The first section here is devoted to a description of a class of functions that is central to the theory of Fourier series.

1. PIECEWISE CONTINUOUS FUNCTIONS

Let a function \( f \) be continuous at all points of a bounded open interval \( a < x < b \) except possibly for a finite set of points \( x_1, x_2, \ldots, x_{n-1} \), where
\[
a < x_1 < x_2 < \cdots < x_{n-1} < b.
\]

If we write \( x_0 = a \) and \( x_n = b \), then \( f \) is continuous on each of the \( n \) open subintervals
\[
x_0 < x < x_1, \quad x_1 < x < x_2, \quad \ldots, \quad x_{n-1} < x < x_n.
\]

It is not necessarily continuous, or even defined, at their endpoints. But if in each of those subintervals, \( f \) has finite limits as \( x \) approaches the endpoints from the interior, \( f \) is said to be piecewise continuous on the interval \( a < x < b \). More precisely, the one-sided limits
\[
(1) \quad f(x_{k-1}+) = \lim_{x \to x_{k-1}^+} f(x) \quad \text{and} \quad f(x_k-) = \lim_{x \to x_k^-} f(x) \quad (k = 1, 2, \ldots, n)
\]
are required to exist.

Note that if the limiting values from the interior of a subinterval are assigned to \( f \) at the endpoints, then \( f \) is continuous on the closed subinterval. Since any function that is continuous on a closed bounded interval is bounded, it follows that \( f \) is bounded on the entire interval \( a \leq x \leq b \). That is, there exists a nonnegative number \( M \) such that \( |f(x)| \leq M \) for all points \( x \) \((a \leq x \leq b)\) at which \( f \) is defined.

**EXAMPLE 1.** Consider the function \( f \) that has the values
\[
f(x) = \begin{cases} 
  x & \text{when } 0 < x < 1, \\
  -1 & \text{when } 1 \leq x < 2, \\
  1 & \text{when } 2 < x < 3,
\end{cases}
\]
and \( f(3) = 0 \). (See Fig. 1.) Although \( f \) is discontinuous at the points \( x = 1 \) and \( x = 2 \) in the interval \( 0 < x < 3 \), it is nevertheless piecewise continuous on
that interval. This is because the one-sided limits from the interior exist at the endpoints of each of the three open subintervals on which \( f \) is continuous. Note, for instance, that the right-hand limit at \( x = 0 \) is \( f(0+) = 0 \) and that the left-hand limit at \( x = 1 \) is \( f(1-) = 1 \).

A function is piecewise continuous on an interval \( a < x < b \) if it is continuous on the closed interval \( a \leq x \leq b \). Continuity on the open interval \( a < x < b \) does not, however, imply piecewise continuity there, as Example 2 illustrates.

**EXAMPLE 2.** The function \( f(x) = \frac{1}{x} \) is continuous on the interval \( 0 < x < 1 \), but it is not piecewise continuous there since \( f(0+) \) fails to exist.

When a function \( f \) is piecewise continuous on an interval \( a < x < b \), the integral of \( f(x) \) from \( x = a \) to \( x = b \) always exists. It is the sum of the integrals of \( f(x) \) over the open subintervals on which \( f \) is continuous:

\[
\int_a^b f(x) \, dx = \int_a^{x_0} f(x) \, dx + \int_{x_0}^{x_1} f(x) \, dx + \cdots + \int_{x_n}^b f(x) \, dx.
\]

The first integral on the right exists since it is defined as the integral over the interval \( a < x \leq x_0 \) of the continuous function whose values are \( f(x) \) when \( a < x < x_0 \) and whose values at the endpoints \( x = a \) and \( x = x_0 \) are \( f(a+) \) and \( f(x_0-) \), respectively. The remaining integrals on the right in equation (2) are similarly defined and therefore exist.

**EXAMPLE 3.** If \( f \) is the function in Example 1 and Fig. 1, then

\[
\int_0^3 f(x) \, dx = \int_0^1 x \, dx + \int_1^2 (-1) \, dx + \int_2^3 1 \, dx = \frac{1}{2} - 1 + 1 = 1.
\]

Observe that the value of the integral of \( f(x) \) over each subinterval is unaffected by the values of \( f \) at the endpoints. The function is, in fact, not even defined at \( x = 0 \) and \( x = 2 \).

If two functions \( f_1 \) and \( f_2 \) are each piecewise continuous on an interval \( a < x < b \), then there is a finite subdivision of the interval such that both functions are continuous on each closed subinterval when the functions are given their limiting values from the interior at the endpoints. Hence linear combinations \( c_1 f_1 + c_2 f_2 \) and products \( f_1 f_2 \) have that continuity on each subinterval and are themselves piecewise continuous on the interval \( a < x < b \). The integrals

\[
\int_a^b [c_1 f_1(x) + c_2 f_2(x)] \, dx \quad \text{and} \quad \int_a^b f_1(x) f_2(x) \, dx
\]

must then exist.

We refer to the class of all piecewise continuous functions defined on an interval \( a < x < b \) as a *function space* and denote it by \( C_p(a, b) \). It is analogous to three-dimensional space, where linear combinations of vectors are well-defined vectors in that space. The analogy will be developed further in Chap. 7.

In this book we shall restrict our attention to functions that are piecewise continuous on bounded intervals; and the notion of piecewise continuity clearly applies regardless of whether the interval is open or closed.

### 2. FOURIER COSINE SERIES

Let \( f \) be any function in \( C_p(0, \pi) \) and assume for the moment that \( f(x) \) has a Fourier cosine series representation

\[
f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \quad (0 < x < \pi),
\]

where \( a_0 \) and \( a_n \) \((n = 1, 2, \ldots)\) are constants. To find these constants, we also assume that series \( (1) \) and any related series that arises can be integrated term by term.

The constant \( a_0 \) is easily found by integrating each side of equation \( (1) \) from \( 0 \) to \( \pi \) and writing

\[
\int_0^\pi f(x) \, dx = \frac{a_0}{2} \int_0^\pi dx + \sum_{n=1}^{\infty} a_n \int_0^\pi \cos nx \, dx,
\]

or

\[
\int_0^\pi f(x) \, dx = \frac{a_0}{2} \pi + \sum_{n=1}^{\infty} a_n \left[ \frac{\sin nx}{n} \right]_0^\pi.
\]

Inasmuch as \( \sin n\pi = 0 \) when \( n \) is an integer, this shows that

\[
a_0 = \frac{2}{\pi} \int_0^\pi f(x) \, dx.
\]

To find \( a_n \) \((n = 1, 2, \ldots)\), we write equation \( (1) \) as

\[
f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \quad (0 < x < \pi),
\]

with a new index of summation, and then multiply each side by \( \cos nx \), where \( n \) is any fixed positive integer. Integration of the resulting equation from 0 to \( \pi \) yields

\[
\int_0^\pi f(x) \cos nx \, dx = \frac{a_0}{2} \int_0^\pi \cos nx \, dx + \sum_{n=1}^{\infty} a_n \int_0^\pi \cos nx \cos nx \, dx.
\]

But

\[
\int_0^\pi \cos nx \cos mx \, dx = \begin{cases} 0 & \text{when } m \neq n, \\ \pi/2 & \text{when } m = n. \end{cases}
\]

and (see Problem 8, Sec. 5)

\[
\int_0^\pi \cos mx \cos nx \, dx = \begin{cases} 0 & \text{when } m \neq n, \\ \pi/2 & \text{when } m = n. \end{cases}
\]
Hence
\[ \int_0^\pi f(x) \cos nx \, dx = \frac{a_n \pi}{2}, \]
or
\[ a_n = \frac{2}{\pi} \int_0^\pi f(x) \cos nx \, dx \quad (n = 1, 2, \ldots). \]

Note that expression (2) for \( a_0 \) can be included with expression (3) when the integer \( n \) is allowed to run from \( n = 0 \), rather than from \( n = 1 \). This is the reason why \( a_0/2 \) was used instead of \( a_0 \) in series (1). Note, too, that \( a_0/2 \) is the mean, or average, value of \( f(x) \) over the interval \( 0 < x < \pi \).

Because we cannot be certain at this time that representation (1) is actually valid for a specific \( f \), we write
\[ f(x) \sim \frac{a_0}{2} + \sum_{n=1}^\infty a_n \cos nx \quad (0 < x < \pi), \]

where the tilde symbol \( \sim \) merely denotes correspondence. Observe that correspondence (4), with coefficients (2) and (3), can be written more compactly as
\[ f(x) \sim \frac{1}{\pi} \int_0^\pi f(x) \cos nx \, dx \]
\[ \sum_{n=1}^\infty \frac{2}{\pi} \int_0^\pi f(x) \cos nx \, dx \, ds, \]
where \( s \) is used for the variable of integration in order to distinguish it from the free variable \( x \).

The fact that \( f \) is piecewise continuous on the interval \( 0 < x < \pi \) ensures the existence of the integrals in expressions (2) and (3) for the coefficients in a cosine series. We shall, in Chap. 2, establish further conditions on \( f \) under which series (4) converges to \( f(x) \) when \( 0 < x < \pi \), in which case correspondence (4) becomes an equality.

If \( f \) is defined on the interval \( 0 \leq x \leq \pi \) and series (4) converges to \( f(x) \) for all \( x \) in that interval, the series also converges to the effective periodic extension, with period \( 2\pi \), of \( f \) on the entire \( x \) axis. That is, it converges to a function \( F(x) \) having the properties
\[ F(x) = f(x) \quad \text{when} \quad 0 \leq x \leq \pi \]
and
\[ F(-x) = F(x), \quad F(x + 2\pi) = F(x) \quad \text{for all} \ x. \]

The reason for this is that each term in series (4) is itself even and periodic with period \( 2\pi \). The graph of the extension \( y = F(x) \) is obtained by reflecting the graph of \( y = f(x) \) in the \( y \) axis, to give a graph for the interval \( -\pi \leq x \leq \pi \), and then repeating that graph on the intervals \( \pi 
\leq \ldots \leq -\pi \), \( 3\pi \leq \ldots \leq 5\pi \), etc., as well as on the intervals \( -3\pi \leq \ldots \leq -\pi \), \( -5\pi \leq \ldots \leq -3\pi \), etc. It follows from these observations that if one is given a function \( f \) that is both even and periodic with period \( 2\pi \), then the cosine series corresponding to \( f(x) \) on the interval \( 0 \leq x < \pi \) represents \( f(x) \) for all \( x \) when that series converges to it on the interval \( 0 \leq x \leq \pi \). Clearly, a cosine series cannot represent a function \( f(x) \) for all \( x \) if \( f(x) \) is not both even and periodic with period \( 2\pi \).

3. EXAMPLES

Examples 1 and 2 here illustrate the material in Sec. 2.

**EXAMPLE 1.** Let us find the coefficients in the Fourier cosine series correspondence
\[ f(x) \sim \frac{a_0}{2} + \sum_{n=1}^\infty a_n \cos nx \quad (0 < x < \pi) \]
when \( f(x) = x \) \((0 < x < \pi)\). It is easy to see that
\[ a_0 = \frac{2}{\pi} \int_0^\pi x \, dx = \pi; \]
and, using integration by parts, we find that
\[ a_n = \frac{2}{\pi} \int_0^\pi \frac{x}{n} \cos nx \, dx = \frac{2}{\pi} \int_0^{\pi} \left( \frac{\sin nx}{n} - \frac{1}{n^2} \int_0^\pi \sin nx \, dx \right) \, dx \quad (n = 1, 2, \ldots). \]
Since
\[ \sin nx = 0 \quad \text{and} \quad \cos nx = (-1)^n \]
when \( n \) is an integer, this reduces to
\[ a_n = \frac{2}{\pi} \frac{(-1)^n}{n^2} - 1 \quad (n = 1, 2, \ldots). \]
Note that \( a_0 \) needed to be found separately in order to avoid division by zero.

For the function \( f(x) \) here, correspondence (1) evidently becomes
\[ x \sim \frac{\pi}{2} + \frac{2}{\pi} \sum_{n=1}^\infty \frac{(-1)^n - 1}{n^2} \cos nx \quad (0 < x < \pi). \]
Since \( (-1)^n - 1 = 0 \) when \( n \) is even, the series can be written more efficiently by summing only the terms that occur when \( n \) is odd. This is accomplished by replacing \( n \) by \( 2n - 1 \) wherever it appears after the summation symbol and in order starting the summation from \( n = 1 \). The result is
\[ x \sim \frac{\pi}{2} + \frac{4}{\pi} \sum_{n=1}^\infty \cos((2n-1)x) \quad (0 < x < \pi). \]

Conditions in Sec. 13 (Chap. 2) will ensure that correspondence (2) is actually an equality when \( 0 \leq x \leq \pi \). The even periodic extension to which the series converges is shown in Fig. 2, which tells us that
\[ |x| = \frac{\pi}{2} + \frac{4}{\pi} \sum_{n=1}^\infty \cos((2n-1)x) \quad (-\pi \leq x \leq \pi). \]
EXAMPLE 2. In this example, we shall find the Fourier cosine series for the function \( f(x) = \sin x \) on the interval \( 0 < x < \pi \). The trigonometric identity

\[
2 \sin A \cos B = \sin(A + B) + \sin(A - B)
\]

enables us to write

\[
a_n = \frac{2}{\pi} \int_0^\pi \sin x \cos nx \, dx = \frac{1}{\pi} \int_0^\pi [\sin(1 + n)x + \sin(1 - n)x] \, dx \quad (n = 0, 1, 2, \ldots).
\]

Hence, when \( n \neq 1 \),

\[
a_n = \frac{1}{\pi} \left[ \left. \frac{-\cos(1 + n)x}{1 + n} - \frac{\cos(1 - n)x}{1 - n} \right|_0^\pi \right] = \frac{2}{\pi} \cdot \frac{1 + (-1)^n}{1 - n^2};
\]

and when \( n = 1 \), the coefficient is

\[
a_1 = \frac{1}{\pi} \int_0^\pi \sin 2x \, dx = 0.
\]

The desired cosine series correspondence is, then,

\[
\sin x \sim \frac{2}{\pi} + \frac{2}{\pi} \sum_{n=2}^\infty \frac{1 + (-1)^n}{1 - n^2} \cos nx \quad (0 < x < \pi).
\]

Observe that \( 1 + (-1)^n = 0 \) when \( n \) is odd (compare with Example 1). To sum the terms occurring when \( n \) is even, we replace \( n \) by \( 2n \) in this correspondence and write

\[
\sin x \sim \frac{2}{\pi} + \frac{4}{\pi} \sum_{n=1}^\infty \frac{\cos 2nx}{4n^2 - 1} \quad (0 < x < \pi).
\]

The function \( \sin x \) will, in fact, satisfy conditions in Sec. 13 ensuring that the correspondence here is an equality for each value of \( x \) in the interval \( 0 < x < \pi \). Thus, at each point on the \( x \) axis, the series converges to the even periodic extension, with period \( 2\pi \), of \( \sin x \) \( (0 \leq x \leq \pi) \). That extension, shown in Fig. 3, is the function \( y = |\sin x| \).

