) as \(n \to \infty\), uniformly on \(|t - r| \leq \alpha\), by the very definition of \(\epsilon_n\). From (2.2) applied to \(\varphi_n\) and \(\varphi_n\) one obtains for \(|t - r| \leq \alpha\)
\[
|\varphi_n(t) - \varphi_n(0)| \leq \frac{(\epsilon_n + \epsilon_n)(e^{\alpha} - 1)}{k}
\]
where \(k\) is the Lipschitz constant. Thus the sequence \(\{\varphi_n\}\) is uniformly convergent on \(|t - r| \leq \alpha\), and therefore there exists a continuous limit
3. The Method of Successive Approximations

The existence proof given in Theorem 1.2 is unsatisfactory in one respect in that there is no constructive method given for obtaining a solution of (E). However, as was pointed out after that proof, if the solution through the given point is known to be unique, then the original polygonal approximate solutions can be used to obtain the solution; no subsequent need be chosen. In particular, if $f$ satisfies a Lipschitz condition, the inequality (2.7) gives a bound for the error in using an $e_0$-approximate solution in place of the actual solution. In the following a very useful method, known as the method of successive approximations, will be considered, and the existence of a solution will be deduced with its aid. Here again one can conveniently compute an upper bound on the error involved in stopping the process after a finite number of steps.

The results will be deduced for the case of the rectangle $R$ defined by

$$R: |t - r| \leq \alpha \quad |x - \xi| \leq b$$

where $(r, \xi)$ is some point in the $(t, x)$ plane, and $a > 0, b > 0$. It will be clear that the analogue of Theorem 1.3 also holds.

If $f \in C$ on $R$, then $f$ is bounded there; let $\max |f| = M$ on $R$, and, as before, $\alpha = \min (a, b/M)$. It is clear that a solution $\phi$ of (E) on $|t - r| \leq \alpha$ for which $\phi(r) = \xi$ must satisfy the integral equation

$$\phi(t) = \xi + \int_r^t f(s, \phi(s)) \, ds \quad (|t - r| \leq \alpha)$$  \hspace{1cm} (3.1)

and conversely, if $\phi$ satisfies (3.1), it satisfies (E) and $\phi(r) = \xi$. The

**successive approximations** for (E) are defined to be the functions $\phi_k, \phi_2, \ldots$, given recursively by the formulas

$$\phi_0(t) = \xi$$

$$\phi_{k+1}(t) = \xi + \int_r^t f(s, \phi_k(s)) \, ds \quad (k = 0, 1, 2, \ldots ; |t - r| \leq \alpha)$$  \hspace{1cm} (3.2)

It is shown below that these functions actually exist on $|t - r| \leq \alpha$.

**Theorem 3.1** (Picard–Lindelöf). If $f \in C(Lip)$ on $R$, then the successive approximations $\phi_k$ exist on $|t - r| \leq \alpha$ as continuous functions, and converge uniformly on this interval to the unique solution $\phi$ of (E) such that $\phi(r) = \xi$.

**Proof.** Consider the interval $[r, \tau + \alpha]$; similar arguments hold for $[\tau - \alpha, \tau]$.

It will be shown that every $\phi_k$ exists on $[r, \tau + \alpha], \phi_k \in C^1$ there, and

$$|\phi_k(t) - \xi| \leq M(t - r) \quad (t \in [r, \tau + \alpha])$$  \hspace{1cm} (3.3)

Obviously $\phi_0$, being the constant $\xi$, satisfies these conditions. Assume $\phi_k$ does the same; then $f(t, \phi_k(t))$ is defined and continuous on $[r, \tau + \alpha]$.

From (3.2) this implies $\phi_{k+1}$ exists on $[r, \tau + \alpha], \phi_{k+1} \in C^1$ there, and $|\phi_{k+1}(t) - \xi| \leq M(t - r)$. Therefore these properties are shared by all the $\phi_k$ by induction. Geometrically, this means that all the $\phi_k$ start at $(r, \xi)$ and stay within a triangular region $T$ between the lines

$$z - \xi = \pm M(t - r)$$

and $t = \tau + \alpha$.

