

We turn now to a careful presentation of the Fourier method for solving boundary value problems involving partial differential equations. Once the basics of the method have been developed, we shall, in Chap. 5, use it to solve a variety of boundary value problems whose solutions entail Fourier series. Then, in subsequent chapters, we shall apply the method to problems with solutions involving other, but closely related, types of representations.

29. LINEAR OPERATORS

If \( u_1 \) and \( u_2 \) are functions and \( c_1 \) and \( c_2 \) are constants, the function \( c_1 u_1 + c_2 u_2 \) is called a linear combination of \( u_1 \) and \( u_2 \). Note that \( u_1 + u_2 \) and \( c_1 u_1 \), as well as the constant function 0, are special cases. A linear space of functions, or function space, is a class of functions, all with a common domain of definition, such that each linear combination of any two functions in that class remains in it; that is, if \( u_1 \) and \( u_2 \) are in the class, then so is \( c_1 u_1 + c_2 u_2 \). An example is the function space \( C_p(a, b) \), introduced in Sec. 1.

A linear operator \( L \) on a given function space transforms each function \( u \) of that space into a function \( Lu \), which need not be in the space, and has the property that for each pair of functions \( u_1 \) and \( u_2 \),

\[
L(c_1 u_1 + c_2 u_2) = c_1 Lu_1 + c_2 Lu_2
\]

whenever \( c_1 \) and \( c_2 \) are constants. In particular,

\[
Lu_1 + Lu_2 = L(u_1 + u_2) \quad \text{and} \quad L(c_1 u_1) = c_1 Lu_1.
\]

The function \( Lu \) may be a constant function; in particular,

\[
L(0) = L(0 \cdot 0) = 0L(0) = 0.
\]

If \( u_3 \) is a third function in the space, then

\[
L(c_1 u_1 + c_2 u_2 + c_3 u_3) = L(c_1 u_1 + c_2 u_2) + L(c_3 u_3) = c_1 Lu_1 + c_2 Lu_2 + c_3 Lu_3.
\]

Proceeding by induction, we find that \( L \) transforms linear combinations of \( N \) functions in this manner:

\[
L \left( \sum_{n=1}^{N} c_n u_n \right) = \sum_{n=1}^{N} c_n Lu_n.
\]

**EXAMPLE 1.** Suppose that both \( u_1 \) and \( u_2 \) are functions of the independent variables \( x \) and \( y \). According to elementary properties of derivatives, a derivative of any linear combination of the two functions can be written as the same linear combination of the individual derivatives. Thus

\[
\frac{\partial}{\partial x}(c_1 u_1 + c_2 u_2) = c_1 \frac{\partial u_1}{\partial x} + c_2 \frac{\partial u_2}{\partial x},
\]

provided that \( \partial u_1 / \partial x \) and \( \partial u_2 / \partial x \) exist. In view of property (4), the class of functions of \( x \) and \( y \) that have partial derivatives of the first order with respect to \( x \) in the \( xy \) plane is a function space. The operator \( \partial / \partial x \) is a linear operator on that space. It is naturally classified as a linear differential operator.

**EXAMPLE 2.** Consider a space of functions \( u(x, y) \) defined on the \( xy \) plane. If \( f(x, y) \) is a fixed function, also defined on the \( xy \) plane, then the operator \( L \) that multiplies each function \( u(x, y) \) by \( f(x, y) \) is a linear operator, where \( Lu = fu \).

If linear operators \( L \) and \( M \), distinct or not, are such that \( M \) transforms each function \( u \) of some function space into a function \( Mu \) to which \( L \) applies and if \( u_1 \) and \( u_2 \) are functions in that space, it follows from equation (1) that

\[
LM(c_1 u_1 + c_2 u_2) = L(c_1 Mu_1 + c_2 Mu_2) = c_1 LMu_1 + c_2 LMu_2.
\]

That is, the product \( LM \) of linear operators is itself a linear operator.

The sum of two linear operators \( L \) and \( M \) is defined by the equation

\[
(L + M)u = Lu + Mu
\]

and is found to be linear by writing

\[
(L + M)(c_1 u_1 + c_2 u_2) = L(c_1 u_1 + c_2 u_2) + M(c_1 u_1 + c_2 u_2) = c_1 (Lu_1 + Mu_1) + c_2 (Lu_2 + Mu_2) = c_1 (L + M)u_1 + c_2 (L + M)u_2.
\]

The sum of any finite number of linear operators is, in fact, linear.
EXAMPLE 3. Let \( L \) denote the linear operator \( \frac{\partial^2}{\partial x^2} \) defined on the space of functions \( u(x, y) \) whose derivatives of the first and second orders with respect to \( x \) exist in a given domain of the \( xy \) plane. The product \( M = f \frac{\partial}{\partial x} \) of the linear operators in Examples 1 and 2 is linear on the same space, and the sum

\[ L + M = \frac{\partial^2}{\partial x^2} + f \frac{\partial}{\partial x} \]

is therefore linear.