4. FOURIER SINE SERIES

We assume here that when \( f \) is in \( C_p(0, \pi) \), there is a Fourier sine series representation

\[
f(x) = \sum_{n=1}^\infty b_n \sin nx \quad (0 < x < \pi),
\]

where the coefficients \( b_n \) \( (n = 1, 2, \ldots) \) are constants. The \( b_n \) can be found in a way similar to that used in Sec. 2 to find the coefficients in a cosine series. This time we write

\[
f(x) = \sum_{m=1}^\infty b_m \sin mx \quad (0 < x < \pi)
\]

and multiply each side by \( \sin nx \), where \( n \) is any fixed positive integer. Assuming that term-by-term integration is valid, we find that

\[
\int_0^\pi f(x) \sin nx \, dx = \sum_{m=1}^\infty b_m \int_0^\pi \sin mx \sin nx \, dx.
\]

Then, because (Problem 9, Sec. 5)

\[
\int_0^\pi \sin mx \sin nx \, dx = \begin{cases} \pi/2 & \text{when } m \neq n, \\ \pi & \text{when } m = n,
\end{cases}
\]

this reduces to

\[
\int_0^\pi f(x) \sin nx \, dx = b_n \frac{\pi}{2}.
\]

That is,

\[
b_n = \frac{2}{\pi} \int_0^\pi f(x) \sin nx \, dx \quad (n = 1, 2, \ldots).
\]

Inasmuch as we have only assumed the validity of representation (1), we use the tilde symbol \( \sim \), as we did in Sec. 2, to denote correspondence:

\[
f(x) \sim \sum_{n=1}^\infty b_n \sin nx \quad (0 < x < \pi).
\]

Expression (2) can, of course, be used to put this correspondence in the form

\[
f(x) \sim \frac{2}{\pi} \sum_{n=1}^\infty b_n \sin nx \int_0^\pi f(s) \sin ns \, ds.
\]
Suppose that \( f(x) \) is defined on the open interval \( 0 < x < \pi \) and that series (3) converges to \( f(x) \) there. Since series (3) clearly converges to zero when \( x = 0 \) and \( x = \pi \), it converges to \( f(x) \) for all \( x \) in the closed interval \( 0 \leq x \leq \pi \) if \( f \) is assigned the values \( f(0) = 0 \) and \( f(\pi) = 0 \). Remarks similar to those in Sec. 2, regarding cosine series, show that series (3) then converges to the odd periodic extension, with period \( 2\pi \), of \( f \) for all values of \( x \). This time, the extension is the function \( F(x) \) defined by means of the equations

\[
F(x) = f(x) \quad \text{when } 0 \leq x \leq \pi
\]

and

\[
F(-x) = -F(x), \quad F(x + 2\pi) = F(x) \quad \text{for all } x.
\]

The extension \( F \) is odd and periodic with period \( 2\pi \) since each term \( b_n \sin nx \) in series (3) has those properties. The graph of \( y = F(x) \) is symmetric with respect to the origin and can be obtained by first reflecting the graph of \( y = f(x) \) in the \( x \) axis, then reflecting the result in the \( y \) axis, and finally repeating the graph found for the interval \(-\pi \leq x \leq \pi \) every \( 2\pi \) units along the entire \( x \) axis. Evidently, a Fourier sine series on the interval \( 0 < x < \pi \) can also be used to represent a given function that is defined for all \( x \) and is both odd and periodic with period \( 2\pi \), provided that the representation is valid when \( 0 \leq x \leq \pi \).

5. EXAMPLES

We now illustrate some methods for finding Fourier sine series.

**EXAMPLE 1.** For the sine series corresponding to the function \( f(x) = x \) on the interval \( 0 < x < \pi \) (Fig. 4), we refer to expression (2), Sec. 4, and use integration by parts to write

\[
b_n = \frac{2}{\pi} \int_0^\pi x \sin nx \, dx = \frac{2}{n} \left( \frac{x \cos nx}{n} \right)_0^\pi + \frac{1}{n} \int_0^\pi \cos nx \, dx = \frac{2(-1)^{n+1}}{n} \quad (n = 1, 2, \ldots)
\]

For \( x^3 \) on the interval \( 0 < x < \pi \), we may write

\[
b_n = \frac{2}{\pi} \int_0^\pi x^3 \sin nx \, dx
\]

\[
= \frac{2}{n} \left( \frac{x^3 \cos nx}{n^2} \right)_0^\pi - \frac{3x^2}{n^2} \left( \frac{x \sin nx}{n} \right)_0^\pi + \left( \frac{6x^4 \cos nx}{n^3} \right)_0^\pi - \left( \frac{3^2 x^2 \sin nx}{n^4} \right)_0^\pi
\]

\[
= 2(-1)^{n+1} \frac{(nx)^2 - 6}{n^3} \quad (n = 1, 2, \ldots)
\]

Hence

\[
x^3 \sim \sum_{n=1}^\infty (-1)^{n+1} \frac{(nx)^2 - 6}{n^3} \sin nx \quad (0 < x < \pi).
\]

Thus

\[
x \sim \sum_{n=1}^\infty (-1)^{n+1} \frac{\sin nx}{n} \quad (0 < x < \pi).
\]

Our theory will show that the series converges to \( f(x) \) when \( 0 < x < \pi \). Hence it converges to the odd periodic function \( y = F(x) \) that is graphed in Fig. 4. The fact that the series converges to zero when \( x = 0, \pm \pi, \pm 3\pi, \pm 5\pi, \ldots \) is in agreement with our theory, which will tell us that it must converge to the mean value of the one-sided limits (Sec. 1) of \( f(x) \) at each of those points.

In the evaluation of integrals representing Fourier coefficients, it is sometimes necessary to apply integration by parts more than once. We now give an example in which this can be accomplished by means of a single formula due to L. Kronecker (1823–1891). We preface the example with a statement of that formula.

Let \( p(x) \) be a polynomial of degree \( m \), and suppose that \( f(x) \) is continuous. Then, except for an arbitrary additive constant,

\[
\int_0^\pi p(x) f(x) \, dx = pF_1 - pF_2 + p^2F_3 - \cdots + (-1)^mp^{(m)}F_{m+1}
\]

where \( p \) is successively differentiated until it becomes zero, where \( F_1 \) denotes an indefinite integral of \( f \), where \( F_2 \) is an indefinite integral of \( F_1 \), etc., and where alternating signs are affixed to the terms. Note that the differentiation of \( p \) begins with the second term, whereas the integration of \( f \) begins with the first term. The formula, which is readily verified by differentiating its right-hand side to obtain \( p(x)f(x) \), could even have been used to evaluate the integral in Example 1, where only one integration by parts was needed.

**EXAMPLE 2.** To illustrate the advantage of formula (2) when successive integration by parts is required, let us find the Fourier sine series for the function \( f(x) = x^3 \) on the interval \( 0 < x < \pi \). With the aid of that formula, we may write

\[
b_n = \frac{2}{\pi} \int_0^\pi x^3 \sin nx \, dx
\]

\[
= \frac{2}{n} \left( \frac{x^3 \cos nx}{n^2} \right)_0^\pi - \frac{3}{n^2} \left( \frac{x^2 \sin nx}{n} \right)_0^\pi + \left( \frac{6x^2 \cos nx}{n^3} \right)_0^\pi - \left( \frac{3^2 x \sin nx}{n^4} \right)_0^\pi
\]

\[
= 2(-1)^{n+1} \frac{(nx)^2 - 6}{n^3} \quad (n = 1, 2, \ldots)
\]

Hence

\[
x^3 \sim \sum_{n=1}^\infty (-1)^{n+1} \frac{(nx)^2 - 6}{n^3} \sin nx \quad (0 < x < \pi).
\]

\[\text{Kronecker actually treated the problem more extensively in papers that originally appeared in the } \textit{Berlin Sitzungsberichte} (1885, 1889).\]
As was the case in Example 1, the series converges to the given function on the interval $0 < x < \pi$. Since $x^2$ is an odd function whose value is zero when $x = 0$, this series represents $x^2$ on the larger interval $-\pi < x < \pi$ too.

We conclude this section by pointing out a computational aid that is useful in finding the coefficients $b_n$ ($n = 1, 2, \ldots$) in the Fourier sine series for a linear combination $c_1 f_1(x) + c_2 f_2(x)$ of two functions $f_1(x)$ and $f_2(x)$ whose sine series are already known. Namely, since the expression

$$b_n = \frac{2}{\pi} \int_0^\pi [c_1 f_1(x) + c_2 f_2(x)] \sin nx \, dx$$

can be written

$$b_n = c_1 \left[ \frac{2}{\pi} \int_0^\pi f_1(x) \sin nx \, dx \right] + c_2 \left[ \frac{2}{\pi} \int_0^\pi f_2(x) \sin nx \, dx \right],$$

it is clear that each $b_n$ is simply the linear combination of the $n$th coefficients in the sine series for the individual functions $f_1(x)$ and $f_2(x)$. Such an observation applies as well in finding coefficients in cosine and other types of series encountered in this and later chapters.

**EXAMPLE 3.** In view of the sine series for $x$ and $x^2$ found in Examples 1 and 2, respectively, the coefficients $b_n$ in the sine series corresponding to the function

$$f(x) = x(\pi^2 - x^2) = \pi^2 x - x^3 \quad (0 < x < \pi)$$

are

$$b_n = \pi^2 \left[ 2 \frac{(-1)^{n+1}}{n} - \left( \frac{2}{\pi} \frac{(2\pi)^2 - 6}{n^3} \right) \right] \quad (n = 1, 2, \ldots).$$

Thus

$$x(\pi^2 - x^2) \sim 12 \sum_{n=1}^\infty \frac{(-1)^{n+1}}{n^3} \sin nx \quad (0 < x < \pi).$$

**PROBLEMS**

For the functions $f$ in Problems 1 through 3, find (a) the Fourier cosine series and (b) the Fourier sine series on the interval $0 < x < \pi$.

1. $f(x) = 1 \quad (0 < x < \pi)$.
   
   **Answers:** (a) 1; (b) $\sum_{n=1}^\infty \frac{2}{n\pi} \frac{1 - (-1)^n}{n} \sin nx = \sum_{n=1}^\infty \frac{\sin(2n-1)x}{2n-1}$.

2. $f(x) = \pi - x \quad (0 < x < \pi)$.
   
   **Answers:** (a) $\frac{\pi}{2} + \sum_{n=1}^\infty \frac{\cos(2n-1)x}{(2n-1)^2}$; (b) $2 \sum_{n=1}^\infty \frac{\sin nx}{n}.$

3. $f(x) = x^2 \quad (0 < x < \pi)$.
   
   **Answers:** (a) $\frac{x^2}{3} + \sum_{n=1}^\infty \frac{(-1)^n}{n^2} \cos nx$; (b) $2\pi \sum_{n=1}^\infty \frac{(-1)^{n+1}}{n \pi} \frac{1 - (-1)^n}{(n\pi)^2} \sin nx$.

4. Find the Fourier cosine series on the interval $0 < x < \pi$ that corresponds to the function $f$ defined by the equations

$$f(x) = \begin{cases} 1 & \text{when } 0 < x < \frac{\pi}{2}, \\ 0 & \text{when } \frac{\pi}{2} < x < \pi. \end{cases}$$

   **Suggestion:** Note that

$$a_n = \frac{2}{\pi} \int_0^{\pi/2} \cos nx \, dx \quad (n = 0, 1, 2, \ldots)$$

and that

$$\sin \frac{(2n-1)x}{2} = \frac{\sin nx \cos \frac{x}{2} - \cos nx \sin \frac{x}{2}}{2} = (-1)^{n+1} \quad (n = 1, 2, \ldots).$$

**Answers:** $\sum_{n=1}^\infty \frac{(-1)^{n+1}}{2n-1} \cos(2n-1)x.$

5. By referring to the sine series for $x$ in Example 1, Sec. 5, and the one found for $x^2$ in Problem 3(b) above, show that

$$x(\pi - x) \sim \frac{1}{\pi} \sum_{n=1}^\infty \frac{\sin(2n-1)x}{(2n-1)^3} \quad (0 < x < \pi).$$

6. Show that

$$x^2 \sim \frac{\pi^4}{5} + \frac{2}{\pi} \sum_{n=1}^\infty \frac{(-1)^n n^2 - 6}{n^3} \cos nx \quad (0 < x < \pi).$$

   **Given that this correspondence is actually an equality when $0 \leq x \leq \pi$, sketch the function represented by the series for all $x$.**

7. Verify Kronecker's formula (2), Sec. 5.

8. Use the trigonometric identity

$$2 \cos A \cos B = \cos(A - B) + \cos(A + B)$$

   to show that

$$\int_0^{\pi/2} \cos nx \sin nx \, dx = \begin{cases} 0 & \text{when } m \neq n, \\ \pi/2 & \text{when } m = n. \end{cases}$$

   **where $m$ and $n$ are positive integers.**

9. Use the trigonometric identity

$$2 \sin A \sin B = \cos(A - B) - \cos(A + B)$$

   to show that

$$\int_0^{\pi/2} \sin nx \sin nx \, dx = \begin{cases} 0 & \text{when } m \neq n, \\ \pi/2 & \text{when } m = n. \end{cases}$$

   **where $m$ and $n$ are positive integers.**
10. With the aid of the integration formula obtained in Problem 9, find the Fourier sine series corresponding to the function \( f(x) = \sin x \) on the interval \( 0 < x < \pi \).