It remains to prove the convergence of the $\phi_k$. Let $\Delta_k$ be defined by

$$\Delta_k(t) = |\phi_{k+1}(t) - \phi_k(t)| \quad (t \in [r, \tau + \alpha])$$

Then from (3.2) by subtraction and the fact that $f \in Lip$ on $R$ with some constant $c > 0$,

$$\Delta_k(t) \leq c \int_r^t \Delta_{k-1}(s) \, ds$$  \hspace{1cm} (3.4)

But (3.3) gives for $k = 1$,

$$\Delta_1(t) = |\phi_2(t) - \phi_0(t)| \leq M(t - r)$$

and an easy induction on (3.4) implies that

$$\Delta_k(t) \leq \frac{M^{c^k+1} (t - r)^{k+1}}{c^{k+1}} \quad (t \in [r, \tau + \alpha])$$

This shows that the terms of the series $\sum_{k=0}^n \Delta_k(t)$ are majorized by those of the power series for $(M/c)\xi$, and therefore the series $\sum_{k=0}^n \Delta_k(t)$ is uni-
formally convergent on \([\tau, \tau + \alpha]\). Thus the series

\[
\varphi_0(t) + \sum_{k=0}^{n} (\varphi_{k+1}(t) - \varphi_k(t))
\]

is absolutely and uniformly convergent on \([\tau, \tau + \alpha]\); consequently the partial sum

\[
\varphi_0(t) + \sum_{k=0}^{n-1} (\varphi_{k+1}(t) - \varphi_k(t)) = \varphi_n(t)
\]

tends uniformly on \([\tau, \tau + \alpha]\) to a continuous limit function \(\varphi\).

It will be shown that the function \(\varphi\) satisfies (3.1), and is hence a solution of (E) on \([\tau, \tau + \alpha]\) for which \(\varphi(\tau) = \xi\). Since all the \(\varphi_k\) are within the region \(T\), so is \(\varphi\). Therefore \(f(\xi, \varphi(s))\) exists for \(s \in [\tau, \tau + \alpha]\). Clearly

\[
|\int_{\tau}^{t} [f(\xi, \varphi(s)) - f(\xi, \varphi_k(s))] \, ds| \leq \int_{\tau}^{t} |f(\xi, \varphi(s)) - f(\xi, \varphi_k(s))| \, ds
\]

the latter inequality being due to the fact that \(f \in \text{Lip on } R\). Now

\[
|\varphi(s) - \varphi_k(s)| \to 0 \quad \text{as} \quad k \to \infty \quad \text{uniformly on} \quad [\tau, \tau + \alpha],
\]

and thus the above inequalities show that (3.2) yields (3.1) as \(k \to \infty\).

The solution \(\varphi\) is unique by Theorem 2.2, and this completes the proof.

An upper bound for the error in approximating the solution \(\varphi\) by the \(n\)th approximation \(\varphi_n\) is easily computed. It is given by

\[
|\varphi(t) - \varphi_n(t)| \leq \sum_{k=n}^{\infty} |\varphi_{k+1}(t) - \varphi_k(t)| \leq M \sum_{k=n+1}^{\infty} \frac{c^k(\tau - \tau)^k}{k!}
\]

\[
\leq M \sum_{k=n+1}^{\infty} \frac{(ca)^k}{k!} < M \frac{(ca)^{n+1}}{(n+1)!} \sum_{k=0}^{\infty} \frac{(ca)^k}{k!} = M \frac{(ca)^{n+1}}{(n+1)!} e^{ca}
\]

4. Continuation of Solutions

Suppose that \(f \in C\) in some domain \(D\) of the \((t, x)\) plane and that (E) has a solution \(\varphi\) which exists on a finite interval \((a, b)\) and passes through some point \((\tau, \xi) \in D, a < \tau < b\). If \(|f|\) is bounded by some constant \(M < \infty\) on \(D\), then it is easy to see that both the limits

\[
\varphi(a + 0) = \lim_{t \to a + 0} \varphi(t) \quad \varphi(b - 0) = \lim_{t \to b - 0} \varphi(t)
\]

exist. This follows at once from the fact that

\[
\varphi(t) = \xi + \int_{\tau}^{t} f(\xi, \varphi(s)) \, ds \quad (t \in (a, b))
\]