30. PRINCIPLE OF SUPERPOSITION

Each nonzero term of a linear homogeneous differential equation in \( u \) consists of a constant or a function of the independent variables alone times one of the derivatives of \( u \) or \( u \) itself. Hence every linear homogeneous differential equation has the form

\[ Lu = 0, \]

where \( L \) is a linear differential operator.

In particular, we recall from Sec. 19 that

\[ Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = 0, \]

where the letters \( A \) through \( F \) denote constants or functions of \( x \) and \( y \) only, is the general second-order linear homogeneous partial differential equation in \( u(x, y) \). It can be written in form (1) when

\[ L = A \frac{\partial^2}{\partial x^2} + B \frac{\partial^2}{\partial y \partial x} + C \frac{\partial^2}{\partial y^2} + D \frac{\partial}{\partial x} + E \frac{\partial}{\partial y} + F. \]

Linear homogeneous boundary conditions also have the form (1). Then the variables appearing as arguments of \( u \) and as arguments of functions that serve as coefficients in the linear operator \( L \) are restricted so that they represent points on the boundary of the domain.

We now state a principle of superposition, which is fundamental to the Fourier method for solving linear boundary value problems.

**Theorem.** Suppose that each function of an infinite set \( u_1, u_2, \ldots \) satisfies a linear homogeneous differential equation or boundary condition \( Lu = 0 \). Then the infinite series

\[ u = \sum_{n=1}^{\infty} c_n u_n, \]

where the \( c_n \) are constants, also satisfies \( Lu = 0 \), provided that the series converges and is differentiable for all derivatives involved in \( L \) and provided that any required continuity condition at the boundary is satisfied by \( Lu \) when \( Lu = 0 \) is a boundary condition.

Superposition is also useful in the theory of ordinary differential equations. For example, from the two solutions \( y = e^x \) and \( y = e^{-x} \) of the linear homogeneous equation \( y'' - y = 0 \), we know that \( y = c_1 e^x + c_2 e^{-x} \) is also a solution. In this book, we shall be concerned mainly with applying the principle of superposition to solutions of partial differential equations.

To prove the theorem, we must deal with the convergence and differentiability of infinite series. Suppose that functions \( u_n \) and constants \( c_n \) are such that series (4) converges to \( u \) throughout some domain of the independent variables, and let \( x \) represent one of those variables. The series is differentiable, or termwise differentiable, with respect to \( x \) if the derivatives \( \partial u_n/\partial x \) and \( \partial u_n/\partial x \) exist and if the series of functions \( c_n \partial u_n/\partial x \) converges to \( \partial u/\partial x \):

\[ \frac{\partial u}{\partial x} = \sum_{n=1}^{\infty} c_n \frac{\partial u_n}{\partial x}. \]

Note that a series must be convergent if it is to be differentiable. If, in addition, series (5) is differentiable with respect to \( x \), then series (4) is differentiable with respect to \( x \).

Let \( L \) be a linear operator where \( Lu \) is a product of a function \( f \) of the independent variables \( u \) or by a derivative of \( u \), or where \( Lu \) is a sum of a finite number of such terms. We now show that if series (4) is differentiable for all the derivatives involved in \( L \) and if each of the functions \( u_n \) in series (4) satisfies the linear homogeneous differential equation \( Lu = 0 \), then series (4) satisfies it.

To accomplish this, we first note that according to the definition of the sum of an infinite series,

\[ f \frac{\partial u}{\partial x} = f \lim_{N \to \infty} \sum_{n=1}^{N} c_n \frac{\partial u_n}{\partial x}, \]

when series (4) is differentiable with respect to \( x \). Thus

\[ f \frac{\partial u}{\partial x} = \lim_{N \to \infty} f \frac{\partial}{\partial x} \sum_{n=1}^{N} c_n u_n. \]

Here the operator \( \partial / \partial x \) can be replaced by other derivatives if the series is so differentiable. Then, by adding corresponding sides of equations similar to equation (6), including one that may not have any derivative at all, we find that

\[ Lu = \lim_{N \to \infty} L \left( \sum_{n=1}^{N} c_n u_n \right). \]

The sum on the right-hand side of equation (7) is a linear combination of the functions \( u_1, u_2, \ldots, u_N \); and if \( Lu_n = 0 \) \((n = 1, 2, \ldots) \), it follows, with the aid of property (3), Sec. 29, that

\[ Lu = \lim_{N \to \infty} \sum_{n=1}^{N} c_n Lu_n = \lim_{N \to \infty} 0 = 0. \]

This is, of course, the desired result.