Answer: \( \sin x \).

6. FOURIER SERIES

Consider a function \( f \) in \( C_p(-\pi, \pi) \) and write

\[
f(x) = g(x) + h(x),
\]

where

\[
g(x) = \frac{f(x) + f(-x)}{2}, \quad \text{and} \quad h(x) = \frac{f(x) - f(-x)}{2}.
\]

The function \( g(x) \) is evidently even, and \( h(x) \) is odd. That is,

\[
g(-x) = g(x) \quad \text{and} \quad h(-x) = -h(x)
\]

for each point \( x \) in the interval \(-\pi < x < \pi\) at which these functions are defined.

According to Secs. 2 and 4,

\[
g(x) \sim a_0 + \sum_{n=1}^{\infty} a_n \cos nx \quad (0 < x < \pi),
\]

where

\[
a_n = \frac{2}{\pi} \int_{0}^{\pi} g(x) \cos nx \, dx \quad (n = 0, 1, 2, \ldots),
\]

and

\[
h(x) \sim \sum_{n=1}^{\infty} b_n \sin nx \quad (0 < x < \pi),
\]

where

\[
b_n = \frac{2}{\pi} \int_{0}^{\pi} h(x) \sin nx \, dx \quad (n = 1, 2, \ldots).
\]

When correspondence (3) is an equality that is valid for \( 0 < x < \pi \), the equation also holds on the interval \(-\pi < x < 0\) since each side of the correspondence is an even function. A similar remark applies to correspondence (5) since each side there is an odd function. Because \( f(x) \) is the sum of \( g(x) \) and \( h(x) \), this suggests that the correspondence

\[
f(x) \sim a_0/2 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \quad (\pi < x < \pi)
\]

may be an equality under certain circumstances.

In view of the first of equations (2), expression (4) for the coefficients \( a_n \) can be written

\[
a_n = \frac{1}{\pi} \left[ \int_{0}^{\pi} f(x) \cos nx \, dx + \int_{0}^{\pi} f(-x) \cos nx \, dx \right].
\]

By making the substitution \( x = -s \) in the second of these two integrals, we find that

\[
a_n = \frac{1}{\pi} \left[ \int_{0}^{\pi} f(x) \cos nx \, dx + \int_{-\pi}^{0} f(s) \cos ns \, ds \right],
\]

or

\[
a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, dx \quad (n = 0, 1, 2, \ldots).
\]

Likewise,

\[
b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx \quad (n = 1, 2, \ldots).
\]

Correspondence (7), when combined with expressions (8) and (9) for the constants \( a_n \) and \( b_n \), becomes

\[
f(x) \sim \frac{1}{2\pi} \int_{-\pi}^{\pi} f(s) \, ds + \frac{1}{\pi} \sum_{n=1}^{\infty} \int_{-\pi}^{\pi} f(s) \cos ns \, ds + \sin nx \int_{-\pi}^{\pi} f(s) \sin ns \, ds.
\]

The trigonometric identity

\[
\cos(A - B) = \cos A \cos B + \sin A \sin B
\]

then enables us to write the correspondence in the form

\[
f(x) \sim \frac{1}{2\pi} \int_{-\pi}^{\pi} f(s) \, ds + \frac{1}{\pi} \sum_{n=1}^{\infty} \int_{-\pi}^{\pi} f(s) \cos n(s - x) \, ds.
\]

Note that the term

\[
\frac{1}{2\pi} \int_{-\pi}^{\pi} f(s) \, ds
\]

here, which is the same as the term \( a_0/2 \) in series (7), is the mean, or average, value of \( f(x) \) over the interval \(-\pi < x < \pi\).

The form (10) of correspondence (7) will be the starting point of the proof in Sec. 12 of our theorem ensuring the convergence of the Fourier series to \( f(x) \) on the interval \(-\pi < x < \pi\).

Series (7), with coefficients (8) and (9), is the Fourier series corresponding to \( f(x) \) on the interval \(-\pi < x < \pi\). Suppose that the series converges to \( f(x) \)
when $-\pi < x < \pi$. Then, in view of the periodicity of its terms, it converges to a function $y = F(x)$ that coincides with $y = f(x)$ on $-\pi < x < \pi$ and whose graph is repeated every $2\pi$ units along the $x$ axis. The function $F(x)$, therefore, is the periodic extension, with period $2\pi$, of $f$. If, on the other hand, $f$ is a given periodic function, with period $2\pi$, series (7) represents $f(x)$ everywhere where it converges to $f(x)$ on the interval $-\pi < x < \pi$. It may be that the given function $f$ in $C_p(-\pi, \pi)$ is even on the interval $-\pi < x < \pi$. That is, $f(-x) = f(x)$ for all such values of $x$. Then
\[ f(-x) \cos(-nx) = f(x) \cos nx \quad (n = 0, 1, 2, \ldots) \]
and
\[ f(-x) \sin(-nx) = -f(x) \sin nx \quad (n = 1, 2, \ldots) \]
when $-\pi < x < \pi$; and we see that $f(x) \cos nx$ and $f(x) \sin nx$ are even and odd, respectively. Because the graph of $y = f(x) \cos nx$ is symmetric with respect to the $y$ axis and the graph of $y = f(x) \sin nx$ is symmetric with respect to the origin, it follows that expressions (8) and (9) reduce to
\[ a_n = \frac{2}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, dx \quad (n = 0, 1, 2, \ldots) \]
and
\[ b_n = 0 \quad (n = 1, 2, \ldots) \]
for $f(x)$ on the interval $0 < x < \pi$. Series (7) thus becomes a Fourier cosine series (Sec. 2). Similarly, if $f$ is odd on the interval $-\pi < x < \pi$, it follows from expressions (8) and (9) that $a_n = 0$ ($n = 0, 1, 2, \ldots$) and
\[ b_n = \frac{2}{\pi} \int_{0}^{\pi} f(x) \sin nx \, dx \quad (n = 1, 2, \ldots) \]
In this case, series (7) becomes a Fourier sine series (Sec. 4) for the function $f(x)$ on $0 < x < \pi$.

7. EXAMPLES

We include here three examples of Fourier series on the interval $-\pi < x < \pi$ that illustrate points made in Sec. 6.

EXAMPLE 1. Let us find the Fourier series corresponding to the function $f(x)$ that is defined on the fundamental interval $-\pi < x < \pi$ as follows:

\[ f(x) = \begin{cases} 0 & \text{when } -\pi < x \leq 0, \\ x & \text{when } 0 < x < \pi. \end{cases} \]

The graph of $y = f(x)$ is indicated by the bold line segments in Fig. 5 that are solid.

According to expression (8), Sec. 6,
\[ a_n = \frac{1}{\pi} \left( \int_{-\pi}^{0} 0 \cos nx \, dx + \int_{0}^{\pi} x \cos nx \, dx \right) = \frac{1}{\pi} \int_{0}^{\pi} x \cos nx \, dx \quad (n = 0, 1, 2, \ldots) \]

This series will be shown to converge to $f(x)$ on the fundamental interval, as well as to the periodic extension $F(x)$ that is indicated in Fig. 5, where the graph of $y = F(x)$ is sketched. As in Example 1, Sec. 5, the series must converge to the mean value of the one-sided limits of the periodic extension at each of the discontinuities $x = \pm \pi, \pm 3\pi, \pm 5\pi, \ldots$. Here the mean values are all $\pi/2$.

EXAMPLE 2. The function $f(x) = |\sin x| (-\pi < x < \pi)$ is even. Hence the Fourier series corresponding to $f(x)$ on the interval $-\pi < x < \pi$ is actually the Fourier cosine series for the function
\[ f(x) = |\sin x| = |\sin x| \quad (0 < x < \pi). \]

That series has already been found in Example 2, Sec. 3; and, by referring to correspondence (3) there, we see that

\[ |\sin x| \sim \frac{2}{\pi} - \frac{4}{\pi} \sum_{n=1}^{\infty} \cos 2nx \quad (-\pi < x < \pi). \]
EXAMPLE 3. Since the function \( f(x) = x (-\pi < x < \pi) \) is odd, the Fourier series for \( f \) on \(-\pi < x < \pi\) is simply the Fourier sine series for that function on \( 0 < x < \pi \). Hence correspondence (1) in Example 1, Sec. 5, is also a correspondence on the larger interval \(-\pi < x < \pi\):

\[
x \sim 2 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin nx \quad (-\pi < x < \pi).
\]

Similarly, correspondence (3) in Example 2, Sec. 5, can be written

\[
x^3 \sim 2 \sum_{n=1}^{\infty} \frac{(-1)^{n+1} (\pi n)^2 - 6}{n^3} \sin nx \quad (-\pi < x < \pi).
\]

PROBLEMS

Find the Fourier series on the interval \(-\pi < x < \pi\) that corresponds to each of the functions in Problems 1 through 6.

1. \( f(x) = \begin{cases} -\pi/2 & \text{when } -\pi < x < 0, \\ \pi/2 & \text{when } 0 < x < \pi. \end{cases} \)
   Answer: \( 2 \sum_{n=1}^{\infty} \frac{\sin(2n-1)x}{2n-1} \).

2. \( f(x) \) is the function such that the graph of \( y = f(x) \) consists of the two bold line segments shown in Fig. 6.
   Answer: \( 2 + 2 \sum_{n=1}^{\infty} \left[ \frac{1}{(n\pi)^2} \cos nx + \frac{(-1)^{n+1}}{n\pi} \sin nx \right] \).

3. \( f(x) = x + \frac{1}{4} x^3 (-\pi < x < \pi) \).
   Suggestion: Use the series (4) for \( x \) in Example 3, Sec. 7, and the one for \( x^3 \) in Problem 3(a), Sec. 5.
   Answer: \( \pi^2/12 + \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \left( \frac{\cos nx}{n^2} - \frac{2 \sin nx}{n} \right) \).

4. \( f(x) = e^{ax} (-\pi < x < \pi) \), where \( a \neq 0 \).
   Suggestion: Use Euler's formula \( e^{ax} = \cos \theta + i \sin \theta \), where \( i = \sqrt{-1} \), to write
   \[
a_n + ib_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) e^{inx} \, dx \quad (n = 1, 2, \ldots).
   \]
   Then, after evaluating this single integral, equate real parts and imaginary parts.
   Answer: \( \sin nx = \frac{2 \sinh nx}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n}{a^2 + n^2} (a \cos nx - n \sin nx) \).

5. \( f(x) = \cosh ax (-\pi < x < \pi) \), where \( a \neq 0 \).
   Suggestion: Use the series found in Problem 4.
   Answer: \( \sinh nx = \frac{2 a^2}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n}{a^2 + n^2} \cos nx \).

6. \( f(x) = \cos ax (-\pi < x < \pi) \), where \( a \neq 0, \pm 1, \pm 2, \ldots \).
   Suggestion: With the aid of Euler's formula, stated in the suggestion with Problem 4, write
   \[
   \cos ax = \frac{e^{ax} + e^{-ax}}{2}.
   \]
   Then use the series already obtained in that earlier problem.
   Answer: \( \frac{2 \sin ax}{\pi} \left[ \frac{1}{2a^2} + \sum_{n=1}^{\infty} \frac{(-1)^n}{a^2 + n^2} \cos nx \right] \).

7. Find the Fourier series on the interval \(-\pi < x < \pi\) for the function \( f \) defined by the equations
   \[
   f(x) = \begin{cases} 0 & \text{when } -\pi \leq x \leq 0, \\ \sin x & \text{when } 0 \leq x \leq \pi. \end{cases}
   \]
   Then, given that the series converges to \( f(x) \) when \(-\pi < x \leq \pi\), describe graphically the function that is represented by the series for all \( x \) \((-\infty < x < \infty)\).
   Suggestion: To find the series, write the function in the form
   \[
   f(x) = \frac{\sin x + |\sin x|}{2} (-\pi \leq x \leq \pi)
   \]
   and then use the results in Problem 10, Sec. 5, and Example 2, Sec. 7.
   Answer: \( \frac{\pi}{4} + \frac{1}{2} \sin x - \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{\cos 2nx}{4n^2 - 1} \).

8. ADAPTATIONS TO OTHER INTERVALS

Let \( f \) denote a piecewise continuous function of \( x \) on an interval \(-c < x < c\) of the \( x \) axis, and define the related function

\[
g(s) = f\left(\frac{cs}{\pi}\right) \quad (-\pi < s < \pi)
\]

\[\text{For a justification of Euler's formula and background on complex-variable methods, see the authors' book (1904), listed in the Bibliography.}\]
of \( s \). The Fourier series corresponding to this new function on \(-\pi < s < \pi\) is, according to Sec. 6,

\[
f(\frac{c s}{\pi}) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} \left( a_n \cos\frac{n\pi s}{c} + b_n \sin\frac{n\pi s}{c} \right) \quad (-\pi < s < \pi)
\]

where

\[
a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(\frac{c s}{\pi}) \cos\frac{n\pi s}{c} ds \quad (n = 0, 1, 2, \ldots)
\]

and

\[
b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(\frac{c s}{\pi}) \sin\frac{n\pi s}{c} ds \quad (n = 1, 2, \ldots).
\]

The function (1) is evidently also piecewise continuous, and we anticipate that correspondence (2) will become an equality when certain further conditions are imposed on \( f \). Thus if we put

\[
s = \frac{n\pi x}{c}
\]

in correspondence (2) and its conditions of validity, we arrive at the series

\[
f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} \left( a_n \cos\frac{n\pi x}{c} + b_n \sin\frac{n\pi x}{c} \right) \quad (-c < x < c).
\]

That same substitution in expressions (3) and (4), moreover, enables us to write

\[
a_n = \frac{1}{c} \int_{-c}^{c} f(x) \cos\frac{n\pi x}{c} dx \quad (n = 0, 1, 2, \ldots)
\]

and

\[
b_n = \frac{1}{c} \int_{-c}^{c} f(x) \sin\frac{n\pi x}{c} dx \quad (n = 1, 2, \ldots).
\]

Series (5) is a Fourier series on the fundamental interval \(-c < x < c\) and becomes series (7) in Sec. 6 when \( c = \pi \). Conditions on \( f \) ensuring that correspondence (5) is, in fact, an equality at points where \( f \) is continuous will be given in Chap. 2. Note that if the series does converge to \( f(x) \) on \(-c < x < c\), the graph of \( y = f(x) \) is repeated every \( 2c \) units along the \( x \) axis.