The above discussion applies as well to linear homogeneous boundary conditions \( Lu = 0 \). In that case, we may require the function \( Lu \) to satisfy a condition
of continuity at points on the boundary so that its values there will represent limiting values as those points are approached from the interior of the domain. This completes the proof of the theorem.

We now illustrate how the superposition theorem is to be used in solving boundary value problems. In our discussion, we shall assume that needed conditions for convergence and differentiability of series are satisfied. We assume, moreover, that any continuity requirements involving boundary conditions are satisfied. The examples here will be used in sections immediately following this one, where two boundary value problems will be completely solved by the Fourier method.

**EXAMPLE 1.** Consider the linear homogeneous heat equation (Sec. 20)

\[ u_t(x, t) = ku_{xx}(x, t) \quad (0 < x < c, t > 0), \tag{8} \]

together with the linear homogeneous boundary conditions

\[ u_t(0, t) = 0, \quad u_t(c, t) = 0 \quad (t > 0). \tag{9} \]

Equation (8) takes the form \( Lu = 0 \) when

\[ L = \frac{\partial^2}{\partial x^2} - \frac{\partial}{\partial t}; \]

and it is straightforward to verify that if

\[ u_0 = 1, \quad u_n = \exp \left( -\frac{n^2\pi^2k}{c^2}t \right) \cos \frac{n\pi x}{c} \quad (n = 1, 2, \ldots), \tag{10} \]

then \( Lu_0 = 0 \) and \( Lu_n = 0 \quad (n = 1, 2, \ldots) \). Thus, by the superposition theorem, \( Lu = 0 \) if \( u \) denotes the infinite series

\[ u = A_0u_0 + \sum_{n=1}^{\infty} A_n u_n, \tag{11} \]

where \( A_n \quad (n = 0, 1, 2, \ldots) \) are constants. In view of expressions (10), then, the series

\[ u(x, t) = A_0 + \sum_{n=1}^{\infty} A_n \exp \left( -\frac{n^2\pi^2k}{c^2}t \right) \cos \frac{n\pi x}{c} \tag{12} \]

satisfies the heat equation (8).

Conditions (9) can, moreover, be written in terms of the operator \( L = \partial / \partial x \) as \( Lu = 0 \), where \( Lu \) is to be evaluated at \( x = 0 \) and at \( x = c \). Hence, by the superposition theorem, series (12) satisfies conditions (9).

This example will be used in Sec. 31, where functions (10) are discovered and where it is shown how the results here can be used to complete the solution of a certain boundary value problem for temperatures in a slab.

**EXAMPLE 2.** It is easy to verify that if \( L \) is the linear operator

\[ L = a^2 \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial t^2} \]

and

\[ y_n = \sin \frac{n\pi x}{c} \cos \frac{n\pi at}{c} \tag{13} \quad (n = 1, 2, \ldots), \]

where \( a \) and \( c \) are positive constants, then \( Ly_n = 0 \quad (n = 1, 2, \ldots) \). Hence it follows from our theorem that \( Ly = 0 \) when \( y \) is the infinite series

\[ y = \sum_{n=1}^{\infty} B_n y_n. \tag{14} \]

That is, the series

\[ y(x, t) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{c} \cos \frac{n\pi at}{c} \tag{15} \]

satisfies the wave equation (Sec. 25)

\[ y_t(x, t) = a^2 y_{xx}(x, t) \quad (0 < x < c, t > 0). \tag{16} \]

Now write \( L = 1 \) and observe that \( Ly_n \quad (n = 1, 2, \ldots) \) has value zero when \( x = 0 \) and when \( x = c \). In view of our superposition theorem, this shows that series (15) also satisfies the boundary conditions

\[ y(0, t) = 0, \quad y(c, t) = 0 \quad (t > 0). \tag{17} \]

On the other hand, if \( L = \partial / \partial t \), each \( Ly_n \quad (n = 1, 2, \ldots) \) has value zero when \( t = 0 \). So, by the superposition theorem, series (15) satisfies the condition

\[ y_t(x, 0) = 0 \quad (0 < x < c). \tag{18} \]

The differential equation (16) and boundary conditions (17) and (18) are part of a boundary value problem for a vibrating string that will be fully solved in Sec. 32, where it is shown how the functions (13) arise.

**PROBLEMS**

1. Show that if an operator \( L \) has the two properties

\[ L(u_1 + u_2) = Lu_1 + Lu_2, \quad L(cu_1) = cLu_1 \]

for all functions \( u_1, u_2 \) in some space and for every constant \( c \), then \( L \) is linear; that is, show that it has property (1), Sec. 29.

2. Use the linear operators \( L = x \) and \( M = \partial / \partial x \) to illustrate the fact that products \( LM \) and \( ML \) are not always the same.

3. Verify that each of the functions

\[ u_0 = y, \quad u_n = \sinh ny \cos nx \quad (n = 1, 2, \ldots) \]

satisfies Laplace's equation

\[ u_{xx}(x, y) + u_{yy}(x, y) = 0 \quad (0 < x < \pi, \ 0 < y < 2) \]

and the three boundary conditions

\[ u_n(0, y) = u_n(\pi, y) = 0, \quad u(x, 0) = 0. \]
Then use the superposition principle in Sec. 30 to show formally, without considering questions of convergence, differentiability, or continuity, that the series
\[ u(x, y) = A_0 y + \sum_{n=1}^{\infty} A_n \sinh n y \cos n x \]
satisfies the same differential equation and boundary conditions.

4. Show that each of the functions
\[ y_1 = \frac{1}{x} \quad \text{and} \quad y_2 = \frac{1}{1 + x} \]
satisfies the nonlinear differential equation
\[ y' + y^2 = 0. \]
Then show that the sum \( y_1 + y_2 \) fails to satisfy that equation. Also show that if \( c \) is a constant, where \( c \neq 0 \) and \( c \neq 1 \), neither \( y_1 \) nor \( y_2 \) satisfies the equation.

5. Let \( u_1 \) and \( u_2 \) satisfy a linear nonhomogeneous differential equation \( Lu = f \), where \( f \) is a function of the independent variables only. Prove that the linear combination \( c_1 u_1 + c_2 u_2 \) fails to satisfy that equation when \( c_1 + c_2 \neq 1 \).

6. Let \( L \) denote a linear differential operator, and suppose that \( f \) is a function of the independent variables. Show that the solutions \( u \) of the equation \( Lu = f \) are of the form \( u = u_1 + u_2 \), where the \( u_1 \) are the solutions of the equation \( Lu_1 = 0 \) and \( u_2 \) is any particular solution of \( Lu_2 = f \). (This is a principle of superposition of solutions for nonhomogeneous differential equations.)

7. Use mathematical induction on the integer \( N \) to verify property (3), Sec. 29, of a linear operator:
\[ L \left( \sum_{n=2}^{N} c_n u_n \right) = \sum_{n=2}^{N} c_n Lu_n. \]
Suggestion: Point out that the property is true when \( N = 1 \) and then show that if it is true when \( N \) is any positive integer \( M \), it must be true for \( N = M + 1 \).

31. A TEMPERATURE PROBLEM

The linear boundary value problem

1. \( u_t(x, t) = ku_{xx}(x, t) \quad (0 < x < c, \ t > 0) \)
2. \( u_x(0, t) = 0, \ u_x(c, t) = 0 \quad (t > 0) \)
3. \( u(x, 0) = f(x) \quad (0 < x < c) \)
is a problem for the temperatures \( u(x, t) \) in an infinite slab of material, bounded by the planes \( x = 0 \) and \( x = c \), if its faces are insulated and the initial temperature distribution is a prescribed function \( f(x) \) of the distance from the face \( x = 0 \). (See Fig. 25.) We assume that the thermal conductivity \( k \) of the material is constant throughout the slab and that no heat is generated within it.

In this section, we illustrate the Fourier method for solving linear boundary value problems by solving the temperature problem just stated. A number of the steps to be taken here are only formal, or manipulative. A verification of the final solution can be found in Chap. 11 (Sec. 95).

To determine nontrivial \( u \neq 0 \) functions that satisfy the homogeneous conditions (1) and (2), we seek separated solutions of those conditions, or functions of the form
\[ u = X(x)T(t) \]
that satisfy them. Note that \( X \) is a function of \( x \) alone and \( T \) is a function of \( t \) alone. Note, too, that \( X \) and \( T \) must be nontrivial \( (X \neq 0, T \neq 0) \).

If \( u = XT \) satisfies equation (1), then
\[ X(x)T'(t) = kX''(x)T(t); \]
and, for values of \( x \) and \( t \) such that the product \( X(x)T(t) \) is nonzero, we can divide by \( kX(x)T(t) \) to separate the variables:
\[ \frac{T'(t)}{kT(t)} = \frac{X''(x)}{X(x)}. \]
Since the left-hand side here is a function of \( t \) alone, it does not vary with \( x \). However, it is equal to a function of \( x \) alone, and so it cannot vary with \( t \). Hence the two sides must have some constant value \( -\lambda \) in common. That is,
\[ \frac{T'(t)}{kT(t)} = -\lambda, \quad \frac{X''(x)}{X(x)} = -\lambda. \]

Our choice of \(-\lambda\), rather than \( \lambda \), for the separation constant is, of course, a minor matter of notation. It is only for convenience later on (Chap. 8) that we have written \(-\lambda\).

This terminology is borrowed from the book by Prins (2003), which is listed in the Bibliography.
If \( u = XT \) is to satisfy the first of conditions (2), then \( X'(0)T(t) \) must vanish for all \( t > 0 \). With our requirement that \( T \neq 0 \), it follows that \( X'(0) = 0 \). Likewise, the second of conditions (2) is satisfied by \( u = XT \) if \( X'(c) = 0 \).