Arguments similar to those used above lead to Fourier cosine and sine series on \( 0 < x < c \):

\[
f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} \frac{a_n}{c} \cos\frac{n\pi x}{c} \quad (0 < x < c),
\]

where

\[
a_n = \frac{2}{c} \int_{0}^{c} f(x) \cos\frac{n\pi x}{c} dx \quad (n = 0, 1, 2, \ldots),
\]

and

\[
f(x) \sim \sum_{n=1}^{\infty} \frac{b_n}{c} \sin\frac{n\pi x}{c} \quad (0 < x < c),
\]

where

\[
b_n = \frac{2}{c} \int_{0}^{c} f(x) \sin\frac{n\pi x}{c} dx \quad (n = 1, 2, \ldots).
\]

The convergence of series (8) and (10) is also treated in Chap. 2.

The following example illustrates how Fourier series on intervals \(-c < x < c\), as well as cosine and sine series on \( 0 < x < c \), can be obtained from known series on \(-\pi < x < \pi\) and \( 0 < x < \pi \). Since we do not yet have theorems ensuring the convergence of Fourier series to the functions in question, we shall continue to use the tilde symbol \( \sim \) to denote mere correspondence and not necessarily equality. Also, anticipating that the correspondences obtained will actually be equalities, we shall continue to include conditions of validity.

**EXAMPLE.** It is a simple matter to obtain the correspondence

\[
x^2 \sim \frac{\pi^2}{3} + 4 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos\frac{n\pi x}{c} \quad (0 < x < c)
\]

from the known one [Problem 5(a), Sec. 5]

\[
x^2 \sim \frac{\pi^2}{3} + 4 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos\frac{n\pi x}{c} \quad (0 < x < \pi).
\]

We let \( x \) be any number in the interval \( 0 < x < c \) and note how it follows that

\[
0 < \frac{n\pi x}{c} < \pi.
\]

Hence it is legitimate to replace \( x \) by \( \frac{n\pi x}{c} \) in correspondence (13) and its condition of validity:

\[
\frac{\pi^2 x^2}{c^2} \sim \frac{\pi^2}{3} + 4 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos\frac{n\pi x}{c} \quad (0 < \frac{n\pi x}{c} < \pi).
\]

Then, multiplying each side by \( c^2/\pi^2 \) and multiplying through the new condition of validity by \( c/\pi \), we arrive at correspondence (12).

Expression (9) could, of course, have been used to find the desired coefficients if correspondence (13) had not been available.
PROBLEMS

1. (a) Use the Fourier sine series in Example 1, Sec. 5, for

\[ f(x) = x \quad (0 < x < \pi) \]

to show that

\[ x \sim \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin n\pi x \quad (0 < x < 1). \]

(b) Obtain the correspondence in part (a) by using expression (11), Sec. 8, for the coefficients in a Fourier sine series on \(0 < x < c\).

2. Show how it follows from the expansions obtained in Problem 1 and the example in Sec. 8 that

\[ x(1 + x) \sim \frac{1}{3} + \frac{2}{\pi} \sum_{n=1}^{\infty} \left( \frac{2}{n\pi} \cos n\pi x - \frac{1}{n} \sin n\pi x \right) \quad (0 < x < 1). \]

3. Use the Fourier sine series found in Problem 3(b), Sec. 5, for

\[ f(x) = x^2 \quad (0 < x < \pi) \]

to obtain the correspondence

\[ x^2 \sim 2c^2 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n\pi} \left[ \frac{2}{n\pi} \cos n\pi x - \frac{1}{n} \sin n\pi x - \frac{1}{n\pi} \right] \quad (0 < x < c). \]

4. (a) Use the Fourier sine series correspondence found in Example 3, Sec. 5, for the function

\[ f(x) = x(\pi^2 - x^2) \quad (0 < x < \pi) \]

to establish the correspondence

\[ x(1 - x^2) \sim \frac{32}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^3} \sin n\pi x \quad (0 < x < 1). \]

(b) Replace \( x \) by \( 1 - x \) on each side of the correspondence in part (a) to show that

\[ x(x-1)(x-2) \sim \frac{12}{\pi^2} \sum_{n=1}^{\infty} \frac{\sin n\pi x}{n^3} \quad (0 < x < 1). \]

5. Show how it follows from the Fourier sine series obtained for

\[ f(x) = x(\pi - x) \quad (0 < x < \pi) \]

in Problem 5, Sec. 5, that

\[ x(2c - x) \sim \frac{32c^2}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} \sin \frac{(2n-1)\pi x}{2c} \quad (0 < x < 2c). \]

6. Use the Fourier series for

\[ f(x) = e^{\pm x} \quad (-\pi < x < \pi), \]

where \( a \neq 0 \), that was found in Problem 4, Sec. 7, to show that

\[ e^x \sim \frac{\sin c}{c} + 2 \sin c \sum_{n=1}^{\infty} \frac{(-1)^n}{c^2 + (n\pi)^2} \left( \frac{c}{n\pi} \cos \frac{n\pi x}{c} - \frac{n\pi x}{c} \sin \frac{n\pi x}{c} \right) \quad (-c < x < c). \]

7. By starting with the Fourier cosine series correspondence obtained for the function

\[ f(x) = \pi - x \quad (0 < x < \pi) \]

in Problem 2(a), Sec. 5, show that

\[ c - x \sim \frac{2c}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} \cos \frac{(4n-2)\pi x}{c} \quad \left(0 < x < \frac{c}{2}\right). \]

8. Use expression (11), Sec. 8, for the coefficients in a Fourier sine series on \(0 < x < c\) to obtain the correspondence

\[ \cos n\pi x \sim \frac{8}{\pi} \sum_{n=1}^{\infty} \frac{n}{4n^2 - 1} \sin 2n\pi x \quad (0 < x < 1). \]

Suggestion: To evaluate the integrals that arise, recall the trigonometric identity

\[ 2 \sin A \cos B = \sin(A + B) + \sin(A - B). \]

9. Show that in Sec. 8 the Fourier series (5), with coefficients (6) and (7), can be written in the compact form

\[ \frac{1}{2c} \int_{-\infty}^{\infty} f(x) \, ds + \frac{1}{c} \sum_{n=1}^{\infty} \int_{-\infty}^{\infty} f(s) \cos \frac{n\pi}{c} (s - x) \, ds. \]

(See Sec. 6, where this form was obtained when \( c = \pi \).)
In this chapter, we shall establish conditions on a function \( f(x) \), defined on the interval \(-\pi < x < \pi\), that ensure a valid Fourier series representation. Corresponding results for Fourier cosine and sine series representations will follow readily. It will be a simple matter to extend the theory to Fourier series on arbitrary intervals \(-c < x < c\), as well as to Fourier cosine and sine series on intervals \(0 < x < c\). Some further aspects of the theory of convergence of Fourier series will be touched on later in the chapter.

9. ONE-SIDED DERIVATIVES

In developing sufficient conditions on a function \( f \) such that its Fourier series on the interval \(-\pi < x < \pi\) converges to \( f(x) \) there, we need to generalize the concept of the derivative

\[
\left. f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} \right. 
\]

of \( f \) at a point \( x = x_0 \).

Suppose that the right-hand limit \( f(x_0 +) \) exists at \( x_0 \) (see Sec. 1). The right-hand derivative of \( f \) at \( x_0 \) is defined as follows:

\[
\left. f'_r(x_0) = \lim_{x \to x_0^+} \frac{f(x) - f(x_0)}{x - x_0} \right. 
\]

provided that the limit here exists. Note that although \( f(x_0) \) need not exist, \( f(x_0 +) \) must exist if \( f'_r(x_0) \) does. When the ordinary, or two-sided, derivative \( f'(x_0) \) exists, \( f \) is continuous at \( x_0 \) and \( f'_r(x_0) = f'(x_0) \).

Similarly, if \( f(x_0 -) \) exists, the left-hand derivative of \( f \) at \( x_0 \) is given by the equation

\[
f'_l(x_0) = \lim_{x \to x_0^-} \frac{f(x) - f(x_0 -)}{x - x_0}
\]

when this limit exists; and if \( f'(x_0) \) exists, \( f'_l(x_0) = f'(x_0) \).

EXAMPLE 1. Let \( f \) denote the continuous function defined by the equations

\[
f(x) = \begin{cases} 
  x^2 & \text{when } x \leq 0, \\
  \sin x & \text{when } x > 0.
\end{cases}
\]

With the aid of l'Hôpital's rule, we see that

\[
f'_r(0) = \lim_{x \to 0^+} \frac{\sin x}{x} = 1;
\]

furthermore,

\[
f'_l(0) = \lim_{x \to 0^-} \frac{x^2}{x} = \lim_{x \to 0} x = 0.
\]

Since these one-sided derivatives have different values, the ordinary derivative \( f'(0) \) cannot exist.

The ordinary derivative \( f'(x_0) \) can fail to exist even when \( f(x_0) \) is defined and \( f'_r(x_0) \) and \( f'_l(x_0) \) have a common value.

EXAMPLE 2. If \( f \) is the step function

\[
f(x) = \begin{cases} 
  0 & \text{when } x < 0, \\
  1 & \text{when } x \geq 0,
\end{cases}
\]

then \( f'_r(0) = f'_l(0) = 0 \). But the derivative \( f'(0) \) does not exist since \( f \) is not continuous at \( x = 0 \).

As is the case with ordinary derivatives, the mere continuity of \( f \) at a point \( x_0 \) does not ensure the existence of either one-sided derivative there.

EXAMPLE 3. The function \( f(x) = \sqrt{x} (x \geq 0) \) has no right-hand derivative at the point \( x = 0 \), although it is continuous there.

A number of properties of ordinary derivatives remain valid for one-sided derivatives. Suppose, for instance, that the right-hand derivatives of two functions \( f \) and \( g \) exist at a point \( x_0 \). Let us find the right-hand derivative of the product \( (fg)(x) = f(x)g(x) \).
at \( x_0 \). Since the difference quotient
\[
\frac{(fg)(x) - (fg)(x_0 +)}{x - x_0} = \frac{f(x)g(x) - f(x_0)g(x_0)}{x - x_0}
\]
can be written
\[
f(x) \frac{g(x) - g(x_0 +)}{x - x_0} + f(x_0 +) \frac{g(x) - g(x_0)}{x - x_0} \frac{g(x_0 +)}{x - x_0},
\]
it follows that
\[
(fg)'(x_0) = f(x_0 +)g'(x_0) + f'(x_0)g(x_0 +).
\]

Likewise, if \( f_1(x_0) \) and \( g_1(x_0) \) exist, the left-hand derivative of the product \((fg)(x_0)\) exists at \( x_0 \).

Finally, we turn to a property of one-sided derivatives that is particularly important in the theory of convergence of Fourier series. It concerns the subspace \( C_p(a, b) \) consisting of all piecewise continuous functions \( f \) on an interval \( a < x < b \) whose derivatives \( f' \) are also piecewise continuous on that interval. Such a function is said to be piecewise smooth because, over the subintervals on which both \( f \) and \( f' \) are continuous, any tangents to the graph of \( y = f(x) \) that turn do so continuously.

**Theorem.** If a function \( f \) is piecewise smooth on an interval \( a < x < b \), then at each point \( x_0 \) in the closed interval \( a \leq x \leq b \) the one-sided derivatives of \( f \), from the interior at the endpoints, exist and are the same as the corresponding one-sided limits of \( f' \):
\[
f'_L(x_0) = f'(x_0 +), \quad f'_R(x_0) = f'(x_0 -) .
\]

(4)

To prove this, we assume for the moment that \( f \) and \( f' \) are actually continuous on the interval \( a < x < b \) and that the one-sided limits of \( f \) and \( f' \) from the interior exist at the endpoints \( x = a \) and \( x = b \). If \( x_0 \) is a point in this open interval, \( f'(x_0) \) exists. Hence \( f'_L(x_0) \) and \( f'_R(x_0) \) exist, and both are equal to \( f'(x_0) \). Because \( f' \) is continuous at \( x_0 \), then equations (4) hold.

The following argument shows that it is also true that \( f'_L(a) \) exists and is equal to \( f'(a+) \). If we let \( s \) denote any number in the interval \( a < x < b \) and define \( f(a) \) to be \( f(a+) \), then \( f \) is continuous on the closed interval \( a \leq x \leq s \) (Fig. 7). Since \( f' \) exists in the open interval \( a < x < s \), the mean value theorem for derivatives applies. That is, there is a number \( c \), where \( a < c < s \), such that
\[
\frac{f(s) - f(a +)}{s - a} = f'(c).
\]

(5)

This is shown geometrically in Fig. 7, where the slopes of the secant line \( S \) and the tangent line \( T \) are the same. Letting \( s \), and therefore \( c \), tend to \( a \) in equation (5), we see that since \( f'(a+) \) exists, the limit of \( f'(c) \) exists and has that value. Consequently, the limit of the difference quotient on the left in equation (5), its value being \( f'_L(a) \). Thus \( f'_L(a) = f'(a+) \). Similarly, \( f'_R(b) = f'(b-) \).

Now any piecewise smooth function \( f \) is continuous, along with its derivative \( f' \), on a finite number of subintervals at whose endpoints the one-sided limits of \( f \) and \( f' \) from the interior exist. If the results of the two preceding paragraphs are applied to each of those subintervals, the theorem is established.

Example 4 illustrates the distinction between one-sided derivatives and one-sided limits of derivatives.

**EXAMPLE 4.** Consider the function \( f \) whose values are
\[
f(x) = \begin{cases} x^2 \sin(1/x) & \text{when } x \neq 0, \\ 0 & \text{when } x = 0. \end{cases}
\]

Since \( 0 \leq |x^2 \sin(1/x)| \leq x^2 \) when \( x \neq 0 \), both one-sided limits \( f(0+) \) and \( f(0-) \) exist and have value zero. Moreover, since \( 0 \leq |x \sin(1/x)| \leq |x| \) when \( x \neq 0 \),
\[
\begin{align*}
f'_R(0) & = \lim_{x \to 0^+} \left( x \sin \frac{1}{x} \right) = 0, \\
f'_L(0) & = \lim_{x \to 0^-} \left( x \sin \frac{1}{x} \right) = 0.
\end{align*}
\]

But, from the expression
\[
f'(x) = 2x \sin \frac{1}{x} - \cos \frac{1}{x}, \quad (x \neq 0),
\]
we see that the one-sided limits \( f'(0+) \) and \( f'(0-) \) do not exist.

Note that although its one-sided derivatives exist everywhere, the function \( f \) is not piecewise smooth on any bounded interval containing the origin. Hence the above theorem is not applicable to this function on such an interval.

**10. A PROPERTY OF FOURIER COEFFICIENTS**

In treating the convergence of Fourier series, we shall find it useful to know that for a function \( f \) in \( C_p(0, \pi) \), the coefficients \( a_n \) and \( b_n \) in the cosine and sine series always tend to zero as \( n \) tends to infinity.

To show that the coefficients
\[
a_n = \frac{2}{\pi} \int_0^\pi f(x) \cos nx \, dx \quad (n = 0, 1, 2, \ldots)
\]
in a cosine series have this property, let \( s_n(x) \) denote the partial sum consisting of
the first $N + 1 (N \geq 1)$ terms in such a series:

$$s_N(x) = \frac{a_0}{2} + \sum_{n=1}^{N} a_n \cos nx.$$  

(2)

Then

$$\int_0^{\pi} |s_N(x) - f(x)|^2 \, dx = \int_0^{\pi} |f(x)|^2 \, dx - 2 \int_0^{\pi} f(x) s_N(x) \, dx$$

$$+ \int_0^{\pi} s_N(x)^2 \, dx.$$  

We need to rewrite the last two integrals on the right here. From equation (2) we have

$$f(x) s_N(x) = \frac{a_0}{2} f(x) + \sum_{n=1}^{N} a_n f(x) \cos nx.$$  

Hence

$$\int_0^{\pi} f(x) s_N(x) \, dx = \frac{a_0}{2} \int_0^{\pi} f(x) \, dx + \sum_{n=1}^{N} a_n \int_0^{\pi} f(x) \cos nx \, dx.$$  

In view of expression (1), then,

$$\int_0^{\pi} f(x) s_N(x) \, dx = \frac{\pi}{2} \left( \frac{a_0^2}{2} + \sum_{n=1}^{N} a_n^2 \right).$$  

(4)

As for the integral on the far right in equation (3), we note from expression (2) that

$$\int_0^{\pi} s_N(x) \, dx = \frac{a_0}{2} \int_0^{\pi} \, dx + \sum_{n=1}^{N} a_n \int_0^{\pi} \cos nx \, dx = \frac{\pi}{2} a_0.$$  

(5)

Also, using $m$ as the index of summation in expression (2), we write

$$s_N(x) \cos nx = \frac{a_0}{2} \cos nx + \sum_{m=1}^{N} a_m \cos mx \cos nx,$$

where $n$ has any one of the values $n = 1, 2, \ldots N$. The integration formula (Sec. 2)

$$\int_0^{\pi} \cos mx \cos nx \, dx = \begin{cases} 0 & \text{when } m \neq n, \\ \pi/2 & \text{when } m = n, \end{cases}$$

where $m$ and $n$ are positive integers, now yields

$$\int_0^{\pi} s_N(x) \cos nx \, dx = \frac{a_0}{2} \int_0^{\pi} \cos nx \, dx + \sum_{n=1}^{N} a_n \int_0^{\pi} \cos mx \cos nx \, dx$$

$$= \frac{\pi}{2} a_n$$  

(6)  

$$(n = 1, 2, \ldots, N).$$

Consequently, by writing

$$|s_N(x)|^2 = \frac{a_0^2}{2} s_N(x) + \sum_{n=1}^{N} a_n s_N(x) \cos nx,$$

integrating each side from 0 to $\pi$, and then referring to expressions (5) and (6), we have

$$\int_0^{\pi} |s_N(x)|^2 \, dx = \frac{\pi}{2} \left( \frac{a_0^2}{2} + \sum_{n=1}^{N} a_n^2 \right).$$  

(7)

It now follows from equations (3), (4), and (7) that

$$\int_0^{\pi} |f(x) - s_N(x)|^2 \, dx = \int_0^{\pi} |f(x)|^2 \, dx - \frac{\pi}{2} \left( \frac{a_0^2}{2} + \sum_{n=1}^{N} a_n^2 \right).$$

Since the value of the integral on the left here is nonnegative, we thus arrive at Bessel's inequality for the coefficients (1):

$$\frac{a_0^2}{2} + \sum_{n=1}^{N} a_n^2 \leq \frac{\pi}{2} \int_0^{\pi} |f(x)|^2 \, dx \quad (N = 1, 2, \ldots).$$  

The desired result,

$$\lim_{N \to \infty} a_n = 0,$$

is an easy consequence of Bessel's inequality (8), as the following argument shows. We observe that the right-hand side of the inequality is independent of the positive integer $N$; and as $N$ increases on the left-hand side, the sums of the squares there form a sequence that is bounded and nondecreasing. Since such a sequence must converge and since this particular sequence is the sequence of the partial sums of the series whose terms are $a_n^2/2$ and $a_n^2$ ($n = 1, 2, \ldots$), that series must converge. Limit (9) then follows from the fact that the $n$th term of a convergent series always tends to zero as $n$ tends to infinity.

A similar procedure can be used (Prob. 3, Sec. 11) to show that the coefficients

$$b_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin nx \, dx \quad (n = 1, 2, \ldots)$$

in the Fourier sine series for $f$ satisfy the Bessel inequality

$$\sum_{n=1}^{N} b_n^2 \leq \frac{2}{\pi} \int_0^{\pi} |f(x)|^2 \, dx \quad (N = 1, 2, \ldots)$$

and that

$$\lim_{n \to \infty} b_n = 0.$$  

Finally, we recall from Sec. 6 that the coefficients $a_n$ and $b_n$ in the Fourier series involving both cosines and sines for a piecewise continuous function $f$ in $C_p(-\pi, \pi)$ are the same as the coefficients in the Fourier cosine and sine series,
respectively, for certain related functions on $0 < x < \pi$. Hence those coefficients themselves tend to zero as $n$ tends to infinity. (See also Problem 5, Sec. 11.)

11. TWO LEMMATA

We preface our theorem on the convergence of Fourier series with two lemmas, or preliminary theorems. The first is a special case of what is known as the Riemann-Lebesgue lemma. That lemma appears later in Chap. 6 (Sec. 46), where it is needed in full generality.

**Lemma 1.** If a function $G(u)$ is piecewise continuous on the interval $0 < u < \pi$, then

$$
\lim_{N \to \infty} \int_0^\pi G(u) \sin \left( \frac{u}{2} + Nu \right) du = 0.
$$

where $N$ denotes positive integers.

Our proof starts with the trigonometric identity

$$
\sin(A + B) = \sin A \cos B + \cos A \sin B,
$$

which tells us that

$$
\sin \left( \frac{u}{2} + Nu \right) = \sin \frac{u}{2} \cos Nu + \cos \frac{u}{2} \sin Nu.
$$

This enables us to write

$$
\int_0^\pi G(u) \sin \left( \frac{u}{2} + Nu \right) du = \frac{\pi}{2} a_N + \pi \frac{b_N}{2},
$$

where

$$
a_N = \frac{2}{\pi} \int_0^\pi \left[ G(u) \sin \frac{u}{2} \right] \cos Nu du
$$

and

$$
b_N = \frac{2}{\pi} \int_0^\pi \left[ G(u) \cos \frac{u}{2} \right] \sin Nu du.
$$

Now the $a_N$ are coefficients in a Fourier cosine series on the interval $0 < u < \pi$, and the $b_N$ are coefficients in a Fourier sine series on that interval. Thus, by limits (9) and (12) in Sec. 10,

$$
\lim_{N \to \infty} a_N = 0 \quad \text{and} \quad \lim_{N \to \infty} b_N = 0.
$$

With limits (3), we need only let $N$ tend to infinity in equation (2) to see that Lemma 1 is true.

Our second lemma involves the Dirichlet kernel

$$
D_N(u) = \frac{1}{2} + \sum_{n=1}^N \cos nu,
$$

where $N$ is any positive integer. Note that $D_N(u)$ is continuous, even, and periodic with period $2\pi$. The Dirichlet kernel plays a central role in our theory, and two other properties will be useful:

$$
\int_0^\pi D_N(u) du = \frac{\pi}{2},
$$

$$
D_N(u) = \frac{\sin \left( \frac{u}{2} + Nu \right)}{2 \sin \frac{u}{2}} \quad (u \neq 0, \pm 2\pi, \pm 4\pi, \ldots).
$$

Property (5) is obvious upon integrating each side of equation (4). Expression (6) can be derived with the aid of a certain trigonometric identity (Problem 6).

**Lemma 2.** Suppose that a function $g(u)$ is piecewise continuous on the interval $0 < u < \pi$ and that the right-hand derivative $g_0'(0)$ exists. Then

$$
\lim_{N \to \infty} \int_0^\pi g(u) D_N(u) du = \frac{\pi}{2} g(0+),
$$

where $D_N(u)$ is the Dirichlet kernel (4).

To prove this, we write

$$
\int_0^\pi g(u) D_N(u) du = I_N + J_N,
$$

where

$$
I_N = \int_0^\pi [g(u) - g(0+)] D_N(u) du
$$

and

$$
J_N = \int_0^\pi g(0+) D_N(u) du.
$$

In view of expression (6) for $D_N(u)$, integral (9) can be put in the form

$$
I_N = \int_0^\pi g(u) - \frac{g(0+)}{2} \sin \left( \frac{u}{2} + Nu \right) du.
$$

Thus

$$
I_N = \int_0^\pi \sin \left( \frac{u}{2} + Nu \right) du,
$$

where the function

$$
G(u) = \frac{g(u) - g(0+)}{2 \sin \frac{u}{2}}
$$

is a quotient of two functions that are piecewise continuous on the interval $0 < u < \pi$. Although the denominator vanishes at the point $u = 0$, one can show that $G(u)$ is itself piecewise continuous on $0 < u < \pi$ by establishing the existence of $G(u)$ at $u = 0$.
of \( G(0+) \). This is done by referring to expression (12) and writing

\[
\lim_{u \to 0} G(u) = \lim_{u \to 0} \frac{g(u) - g(0+)}{u} = \lim_{u \to 0} \frac{u}{2 \sin \frac{u}{2}}.
\]

The first of the limits on the right here is, of course, \( g'_0(0) \); and an application of \( \text{Hôpital's rule} \) reveals that the second limit is unity. According to Lemma 1, then, the limit of the right-hand side of equation (11) is zero as \( N \) tends to infinity. That is,

\[
(13) \quad \lim_{N \to \infty} J_N = 0.
\]

With property (5) of the Dirichlet kernel, one can see from expression (10) for \( J_N \) that

\[
J_N = g'(0+) \int_0^\pi D_N(u) \, du = \frac{\pi}{2} g'(0+).
\]

Hence

\[
(14) \quad \lim_{N \to \infty} J_N = \frac{\pi}{2} g'(0+).
\]

The desired result (7) now follows from equation (8) together with limits (13) and (14).

**PROBLEMS**

1. With the aid of \( \text{Hôpital's rule} \), find \( f'(0+) \) and \( f'_0(0) \) when

\[
f(x) = e^{\frac{x^2}{2}} \quad (x \neq 0)\nonumber
\]

**Answer:** \( f'(0+) = 1 \), \( f'_0(0) = \frac{1}{2} \).

2. Show that the function defined by the equations

\[
f(x) = \begin{cases} x \sin(1/x) & \text{when } x \neq 0, \\ 0 & \text{when } x = 0 \end{cases}
\]

is continuous at \( x = 0 \) but that neither \( f'_0(0) \) nor \( f''_0(0) \) exists. This provides another illustration (see Example 3, Sec. 9) of the fact that the continuity of a function \( f \) at a point \( x_0 \) is not a sufficient condition for the existence of one-sided derivatives of \( f \) at \( x_0 \).

3. By following the steps used in Sec. 10 to find Bessel's inequality for the coefficients \( a_n \) in the Fourier cosine series for a function \( f \) in \( C_2(0, \pi) \), derive the Bessel inequality

\[
\sum_{n=1}^{N} b_n^2 \leq \frac{2}{\pi} \int_0^\pi [f(x)]^2 \, dx \quad (N = 1, 2, \ldots)
\]

for the coefficients \( b_n \) in the sine series for \( f \). Then use this result to show that

\[
\lim_{N \to \infty} b_n = 0.
\]

4. In Chap. 1 (Sec. 6) we expressed a function \( f(x) \) in \( C_2(-\pi, \pi) \) as a sum

\[
f(x) = g(x) + h(x)
\]

where

\[
g(x) = \frac{f(x) + f(-x)}{2} \quad \text{and} \quad h(x) = \frac{f(x) - f(-x)}{2}
\]

We then saw that the coefficients \( a_n \) and \( b_n \) in the Fourier series

\[
a_n + \sum_{n=1}^{N} \left(a_n \cos nx + b_n \sin nx\right)
\]

for \( f(x) \) on \(-\pi < x < \pi\) are the same as the coefficients in the Fourier cosine and sine series for \( g(x) \) and \( h(x) \), respectively, on \(0 < x < \pi\).