Thus \( u = XT \) satisfies conditions (1) and (2) when \( X \) and \( T \) satisfy these two homogeneous problems:

\[
\begin{align*}
X''(x) + \lambda X(x) &= 0, & X'(0) &= 0, & X'(c) &= 0, \\
T'(t) + \lambda k T(t) &= 0, & \text{where the parameter } \lambda & \text{ has the same value in both problems. To find nontrivial solutions of this pair of problems, we first note that problem (6) has no boundary conditions. Hence it has nontrivial solutions for all values of } \lambda. \text{ Since problem (5) has two boundary conditions, it may have nontrivial solutions only for particular values of } \lambda. \text{ Problem (5) is called a Sturm-Liouville problem. The general theory of such problems is developed in Chap. 8, where it is shown that } \lambda & \text{ must be real-valued in order for there to be nontrivial solutions.}
\end{align*}
\]

If \( \lambda = 0 \), the differential equation in problem (5) becomes \( X''(x) = 0 \). Its general solution is \( X(x) = Ax + B \), where \( A \) and \( B \) are constants. Since \( X''(x) = A \), the boundary conditions \( X'(0) = 0 \) and \( X'(c) = 0 \) require that \( A = 0 \). So \( X(x) = B \); and, except for a constant factor, problem (5) has the solution \( X(x) = 1 \) if \( \lambda = 0 \). Note that any nonzero value of \( B \) might have been selected here.

If \( \lambda > 0 \), we can write \( \lambda = \alpha^2 \) (\( \alpha > 0 \)). The differential equation in problem (5) then takes the form \( X''(x) + \alpha^2 X(x) = 0 \), its general solution being

\[
X(x) = C_1 \cos \alpha x + C_2 \sin \alpha x.
\]

Writing

\[
X'(x) = -C_1 \alpha \sin \alpha x + C_2 \alpha \cos \alpha x
\]

and keeping in mind that \( \alpha \) is positive and, in particular, nonzero, we see that the condition \( X''(0) = 0 \) implies that \( C_2 = 0 \). Also, from the condition \( X'(c) = 0 \), it follows that \( C_1 \alpha \sin \alpha c = 0 \). Now if \( X(x) \) is to be a nontrivial solution of problem (5), \( C_1 \neq 0 \). Hence \( \alpha \) must be a positive root of the equation \( \sin \alpha c = 0 \). That is,

\[
\alpha = \frac{n\pi}{c} \quad (n = 1, 2, \ldots).
\]

So, except for the constant factor \( C_1 \),

\[
X(x) = \cos \frac{n\pi x}{c} \quad (n = 1, 2, \ldots).
\]

If \( \lambda < 0 \), we write \( \lambda = -\alpha^2 \) (\( \alpha > 0 \)). This time, the differential equation in problem (5) has the general solution

\[
X(x) = C_1 e^{\alpha x} + C_2 e^{-\alpha x}.
\]

Since

\[
X'(x) = C_1 \alpha e^{\alpha x} - C_2 \alpha e^{-\alpha x},
\]

the condition \( X''(0) = 0 \) implies that \( C_2 = C_1 \). Hence

\[
X(x) = C_1 (e^{\alpha x} + e^{-\alpha x}).
\]

But the condition \( X'(c) = 0 \) requires that \( C_1 \sin \alpha c = 0 \); and, since \( \sin \alpha c \neq 0 \), it follows that \( C_1 = 0 \). So problem (5) has only the trivial solution \( X(x) = 0 \) if \( \lambda < 0 \).

The values

\[
\lambda_0 = 0, \quad \lambda_n = \left( \frac{n\pi}{c} \right)^2 \quad (n = 1, 2, \ldots)
\]

of \( \lambda \) for which problem (5) has nontrivial solutions are called eigenvalues of that problem, and the solutions

\[
X_0(x) = 1, \quad X_n(x) = \cos \frac{n\pi x}{c} \quad (n = 1, 2, \ldots)
\]

are the corresponding eigenfunctions.

Turning to the differential equation (6), we need to determine its solutions \( T_0(t) \) and \( T_n(t) \) \((n = 1, 2, \ldots)\) corresponding to each of the eigenvalues \( \lambda_0 \) and \( \lambda_n \). These solutions are found to be constant multiples of

\[
T_0(t) = 1, \quad T_n(t) = \exp \left( -\frac{n^2 \pi^2 k}{c^2} t \right) \quad (n = 1, 2, \ldots)
\]

Hence each of the products

\[
u_0 = X_0(t)T_0(t) = 1 \quad \text{and} \quad \nu_n = X_n(t)T_n(t) = \exp \left( -\frac{n^2 \pi^2 k}{c^2} t \right) \cos \frac{n\pi x}{c} \quad (n = 1, 2, \ldots)
\]

satisfies the homogeneous conditions (1) and (2). The procedure just used to obtain them is called the method of separation of variables.