(a) By referring to the Bessel inequalities (8) and (11) in Sec. 10, write

\[
\frac{a_n^2}{2} + \sum_{n=1}^{N} b_n^2 \leq \frac{2}{\pi} \int_0^\pi [g(x)]^2 \, dx \quad (N = 1, 2, \ldots)
\]

and

\[
\frac{b_n^2}{2} \leq \frac{2}{\pi} \int_0^\pi [h(x)]^2 \, dx \quad (N = 1, 2, \ldots).
\]

Then point out how it follows that

\[
\frac{a_n^2}{2} + \sum_{n=1}^{N} (a_n^2 + b_n^2) \leq \frac{1}{\pi} \left\{ \int_0^\pi [f(x)]^2 \, dx + \int_0^\pi [-f(-x)]^2 \, dx \right\} \quad (N = 1, 2, \ldots).
\]

(b) By making the substitution \( x = -s \) in the last integral in part (a), obtain the Bessel inequality

\[
\frac{a_n^2}{2} + \sum_{n=1}^{N} (a_n^2 + b_n^2) \leq \frac{1}{\pi} \int_0^\pi [-f(s)]^2 \, dx \quad (N = 1, 2, \ldots).
\]

5. Show how it follows from the Bessel inequality in Problem 4(b) that

\[
\lim_{n \to \infty} a_n = 0 \quad \text{and} \quad \lim_{n \to \infty} b_n = 0,
\]

where \( a_n \) and \( b_n \) are the coefficients in the Fourier series

\[
a_n + \sum_{n=1}^{N} (a_n \cos nx + b_n \sin nx)
\]

for a piecewise continuous function in \( C_2(-\pi, \pi) \).

6. Derive the expression

\[
D_N(u) = \frac{\sin \left(\frac{u^2}{2} + Nuu\right)}{2 \sin \frac{u^2}{2}}
\]

for the Dirichlet kernel (Sec. 11)

\[
D_N(u) = \frac{1}{2} + \sum_{n=1}^{N} \cos nu
\]
by writing\[ A = \frac{u}{2} \quad \text{and} \quad B = nu \]
in the trigonometric identity\[ 2 \sin A \cos B = \sin (A + B) + \sin (A - B) \]
and then summing each side of the resulting equation from \( n = 1 \) to \( n = N \).

Suggestion: Note that\[ \sum_{n=1}^{N} \sin \left( \frac{u}{2} - nu \right) = -\sum_{n=0}^{N-1} \sin \left( \frac{u}{2} + nu \right). \]

12. A FOURIER THEOREM

A theorem that gives conditions under which a Fourier series

\[
\frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx),
\]

with coefficients

\[
a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, dx \quad (n = 0, 1, 2, \ldots)
\]

and

\[
b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx \quad (n = 1, 2, \ldots),
\]

converges to \( f(x) \) is called a Fourier theorem. One such theorem will now be established. Although it is stated for periodic functions of period \( 2\pi \), it also applies to functions defined only on the fundamental interval \( -\pi < x < \pi \); for, as is done in the corollary following this theorem and its proof, we need only consider the periodic extensions, with period \( 2\pi \), of such functions.

**Theorem.** Let \( f \) denote a function that is piecewise continuous on the interval \( -\pi < x < \pi \) and periodic, with period \( 2\pi \), on the entire \( x \) axis. Its Fourier series (1), with coefficients (2) and (3), converges to the mean value

\[
\frac{f(x+) + f(x-)}{2}
\]

of the one-sided limits of \( f \) at each point \( x \) \((-\infty < x < \infty)\) where both of the one-sided derivatives \( f'_L(x) \) and \( f'_R(x) \) exist.

Note that if \( f \) is actually continuous at \( x \), the quotient (4) becomes \( f(x) \).

Hence

\[
f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)
\]
at \( x \), provided that both \( f'_L(x) \) and \( f'_R(x) \) exist.

The fact that \( f \) is piecewise continuous on \( -\pi < x < \pi \) ensures that the integrals (2) and (3) always exist; and we begin our proof of the theorem by writing series (1) as (see Sec. 6)

\[
\frac{1}{2\pi} \int_{-\pi}^{\pi} f(s) \, ds + \frac{1}{\pi} \sum_{n=1}^{\infty} \int_{-\pi}^{\pi} f(s) \cos nx \, ds,
\]

with those coefficients incorporated into it. Then, if \( S_N(x) \) denotes the partial sum consisting of the sum of the first \( N + 1 \) \((N \geq 1)\) terms of the series,

\[
S_N(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(s) \, ds + \frac{1}{\pi} \sum_{n=1}^{N} \int_{-\pi}^{\pi} f(s) \cos nx \, ds.
\]

Using the Dirichlet kernel (Sec. 11)

\[
D_N(u) = \frac{1}{2} + \sum_{n=1}^{N} \cos nu,
\]

we can put equation (5) in the form

\[
S_N(x) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(s) D_N(s-x) \, ds.
\]

The periodicity of the integrand here allows us to change the interval of integration to any interval of length \( 2\pi \) without altering the value of the integral (see Problem 9, Sec. 13). Thus

\[
S_N(x) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(s) D_N(s-x) \, ds,
\]

where point \( x \) is at the center of the interval we have chosen. It now follows from equation (6) that

\[
S_N(x) = \frac{1}{\pi} \int_{-\pi-x}^{\pi-x} f(s) D_N(s-x) \, ds,
\]

where

\[
I_N(x) = \int_{x}^{x+\pi} f(s) D_N(s-x) \, ds \quad \text{and}
\]

\[
J_N(x) = \int_{x-\pi}^{x} f(s) D_N(s-x) \, ds.
\]

If we replace the variable of integration \( s \) in integral (8) by the new variable \( u = s - x \), that integral becomes

\[
I_N(x) = \int_{0}^{\pi} f(x+u) D_N(u) \, du.
\]

Since \( f \) is piecewise continuous on the fundamental interval \( -\pi < x < \pi \) and also periodic, it is piecewise continuous on any bounded interval of the \( x \) axis. So, for a fixed value of \( x \), the function \( g(u) = f(x+u) \) in expression (10) is piecewise
continuous on any bounded interval of the $u$ axis, and in particular, on the interval $0 < u < \pi$. Let us assume that the right-hand derivative $f'_R(x)$ exists. After observing that

$$ g(0+) = \lim_{u \to 0^+} g(u) = \lim_{u \to 0^+} f(x + u) = \lim_{v \to x} f(v) = f(x+) $$

one can show that the right-hand derivative of $g$ at $u = 0$ exists:

$$ g'_R(0) = \lim_{u \to 0^+} \frac{g(u) - g(0^+)}{u} = \lim_{u \to 0^+} \frac{f(x + u) - f(x^+)}{u} = \lim_{v \to x} \frac{f(v) - f(x^+)}{v - x} = f'_R(x). $$

According to Lemma 2 in Sec. 11, then,

$$ \lim_{N \to \infty} J_N(x) = \frac{\pi}{2} g(0^+) = \frac{\pi}{2} f(x^+). $$

If, on the other hand, we make the substitution $u = x - s$ in integral (9) and recall from our discussion in Sec. 11 that $D_N(u)$ is an even function of $u$, we find that

$$ \lim_{N \to \infty} J_N(x) = \int_0^\pi f(x - u) D_N(u) \, du. $$

This time, we assume that the left-hand derivative $f'_L(x)$ exists; and we note that the function $g(u) = f(x - u)$ in expression (12) is piecewise continuous on the interval $0 < u < \pi$. Furthermore,

$$ g(0+) = \lim_{u \to 0^-} g(u) = \lim_{u \to 0^-} f(x - u) = \lim_{v \to x} f(v) = f(x-) $$

and

$$ g'_L(0) = \lim_{u \to 0^-} \frac{g(u) - g(0^+)}{u} = \lim_{u \to 0^-} \frac{f(x - u) - f(x^+)}{u} = \lim_{v \to x} \frac{f(v) - f(x^+)}{v - x} = -f'_L(x). $$

So once again by Lemma 2 in Sec. 11,

$$ \lim_{N \to \infty} J_N(x) = \frac{\pi}{2} g(0^+) = \frac{\pi}{2} f(x^+). $$

Finally, we may conclude from equation (7) and limits (11) and (13) that

$$ \lim_{N \to \infty} S_N(x) = \frac{f(x^+) + f(x^-)}{2}; $$

and the theorem is proved.

This theorem is especially suited to functions $f$ that are piecewise smooth on the fundamental interval $-\pi < x < \pi$. We recall from Sec. 9 that $f$ is piecewise smooth if both $f$ and $f'$ are piecewise continuous.

---

**Corollary.** Let $f$ denote a function that is piecewise smooth on the interval $-\pi < x < \pi$, and let $F$ denote the periodic extension, with period $2\pi$, of $f$. At each point $x (-\infty < x < \infty)$, the Fourier series for $f$ on $-\pi < x < \pi$ converges to the mean value of the one-sided limits of $F(x^+)$ and $F(x^-)$, namely

$$ \frac{F(x^+) + F(x^-)}{2}. $$

The proof of this corollary relies on the theorem in Sec. 9, which tells us that when $f$ is piecewise smooth on $-\pi < x < \pi$, its one-sided derivatives, from the interior at the endpoints $x = \pm \pi$, exist everywhere in the closed interval $-\pi \leq x \leq \pi$. Hence if $F$ denotes the periodic extension of $f$, with period $2\pi$, the one-sided derivatives of $F$ exist at each point $x (-\infty < x < \infty)$. According to the theorem just proved, then, the Fourier series for $f$ on $-\pi < x < \pi$ converges everywhere to the mean value of the one-sided limits of $F$.

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**13. DISCUSSION OF THE THEOREM AND ITS COROLLARY**

It should be emphasized that the conditions in the theorem in Sec. 12, as well as the corollary there, are only sufficient, and there is no claim that they are necessary conditions. More general conditions are given in a number of the references listed in the Bibliography. Indeed, there are functions that even become unbounded at certain points but nevertheless have valid Fourier series representations.

The corollary in Sec. 12 will be adequate for most of the applications in this book, where the functions are generally piecewise smooth. We note that if $f$ and $F$ denote the functions in the corollary, then

$$ F(x^+) = f(x^+) \quad \text{and} \quad F(x^-) = f(x^-) \quad \text{when} -\pi < x < \pi. $$

Consequently, when $-\pi < x < \pi$, the corollary tells us that the Fourier series for $f$ on the interval $-\pi < x < \pi$ converges to the number

$$ \frac{f(x^+) + f(x^-)}{2}, $$

which becomes $f(x)$ if $x$ is a point of continuity of $f$.

At the endpoints $x = \pm \pi$, however, the series converges to

$$ \frac{f(-\pi^+) + f(\pi^-)}{2}. $$

To see that this is so, consider first the point $x = \pi$. Since

$$ F(\pi^+) = f(-\pi^-) \quad \text{and} \quad F(\pi^-) = f(\pi^-), $$

as is evident from Fig. 8, the quotient

$$ \frac{F(x^+) + F(x^-)}{2} $$

See, for instance, the book by Tolstov (1976, pp. 91-94), which is listed in the Bibliography.
in the corollary becomes the quotient (2) when \( x = \pi \). Because of the periodicity of the series, it also converges to the quotient (2) when \( x = -\pi \).

**EXAMPLE 1.** In Example 1, Sec. 7, we obtained the Fourier series

\[
\frac{\pi}{4} + \sum_{n=1}^{\infty} \left( \frac{(-1)^n}{n^2} \cos nx + \frac{(-1)^n+1}{n} \sin nx \right)
\]

on the interval \(-\pi < x < \pi\) for the function \( f \) defined by the equations

\[
f(x) = \begin{cases} 
0 & \text{when } -\pi < x \leq 0, \\
x & \text{when } 0 < x < \pi.
\end{cases}
\]

Since

\[
f'(x) = \begin{cases} 
0 & \text{when } -\pi < x < 0, \\
1 & \text{when } 0 < x < \pi,
\end{cases}
\]

\( f \) is clearly piecewise smooth on the fundamental interval \(-\pi < x < \pi\). In view of the continuity of \( f \) when \(-\pi < x < \pi\), the series converges to \( f(x) \) at each point \( x \) in that open interval. Since \( f(-\pi+) = 0 \) and \( f(\pi-) = \pi \), it converges to \( \pi/2 \) at the endpoints \( x = \pm \pi \). The series, in fact, converges to \( \pi/2 \) at each of the points \( x = \pm \pi, \pm 3\pi, \pm 5\pi, \ldots \), as indicated in Fig. 5 (Sec. 7), where the sum of the series for all \( x \) is described graphically.

In particular, since series (3) converges to \( \pi/2 \) when \( x = \pi \), we have the identity

\[
\frac{\pi}{4} + \sum_{n=1}^{\infty} \frac{(-1)^n-1}{n^2} = \frac{\pi}{2}
\]

which can be written

\[
\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \frac{\pi^2}{8}.
\]

This illustrates how Fourier series can sometimes be used to find the sums of convergent series encountered in calculus. Setting \( x = 0 \) in series (3) also yields this particular summation formula.

The corollary in Sec. 12 tells us that a function \( f \) in the space \( C'_p(-\pi, \pi) \) of piecewise smooth functions on the interval \(-\pi < x < \pi\) has a valid Fourier series representation on that interval, or one that is equal to \( f(x) \) at all but possibly a finite number of points there. It also ensures that a function \( f \) in the space \( C'_p(0, \pi) \) has valid Fourier cosine and sine series representations on the interval \( 0 < x < \pi \). This is because, according to Sec. 6, the cosine series for a function \( f \) on the interval \( 0 < x < \pi \) is the same as the Fourier series corresponding to the even extension of \( f \) on the interval \(-\pi < x < \pi \) and the sine series for \( f \) on the interval \( 0 < x < \pi \) is the Fourier series for the odd extension of \( f \). In view of the even periodic function represented by the cosine series, that series converges to \( f(0+) \) at the point \( x = 0 \) and to \( f(\pi-) \) at \( x = \pi \). The sum of the sine series is, of course, zero when \( x = 0 \) and when \( x = \pi \).

**EXAMPLE 2.** In Example 2, Sec. 3, we found the Fourier cosine series corresponding to the function \( f(x) = \sin x \) on the interval \( 0 < x < \pi \):

\[
\sin x \sim \frac{2}{\pi} - \frac{4}{\pi} \sum_{n=1}^{\infty} \cos 2nx
\]

Since \( \sin x \) is piecewise smooth on \( 0 < x < \pi \) and continuous on the closed interval \( 0 \leq x \leq \pi \), correspondence (4) is evidently an equality when \( 0 \leq x \leq \pi \).

Our final example here illustrates how the theorem in Sec. 12 can be useful when the corollary there fails to apply.