Now, as already shown in Example 1, Sec. 30, the superposition principle in that section tells us that the generalized linear combination

\[
u(x, t) = A_0 + \sum_{n=1}^\infty A_n \exp \left( -\frac{n^2 \pi^2 k}{c^2} t \right) \cos \frac{n\pi x}{c} \quad (n = 1, 2, \ldots)
\]

of the functions (10) and (11) satisfies conditions (1) and (2). The constants \( A_n \) \((n = 0, 1, 2, \ldots)\) in expression (12) are readily obtained from the nonhomogeneous condition (3), namely \( u(x, 0) = f(x) \). More precisely, by writing \( t = 0 \) in expression (12), we have

\[
f(x) = \frac{2A_0}{c} + \sum_{n=1}^\infty A_n \cos \frac{n\pi x}{c} \quad (0 < x < c)
\]

Since this is a Fourier cosine series on \( 0 < x < c \) (see Sec. 14), it follows that

\[
A_0 = \frac{1}{c} \int_0^c f(x) \, dx
\]
and
\[ A_n = \frac{2}{c} \int_0^c f(x) \cos \frac{nx}{c} \, dx \quad (n = 1, 2, \ldots). \]

The formal solution of our temperature problem is now complete. It consists of expression (12) together with coefficients (13) and (14). Note that the steady-state temperatures, occurring when \( t \) tends to infinity, are \( A_n \). That constant temperature is evidently the mean, or average, value of the initial temperatures \( f(x) \) over the interval \( 0 < x < c \).

**EXAMPLE.** Suppose that the thickness \( c \) of the slab is unity and that the initial temperatures are \( f(x) = x \) \((0 \leq x \leq 1)\). Here
\[ A_n = \frac{1}{2} \int_0^1 x \, dx = \frac{1}{2}. \]
Using integration by parts, or Kronecker’s method (Sec. 5), and observing that
\[ \sin n\pi = 0 \quad \text{and} \quad \cos n\pi = (-1)^n, \]
when \( n \) is an integer, we find that
\[ A_n = \frac{1}{2} \int_0^1 x \cos nx \, dx = \frac{1}{2} \left[ \frac{x \sin nx}{n} + \frac{\cos nx}{n^2} \right]_0^1 = \frac{1}{2} \frac{(-1)^n - 1}{n^2} \quad (n = 1, 2, \ldots). \]

When \( c = 1 \) and these values for \( A_n \) \((n = 0, 1, 2, \ldots)\) are used, expression (12) becomes
\[ u(x, t) = \frac{1}{2} + \frac{2}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^n - 1}{n^2} \exp(-n^2\pi^2kt) \cos nx, \]
or
\[ u(x, t) = \frac{1}{2} - \frac{4}{\pi^2} \sum_{n=1}^{\infty} \frac{\exp(-(2n-1)^2\pi^2kt)}{(2n-1)^2} \cos(2n-1)x. \]

**PROBLEMS**

1. In Problem 3, Sec. 30, the functions
\[ u_0 = y, \quad u_n = \sinh ny \cos nx \quad (n = 1, 2, \ldots) \]
were shown to satisfy Laplace’s equation
\[ u_{xx}(x, y) + u_{yy}(x, y) = 0 \quad (0 < x < \pi, \ 0 < y < 2) \]
and the homogeneous boundary conditions
\[ u_t(0, y) = u_0(x, y) = 0, \quad u_t(x, 0) = 0. \]

After writing \( u = X(x)T(t) \) and separating variables, use the solutions of the Sturm-Liouville problem in Sec. 31 to show how the functions \( u_0 \) and \( u_n \) \((n = 1, 2, \ldots)\) can be discovered. Then, by proceeding formally, derive the following solution of the boundary value problem resulting when the nonhomogeneous condition \( u(x, 2) = f(x) \) is included:
\[ u(x, y) = A_0 y + \sum_{n=1}^{\infty} A_n \sinh ny \cos nx, \]
where
\[ A_n = \frac{1}{2\pi} \int_0^\pi f(x) \, dx, \quad A_0 = \frac{2}{\pi} \int_0^\pi f(x) \cos nx \, dx \quad (n = 1, 2, \ldots). \]

2. Suppose that in Sec. 31 we had written
\[ \frac{T'(t)}{T(t)} = k \frac{X''(x)}{X(x)} \quad \text{instead of} \quad \frac{T'(t)}{T(t)} = k \frac{X''(x)}{X(x)}. \]
Continuing with
\[ \frac{T'(t)}{T(t)} = k \frac{X''(x)}{X(x)} = -\lambda, \]
where \( \lambda \) is a separation constant, show how the functions \( X \) and \( T \) in Sec. 31 still follow. (This illustrates how it is generally simpler to keep the physical constants out of the Sturm-Liouville problem, as we did in Sec. 31.)

3. For each of the following partial differential equations in \( u = u(x, t) \), determine if it is possible to write \( u = X(x)T(t) \) and separate variables to obtain ordinary differential equations in \( X \) and \( T \). If it can be done, find those ordinary differential equations.
   (a) \( uu_t - xu_{xx} = 0 \);
   (b) \( (x + t)u_{xx} - u_t = 0 \);
   (c) \( xu_{xx} + u_t + uu_{xx} = 0 \);
   (d) \( uu_t - u_{xx} - u_t = 0 \).

**32. A VIBRATING STRING PROBLEM**

To illustrate further the Fourier method, we now consider a boundary value problem for displacements in a vibrating string. This time, the nonhomogeneous condition will require us to expand the function \( f(x) \) into a sine series, rather than a cosine series.

Let us find an expression for the transverse displacements \( y(x, t) \) in a string, stretched between the points \( x = 0 \) and \( x = c \) on the \( x \) axis and without external forces acting along it, if the string is initially displaced into a position \( y = f(x) \) and released at rest from that position. The function \( y(x, t) \) must satisfy the wave equation (Sec. 25)
\[ y_{tt}(x, t) = a^2 y_{xx}(x, t) \quad (0 < x < c, \ t > 0). \]
It must also satisfy the boundary conditions
\[ y(0, t) = 0, \quad y(c, t) = 0, \quad y_t(x, 0) = 0. \]
(3) \[ y(x, 0) = f(x) \quad (0 \leq x \leq c), \]

where the prescribed displacement function \( f \) is continuous on the interval \( 0 \leq x \leq c \) and where \( f(0) = f(c) = 0 \).

We assume a product solution

(4) \[ y = X(x)T(t) \]

of the homogeneous conditions (1) and (2) and substitute it into those conditions. This leads to the two homogeneous problems

(5) \[ X''(x) + \lambda X(x) = 0, \quad X(0) = 0, \quad X(c) = 0, \]

(6) \[ T''(t) + \lambda a^2 T(t) = 0, \quad T'(0) = 0. \]

Problem (5) is another instance of a Sturm-Liouville problem. The method of solution that was used to solve the one in Sec. 31 can be applied here. It turns out (Problem 2) that the eigenvalues are

(7) \[ \lambda_n = \left( \frac{m \pi}{c} \right)^2 \quad (n = 1, 2, \ldots) \]

and that the corresponding eigenfunctions are

(8) \[ X_n(x) = \sin \frac{n \pi x}{c} \quad (n = 1, 2, \ldots). \]

When \( \lambda = \lambda_n \), problem (6) becomes

\[ T''(t) + \left( \frac{mn a^2}{c} \right)^2 T(t) = 0, \quad T'(0) = 0; \]

and it follows that except for a constant factor, the solution is

(9) \[ T_n(t) = \cos \frac{mn a t}{c} \quad (n = 1, 2, \ldots). \]

Consequently, each of the products

(10) \[ y_n = X_n(x)T_n(t) = \sin \frac{n \pi x}{c} \cos \frac{mn a t}{c} \quad (n = 1, 2, \ldots) \]

satisfies the homogeneous conditions (1) and (2).

According to Example 2 in Sec. 30, the generalized linear combination

(11) \[ y(x, t) = \sum_{n=1}^{\infty} B_n \sin \frac{n \pi x}{c} \cos \frac{mn a t}{c} \]

also satisfies the homogeneous conditions (1) and (2), provided that the constants \( B_n \) can be restricted so that the infinite series is suitably convergent and differentiable. That series will satisfy the nonhomogeneous condition (3) if the \( B_n \) are such that

(12) \[ f(x) = \sum_{n=1}^{\infty} B_n \sin \frac{n \pi x}{c} \quad (0 < x < c). \]

Because representation (12) is a Fourier sine series representation on the interval \( 0 < x < c \), we know from Sec. 14 that

(13) \[ B_n = \frac{2}{c} \int_0^c f(x) \sin \frac{n \pi x}{c} \, dx \quad (n = 1, 2, \ldots). \]

The formal solution of our boundary value problem is, therefore, series (11) with coefficients (13). Note that it converges to zero at the endpoints \( x = 0 \) and \( x = c \) of the interval \( 0 < x < c \).

**EXAMPLE.** Suppose that the string has length \( c = 2 \) and that its midpoint is initially raised to a height \( h \) above the horizontal axis. The rest position from which the string is released thus consists of two line segments (Fig. 26).