**EXAMPLE 3.** The odd function

\[
f(x) = \sqrt{x} \quad (-\pi < x < \pi)
\]

is piecewise continuous on the interval \(-\pi < x < \pi \). But since

\[
f'(x) = \frac{1}{2x^{3/2}}
\]

when \( x \neq 0 \), it is clear that the one-sided limits \( f'(0+) \) and \( f'(0-) \) do not exist. Hence \( f \) is not piecewise smooth on \(-\pi < x < \pi \), and the corollary in Sec. 12 does not apply.

If, however, \( F \) denotes the periodic extension, with period \( 2\pi \), of the piecewise continuous function (5), the theorem in Sec. 12 can be applied to that extension. To be precise, since the one-sided derivatives of \( F \) exist everywhere in the interval \(-\pi < x < \pi \) except at \( x = 0 \), we find that the Fourier series for \( F \) on \(-\pi < x < \pi \) converges to \( F(x) \) when \(-\pi < x < 0 \) and when \( 0 < x < \pi \). That series representation is also valid at \( x = 0 \) since \( F \) is odd and the series is actually a Fourier sine series on \( 0 < x < \pi \), which converges to zero when \( x = 0 \). Since \( f(x) = F(x) \) when \(-\pi < x < \pi \), we may conclude that the Fourier series for \( f \) on that interval is valid for all such \( x \).
PROBLEMS

1. State why the Fourier sine series in Example 1, Sec. 5, for the function

   \[ f(x) = x \quad (0 < x < \pi) \]

   is a valid representation for \( x \) on the interval \( -\pi < x < \pi \). Thus verify fully that the series converges for all \( x \) \((-\infty < x < \infty)\) to the function whose graph is shown in Fig. 4 (Sec. 5).

2. For each of the following functions, point out why its Fourier series on the interval \(-\pi < x < \pi\) is convergent when \(-\pi \leq x \leq \pi\), and state the sum of the series when \( x = \pi \):
   
   (a) the function
   
   \[ f(x) = \begin{cases} -\pi/2 & \text{when } -\pi < x < 0, \\ \pi/2 & \text{when } 0 < x < \pi, \end{cases} \]
   
   whose series was found in Problem 1, Sec. 7;
   
   (b) the function
   
   \[ f(x) = e^{ax} \quad (a \neq 0), \]
   
   whose series was found in Problem 4, Sec. 7.

   Answers: (a) sum = 0; (b) sum = \( \cosh \pi a \).

3. By writing \( x = 0 \) and \( x = \pi/2 \) in the representation

   \[ \sin x = 2 \pi \sum_{n=1}^{\infty} \frac{\cos 2nx}{4n^2 - 1} \quad (0 \leq x \leq \pi), \]

   established in Example 2, Sec. 13, obtain the following summations:

   \[ \sum_{n=1}^{\infty} \frac{1}{4n^2 - 1} = \frac{1}{2}, \quad \sum_{n=1}^{\infty} \frac{(-1)^n}{4n^2 - 1} = \frac{\pi}{2} - \frac{1}{4}. \]

4. Point out why the Fourier series in Problem 7, Sec. 7, for the function

   \[ f(x) = \begin{cases} 0 & \text{when } -\pi \leq x \leq 0, \\ \sin x & \text{when } 0 < x \leq \pi \end{cases} \]

   converges to \( f(x) \) everywhere in the interval \(-\pi \leq x \leq \pi\).

5. State why the correspondence

   \[ x \sim \frac{\pi}{2} \frac{4}{3} \sum_{n=1}^{\infty} \frac{\cos(2n-1)x}{(2n-1)^2} \quad (0 < x < \pi), \]

   obtained in Example 1, Sec. 3, is actually an equality on the closed interval \( 0 \leq x \leq \pi \).

   Thus show that

   \[ \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \frac{\pi^2}{8}. \]

   (Compare with Example 1, Sec. 13.)

6. (a) Use the correspondence

   \[ x^2 \sim \frac{\pi^2}{3} + 4 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos nx \quad (0 < x < \pi), \]

   found in Problem 3(a), Sec. 5, to show that

   \[ \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} = \frac{\pi^2}{12}, \quad \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}. \]

   (b) By writing \( x = \pi \) in the correspondence (Problem 6, Sec. 5)

   \[ x^4 \sim \frac{\pi^4}{5} + 8 \sum_{n=1}^{\infty} \frac{(-1)^n(n\pi)^2 - 6}{n^2} \cos nx \quad (0 < x < \pi) \]

   and referring to the second summation obtained in part \( (a) \), show that

   \[ \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^4}{90}. \]

7. With the aid of the correspondence (Problem 6, Sec. 7)

   \[ \cos ax \sim \frac{2a}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos nx \quad (-\pi < x < \pi), \]

   where \( a \neq 0, \pm 1, \pm 2, \ldots \), show that

   \[ \frac{a\pi}{\sin a\pi} = 1 + 2a^2 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \quad (a \neq 0, \pm 1, \pm 2, \ldots). \]

8. Without actually finding the Fourier series for the even function \( f(x) = \sqrt{2x} \) on \(-\pi < x < \pi\), point out how the theorem in Sec. 12 ensures the convergence of that series to \( f(x) \) when \( -\pi \leq x < 0 \) and when \( 0 < x \leq \pi \) but not when \( x = 0 \).

9. Let \( f \) denote a function that is piecewise continuous on an interval \(-c < x < c\) and periodic with period \( 2c \). Show that for any number \( a \),

   \[ \int_{-c}^{c} f(x) \, dx = \int_{-c}^{c} f(x) \, dx + \int_{c}^{\infty} f(s) \, ds. \]

   Suggestion: Write

   \[ \int_{-c}^{c} f(x) \, dx = \int_{-c}^{c} f(x) \, dx + \int_{c}^{\infty} f(s) \, ds \]

   and then make the substitution \( x = s - 2c \) in the second integral on the right-hand side of this equation.

14. CONVERGENCE ON OTHER INTERVALS

Section 8 began with a discussion of Fourier series corresponding to piecewise continuous functions \( f \) on arbitrary intervals \(-c < x < c\). To treat the convergence of such series, we include here a few remarks about the function

(1) \[ g(s) = f \left( \frac{cs}{\pi} \right) \quad (-\pi < s < \pi) \]

that was used in Sec. 8.
Let us write the function (1) as

\[
g(s) = f(x) \quad \text{where} \quad x = \frac{cs}{\pi} \quad (-\pi < s < \pi).
\]

It is clear that the equation \( x = cs/\pi \), or \( s = \pi x/c \), establishes a one-to-one correspondence between points in the interval \(-\pi < s < \pi\) and points in the interval \(-c < x < c\). Suppose now that \( f \) is piecewise smooth on the interval \(-c < x < c\) and that \( f(x) \) at each point \( x \) where \( f \) is discontinuous is the mean value of the one-sided limits \( f(x+) \) and \( f(x-) \), as is the case when \( f \) is continuous at \( x \).

One can see from equations (2) that if a specific point \( x_0 \) corresponds to a specific point \( s_0 \), then

\[
g(s_0 +) = f(x_0 +), \quad g(s_0 -) = f(x_0 -).
\]

Since \( f(x) \) is always the mean value of \( f(x+) \) and \( f(x-) \), it follows from these relations between one-sided limits that the number \( g(s) = f(x) \) is always the mean value of \( g(s+) \) and \( g(s-) \). In particular, if \( g \) is continuous at \( s \), then \( g \) is piecewise continuous on the interval \(-c < x < c\), then, \( g \) is piecewise continuous on the interval \(-\pi < s < \pi\). The derivative \( f' \) is also piecewise continuous, and a similar argument shows that \( g' \) is piecewise continuous. So \( g \) is piecewise smooth on the interval \(-\pi < s < \pi\). According to the corollary in Sec. 12, the Fourier series for \( g \) on the interval \(-\pi < s < \pi\) converges to \( g(s) \) for each \( s \) in that interval. Moreover, since it was relation (1) that gave us the new correspondence for \( f \) on \(-c < x < c\) in Sec. 8, the correspondence is, in fact, an equality. That is,

\[
f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left( a_n \cos \frac{n\pi x}{c} + b_n \sin \frac{n\pi x}{c} \right).
\]

where

\[
a_n = \frac{1}{c} \int_{-c}^{c} f(x) \cos \frac{n\pi x}{c} \, dx \quad (n = 0, 1, 2, \ldots),
\]

\[
b_n = \frac{1}{c} \int_{-c}^{c} f(x) \sin \frac{n\pi x}{c} \, dx \quad (n = 1, 2, \ldots).
\]

We state this result as a theorem that is sufficient for our applications. The theorem and the one that follows it apply to any function \( f \) that has the following properties:

(i) The function \( f \) is piecewise smooth on the stated interval.

(ii) The value \( f(x) \) at each point of discontinuity of \( f \) in that interval is the mean value of the one-sided limits \( f(x+) \) and \( f(x-) \).

**Theorem 1.** Let \( f \) denote a function that has properties (i) and (ii) on an interval \(-c < x < c\). The Fourier series representation (3), with coefficients (4) and (5), is valid for each \( x \) in that interval.

Note that series (3) also represents the periodic extension, with period \( 2c \), of the function \( f \). That is, it converges to a function \( F(x) \) whose graph coincides with the graph of \( f(x) \) on \(-c < x < c\) and is repeated every \( 2c \) units along the \( x \) axis. The series has the expected sums at the endpoints \( x = \pm c \). The sum at \( x = c \) is, for instance, the mean value of \( F(c+) \) and \( F(c-) \).

If we restrict function (1) to the interval \( 0 < x < c \), Fourier cosine and sine series representations on \( 0 < x < c \) follow from representations on \( 0 < x < \pi \) that involve only cosines and sines, respectively.

**Theorem 2.** Let \( f \) denote a function that has properties (i) and (ii) on an interval \( 0 < x < c \). The Fourier cosine series representation

\[
f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{c},
\]

with coefficients

\[
a_n = \frac{2}{c} \int_{0}^{c} f(x) \cos \frac{n\pi x}{c} \, dx \quad (n = 0, 1, 2, \ldots),
\]

is valid for all \( x \) in that interval. The same is true of the Fourier sine series representation

\[
f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{c},
\]

with coefficients

\[
b_n = \frac{2}{c} \int_{0}^{c} f(x) \sin \frac{n\pi x}{c} \, dx \quad (n = 1, 2, \ldots).
\]

Series (6) represents, of course, the even periodic extension, with period \( 2c \), of \( f \); and series (8) represents the odd periodic extension, with period \( 2c \), of \( f \).

**PROBLEMS**

1. Use formulas (4) and (5), Sec. 14, as well as Theorem 1 in that section, to show that if

\[
f(x) = \begin{cases} 0 & \text{when } -3 < x < 0, \\ 1 & \text{when } 0 < x < 3,
\end{cases}
\]

and if \( f(0) = 1/2 \), then

\[
f(x) = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{1}{2n-1} \sin \frac{(2n-1)\pi x}{3} \quad (-3 < x < 3).
\]

Describe graphically the function that is represented by this series for all values of \( x \) \((-\infty < x < \infty)\).

2. Let \( f \) denote the function whose values are

\[
f(x) = \begin{cases} 0 & \text{when } -2 < x < 1, \\ 1 & \text{when } 1 < x < 2.
\end{cases}
\]
and

\[ f(-2) = f(1) = f(2) = \frac{1}{2}. \]

Use formulas (4) and (5) in Sec. 14, together with Theorem 1 there, to show that

\[ f(x) = \frac{1}{4} - \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{1}{n^2} \left[ \sin \frac{n\pi x}{2} - \cos \frac{n\pi x}{2} + \frac{\pi}{2} \cos n\pi - \cos n\pi \right] \sum_{n=1}^{2} \sin \frac{n\pi x}{2} \]

for each \( x \) in the closed interval \(-2 \leq x \leq 2\).

3. Let \( M(c, t) \) denote the square wave (Fig. 9) defined by the equations

\[
M(c, t) = \begin{cases} 
1 & \text{when } 0 < t < c, \\
-1 & \text{when } c < t < 2c, 
\end{cases}
\]

and \( M(c, t + 2c) = M(c, t) \) when \( t > 0 \). Show that

\[
M(c, t) = \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{1}{2n-1} \sin \frac{(2n-1)c}{c} \quad (t \neq c, 2c, 3c, \ldots).
\]

![Figure 9](image)

4. Let \( F \) denote the periodic function, of period \( c \), where

\[
F(x) = \begin{cases} 
\frac{c-x}{4} & \text{when } 0 \leq x \leq \frac{c}{2}, \\
\frac{x-3c}{4} & \text{when } \frac{c}{2} < x < c. 
\end{cases}
\]

(a) Describe the function \( F(x) \) graphically, and show that it is, in fact, the even periodic extension, with period \( c \), of the function

\[
f(x) = \frac{c-x}{4} \quad (0 \leq x \leq \frac{c}{2}).
\]

(b) Use the result in part (a) and the Fourier cosine series correspondence found in Problem 7, Sec. 8, to show that

\[
F(x) = \frac{2c}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} \cos \frac{(2n-1)c}{c} \quad (-\infty < x < \infty).
\]

5. Let \( f \) denote the periodic function, of period 2, where

\[
f(x) = \begin{cases} 
\cos \frac{\pi x}{2} & \text{when } 0 < x < 1, \\
0 & \text{when } 1 < x < 2.
\end{cases}
\]

and where

\[
f(0) = \frac{1}{2} \quad \text{and} \quad f(1) = -\frac{1}{2}.
\]

By referring to the correspondence

\[
\cos \frac{\pi x}{2} \sim \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{n \sin 2\pi x}{2n^2 - 1} \quad (0 < x < 1),
\]

obtained in Problem 7, Sec. 8, show that

\[
f(x) = \frac{1}{2} \cos \frac{\pi x}{2} + \sum_{n=1}^{\infty} \frac{n \sin 2\pi x}{2n^2 - 1} \quad (-\infty < x < \infty).
\]

6. Suppose that a function \( f \) is piecewise smooth on an interval \( 0 < x < c \), and let \( F \) denote this extension of \( f \) on the interval \( 0 < x < 2c \):

\[
F(x) = \begin{cases} 
f(x) & \text{when } 0 < x < c, \\
\frac{f(2c-x)}{c} & \text{when } c < x < 2c.
\end{cases}
\]

[The graph of \( y = F(x) \) is evidently symmetric with respect to the line \( x = c \).] Show that the coefficients \( b_n \) in the Fourier sine series for \( F \) on the interval \( 0 < x < 2c \) can be written

\[
b_n = \frac{1}{c} \int_{0}^{c} f(x) \sin \frac{n\pi x}{2c} \, dx \quad (n = 1, 2, \ldots).
\]

Thus show that

\[
f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{2c},
\]

where

\[
b_n = \frac{2}{\pi} \int_{0}^{c} f(x) \sin \frac{2n\pi x}{2c} \, dx \quad (n = 1, 2, \ldots),
\]

for each point \( x (0 < x < c) \) at which \( f \) is continuous.