![Figure 26](image)

The function \( f \), which describes the initial position of this plucked string, is given by the equations

(14) \[ f(x) = \begin{cases} kh & \text{when } 0 \leq x < 1, \\ -h(x - 2) & \text{when } 1 < x \leq 2. \end{cases} \]

and the coefficients \( B_n \) in the Fourier sine series for that function on the interval \( 0 < x < 2 \) can be written

\[ B_n = \frac{2}{c} \int_0^c f(x) \sin \frac{n \pi x}{c} \, dx = h \int_0^1 x \sin \frac{n \pi x}{c} \, dx - h \int_1^2 (x - 2) \sin \frac{n \pi x}{c} \, dx. \]

After integrating by parts, or using Kronecker's method (Sec. 5), and simplifying, we find that

\[ B_n = \frac{8h}{\pi^2} \left( \frac{1}{n^2} \sin \frac{n \pi}{2} \right) \quad (n = 1, 2, \ldots). \]

Series (11) then becomes

(15) \[ y(x, t) = \frac{8h}{\pi^2} \sum_{n=1}^{\infty} \sin \frac{n \pi x}{2} \sin \frac{n \pi a t}{2} \cos \frac{n \pi}{2}. \]

Since

\[ \sin \frac{n \pi}{2} = 0 \]

when \( n \) is even and since

\[ \sin \left( \frac{(2n-1)\pi}{2} \right) = -\cos \pi = (-1)^{n+1} \quad (n = 1, 2, \ldots), \]
expression (15) for the displacements of points on the string in question can also be written

\[ y(x, t) = \sum_{n=1}^{\infty} \frac{8h}{\pi^2 n^2} \frac{(-1)^{n+1}}{(2n-1)^2} \sin \left( \frac{(2n-1)\pi x}{2} \right) \cos \left( \frac{(2n-1)\pi t}{2} \right). \]

### PROBLEMS

1. By assuming a product solution \( y = X(x)T(t) \), obtain conditions (5) and (6) on \( X \) and \( T \) in Sec. 32 from the homogeneous conditions (1) and (2) of the string problem there.

2. Derive the eigenvalues and eigenfunctions, stated in Sec. 32, of the Sturm-Liouville problem

\[ X''(x) + \lambda X(x) = 0, \quad X(0) = 0, \quad X(c) = 0. \]

3. Point out how it follows from expression (11), Sec. 32, that, for each fixed \( x \), the displacement function \( y(x, t) \) is periodic in \( t \) with period

\[ T_0 = \frac{2c}{a}. \]

### 33. HISTORICAL DEVELOPMENT

Mathematical sciences experienced a burst of activity following the invention of calculus by Newton (1642–1727) and Leibnitz (1646–1716). Among topics in mathematical physics that attracted the attention of great scientists during that period were boundary value problems in vibrations of strings, elastic bars, and columns of air, all associated with mathematical theories of musical vibrations. Early contributors to the theory of vibrating strings included the English mathematician Brook Taylor (1685–1731), the Swiss mathematicians Daniel Bernoulli (1700–1782) and Leonhard Euler (1707–1783), and Jean d’Alembert (1717–1783) in France.

By the 1750s d’Alembert, Bernoulli, and Euler had advanced the theory of vibrating strings to the stage where the wave equation \( \frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2} \) was known and a solution of a boundary value problem for strings had been found from the general solution of that equation. Also, the concept of fundamental modes of vibration led those men to the notion of superposition of solutions, to a solution of the form (11), Sec. 32, where a series of trigonometric functions appears, and thus to the matter of representing arbitrary functions by trigonometric series. Euler later found expressions for the coefficients in those series. But the general concept of a function had not been clarified, and a lengthy controversy took place over the question of representing arbitrary functions on a bounded interval by such series. The question of representation was finally settled by the German mathematician Peter Gustav Lejeune Dirichlet (1805–1859) about 70 years later.

The French mathematical physicist Jean Baptiste Joseph Fourier (1768–1830) presented many instructive examples of expansions in trigonometric series in connection with boundary value problems in the conduction of heat. His book *Théorie analytique de la chaleur* published in 1822, is a classic in the theory of heat conduction. It was actually the third version of a monograph that he originally submitted to the Institut de France on December 21, 1807. He effectively illustrated the basic procedures of separation of variables and superposition, and his work did much toward arousing interest in trigonometric series representations.

But Fourier’s contributions to the representation problem did not include conditions of validity; he was interested in applications and methods. As noted above, Dirichlet was the first to give such conditions. In 1829 he firmly established general conditions on a function sufficient to ensure that it can be represented by a series of sine and cosine functions.\(^1\)

Representation theory has been refined and greatly extended since the time of Dirichlet. It is still growing.

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\(^1\) See the footnote with the example in Sec. 31.

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\(^2\) For supplementary reading on the history of these series, see the articles by Langer (1947) and Van Vleck (1914), listed in the Bibliography.