Suggestion: Write

\[
b_n = \frac{1}{c} \left[ \int_{0}^{c} f(x) \sin \frac{n\pi x}{2c} \, dx + \int_{c}^{2c} f(2c-x) \sin \frac{n\pi x}{2c} \, dx \right]
\]

and make the substitution \( x = 2c-s \) in the second of these integrals.

7. Use the result in Problem 6 to establish the representation

\[
x = \frac{8c}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n-1} \sin \frac{(2n-1)c}{c} \quad (-c \leq x \leq c).
\]

8. After writing the Fourier series representation (3), Sec. 14, as

\[
f(x) = \frac{a_0}{2} + \lim_{n \to \infty} \sum_{n=1}^{N} \left( a_n \cos \frac{n\pi x}{c} + b_n \sin \frac{n\pi x}{c} \right),
\]
use the exponential forms:
\[
\cos \theta = \frac{e^{i\theta} + e^{-i\theta}}{2}, \quad \sin \theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}
\]
of the cosine and sine functions to put that representation in exponential form:
\[
f(x) = \lim_{N \to \infty} \sum_{n=-N}^{N} A_n \exp\left(\frac{i 2\pi n x}{c}\right).
\]
where
\[
A_0 = \frac{a_0}{2}, \quad A_n = \frac{a_n - ib_n}{2}, \quad A_{-n} = \frac{a_n + ib_n}{2}. \quad (n = 1, 2, \ldots).
\]
Then use expressions (4) and (5), Sec. 14, for the coefficients \(a_n\) and \(b_n\) to obtain the single formula
\[
A_n = \frac{1}{2c} \int_{-c}^{c} f(x) \exp\left(-i \frac{2\pi n x}{c}\right) dx \quad (n = 0, \pm 1, \pm 2, \ldots).
\]

15. A LEMMA

We prove here an important lemma to be used in Sec. 16. That section, regarding the absolute and uniform convergence of Fourier series, and Secs. 17 and 18, dealing with differentiation and integration of such series, will be used only occasionally later on and will be specifically cited as needed. Hence the reader may at this time pass directly to Chap. 3 without serious disruption.

For convenience, we treat only Fourier series for which the fundamental interval is \(-\pi < x < \pi\). Adaptations of our results to series on any fundamental interval \(-c < x < c\) can be made by the method used in Sec. 8.

**Lemma.** Let \(f\) denote a function such that

(i) \(f\) is continuous on the interval \(-\pi \leq x \leq \pi\);

(ii) \(f(-\pi) = f(\pi)\);

(iii) its derivative \(f'\) is piecewise continuous on the interval \(-\pi < x < \pi\).

If \(a_n\) and \(b_n\) are the Fourier coefficients

(1) \[a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, dx, \quad b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx\]

for \(f\), the series

(2) \[\sum_{n=1}^{\infty} \sqrt{a_n^2 + b_n^2}\]

converges.

---

The class of functions satisfying conditions (i) through (iii) here is, of course, a subspace of the space of piecewise smooth functions on the interval \(-\pi < x < \pi\).

We begin the proof of the lemma with the observation that the Fourier coefficients

(3) \[a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f'(x) \cos nx \, dx, \quad b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f'(x) \sin nx \, dx\]

for \(f'\) exist because of the piecewise continuity of \(f'\). Note that

\[a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f'(x) \, dx = \frac{1}{\pi} \left[f(\pi) - f(-\pi)\right] = 0.
\]

Also, since \(f\) is continuous and \(f(-\pi) = f(\pi)\), integration by parts reveals that when \(n = 1, 2, \ldots\),

\[a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} \cos nx \, f'(x) \, dx = \frac{1}{\pi} \left[(\cos nx) f(x)\right]_{-\pi}^{\pi} + n \int_{-\pi}^{\pi} f(x) \sin nx \, dx = nb_n.
\]

Likewise,

\[b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} \sin nx \, f'(x) \, dx = \frac{1}{\pi} \left[(\sin nx) f(x)\right]_{-\pi}^{\pi} - n \int_{-\pi}^{\pi} f(x) \cos nx \, dx = -na_n;
\]

and we find that

(4) \[a_n = -\frac{\beta_n}{n}, \quad b_n = \frac{\alpha_n}{n} \quad (n = 1, 2, \ldots).
\]

In view of relations (4), the sum \(s_N\) of the first \(N\) terms of the infinite series (2) becomes

(5) \[s_N = \sum_{n=1}^{N} \sqrt{a_n^2 + b_n^2} = \sum_{n=1}^{N} \frac{1}{\pi} \sqrt{a_n^2 + b_n^2}.
\]

**Cauchy's inequality**

\[
\left(\sum_{n=1}^{N} p_n q_n\right)^2 \leq \left(\sum_{n=1}^{N} p_n^2\right) \left(\sum_{n=1}^{N} q_n^2\right),
\]

which applies to any two sets of real numbers \(p_n (n = 1, 2, \ldots, N)\) and \(q_n (n = 1, 2, \ldots, N)\) (see Problem 6, Sec. 18, for a derivation), can now be used to write

(6) \[s_N^2 \leq \left(\sum_{n=1}^{N} \frac{1}{n^2}\right) \left(\sum_{n=1}^{N} (a_n^2 + b_n^2)\right) \quad (N = 1, 2, \ldots).
\]
The sequence of sums
\[ \sum_{n=1}^{N} \frac{1}{n^2} \quad (N = 1, 2, \ldots) \]
here is clearly bounded since each sum is a partial sum of the convergent series
whose terms are \(1/n^2\) [see Problem 6(a), Sec. 13]. The sequence
\[ \sum_{n=1}^{N} (a_n^2 + b_n^2) \quad (N = 1, 2, \ldots) \]
is also bounded since \(a_n\) (\(n = 0, 1, 2, \ldots\)) and \(b_n\) (\(n = 1, 2, \ldots\)) are the Fourier
coefficients for \(f\)' on the interval \(- \pi < x < \pi\) and must, therefore, satisfy Bessel's
inequality:
\[ \sum_{n=1}^{N} (a_n^2 + b_n^2) \leq \frac{1}{\pi} \int_{-\pi}^{\pi} |f'(x)|^2 \, dx \quad (N = 1, 2, \ldots). \]

[See Problem 4(b), Sec. 11.] It now follows from inequality (6) that the sequence
\(s_n^2\) (\(N = 1, 2, \ldots\)) is both bounded and nondecreasing. Hence it converges, and this
means that the sequence \(s_N\) (\(N = 1, 2, \ldots\)) converges. Thus series (2) converges.

16. ABSOLUTE AND UNIFORM CONVERGENCE OF FOURIER SERIES

We turn now to the absolute and uniform convergence of Fourier series. We begin
by recalling some facts about uniformly convergent series of functions.1

Let \(s(x)\) denote the sum of an infinite series of functions \(f_n(x)\), where the
series is convergent for all \(x\) in some interval \(a \leq x \leq b\). Thus
\[ s(x) = \sum_{n=1}^{\infty} f_n(x) = \lim_{N \to \infty} s_N(x) \quad (a \leq x \leq b), \]
where \(s_N(x)\) is the partial sum consisting of the sum of the first \(N\) terms of the
series. The series converges uniformly with respect to \(x\) if the absolute value of
its remainder \(r_N(x) = s(x) - s_N(x)\) can be made arbitrarily small for all \(x\) in
the interval by taking \(N\) sufficiently large; that is, for each positive number \(\varepsilon\),
there exists a positive integer \(N_{\varepsilon}\), independent of \(x\), such that
\[ |s(x) - s_N(x)| < \varepsilon \quad \text{whenever} \quad N > N_{\varepsilon} \quad (a \leq x \leq b). \]

A sufficient condition for uniform convergence is given by the Weierstrass
M-test. Namely, if there is a convergent series
\[ \sum_{n=1}^{\infty} M_n \]
of positive constants such that
\[ |f_n(x)| \leq M_n \quad (a \leq x \leq b) \]
for each \(n\), then series (1) is uniformly convergent on the stated interval.

We include here a few properties of uniformly convergent series that are
often useful. If the functions \(f_n\) are continuous and if series (1) is uniformly
convergent, then the sum \(s(x)\) of that series is a continuous function. Also,
the series can be integrated term by term over the interval \(a \leq x \leq b\) to give the
integral of \(s(x)\) from \(x = a\) to \(x = b\). If the functions \(f_n\) and their derivatives \(f'_n\)
are continuous, if series (1) converges, and if the series whose terms are \(f'_n\)
is uniformly convergent, then \(s'(x)\) is found by differentiating series (1) term
by term.

**Theorem.** Let \(f\) denote a function such that

(i) \(f\) is continuous on the interval \(- \pi \leq x \leq \pi\);

(ii) \(f(-\pi) = f(\pi)\);

(iii) its derivative \(f'\) is piecewise continuous on the interval \(- \pi < x < \pi\).

The Fourier series
\[ \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[ a_n \cos nx + b_n \sin nx \right] \]
for \(f\), with coefficients
\[ a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, dx, \quad b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx, \]
converges absolutely and uniformly to \(f(x)\) on the interval \(- \pi \leq x \leq \pi\).

To prove this, we first note that the conditions on \(f\) ensure the continuity of
the periodic extension of \(f\) for all \(x\). Hence it follows from the corollary in Sec. 12
that series (5) converges to \(f(x)\) everywhere in the interval \(- \pi \leq x \leq \pi\). Observe
how it follows from the inequalities
\[ |a_n| \leq \sqrt{a_n^2 + b_n^2} \quad \text{and} \quad |b_n| \leq \sqrt{a_n^2 + b_n^2} \]
that
\[ |a_n \cos nx + b_n \sin nx| \leq |a_n| + |b_n| \leq 2 \sqrt{a_n^2 + b_n^2} \quad (n = 1, 2, \ldots). \]

Since the series
\[ \sum_{n=1}^{\infty} \sqrt{a_n^2 + b_n^2} \]
converges, according to the lemma in Sec. 15, the comparison test and the
Weierstrass M-test thus apply to show that the convergence of series (5) is
absolute and uniform on the interval \(- \pi \leq x \leq \pi\), as stated.

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1 See, for instance, the book by Kaplan (2003, chap. 6) or the one by Taylor and Mann (1983, chap. 20), both listed in the Bibliography.
Modifications of the statements in both the lemma in Sec. 15 and the above theorem are apparent. For instance, it follows from the theorem that the Fourier cosine series on $0 < x < \pi$ for a function $f$ that is continuous on the closed interval $0 \leq x \leq \pi$ converges absolutely and uniformly to $f(x)$ when $0 \leq x \leq \pi$ if $f$ is piecewise continuous on $0 < x < \pi$. For the sine series, however, the additional conditions $f(0) = f(\pi) = 0$ are needed.

Since a uniformly convergent series of continuous functions always converges to a continuous function, a Fourier series for a function $f$ cannot converge uniformly on an interval that contains a point at which $f$ is discontinuous. Hence the continuity of $f$, assumed in the theorem, is necessary for the series there to converge uniformly.

Suppose that $x_0$ is a point at which a piecewise smooth function $f$ is discontinuous. The nature of the deviation near $x_0$ of the values of the partial sums of a Fourier series for $f$ from the values of $f$ is commonly referred to as the Gibbs phenomenon and is illustrated below.\(^1\)

**EXAMPLE.** Consider the piecewise smooth function defined by the equations

$$f(x) = \begin{cases} 
-\pi/2 & \text{when } -\pi < x < 0, \\
\pi/2 & \text{when } 0 < x < \pi,
\end{cases}$$

and $f(0) = 0$. According to Problem 1, Sec. 7, and Theorem 1 in Sec. 14, the Fourier (sine) series

$$\sum_{n=1}^{\infty} \frac{\sin((2n-1)x)}{2n-1} \quad (-\pi < x < \pi)$$

for $f$ converges to $f(x)$ everywhere in the interval $-\pi < x < \pi$.

Let $S_N(x)$ denote the sum of the first $N$ terms of this series. The sequence $S_N(x)$ ($N = 1, 2, \ldots$) thus converges to $f(x)$ when $-\pi < x < \pi$. In particular, it converges to the number $\pi/2 = 1.57 \ldots$ when $0 < x < \pi$. But, as shown in Problem 7, Sec. 18, there is a fixed number $\sigma = 1.85 \ldots$ such that

$$S_N\left(\frac{\pi}{2N}\right)$$

tends to $\sigma$. See Fig. 10, which indicates how "spikes" in the graphs of the partial sums $y = S_N(x)$, moving to the left as $N$ increases, are formed, their tips tending to the point $\sigma$ on the $y$ axis. The behavior of the partial sums is similar on the interval $-\pi < x < 0$.

This illustrates that special care must be taken when a function is approximated by a partial sum of its Fourier series near a point of discontinuity.

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\(^1\)For a detailed analysis of this phenomenon, see the book by Carslaw (1952, chap. 9), which is listed in the Bibliography.

17. DIFFERENTIATION OF FOURIER SERIES

Not all Fourier series are differentiable, as Example 1 illustrates.

**EXAMPLE 1.** According to Theorem 1 in Sec. 14, the Fourier series in Example 3, Sec. 7, for the function $f(x) = x (-\pi < x < \pi)$ converges to $f(x)$ at each point in the interval $-\pi < x < \pi$:

$$(1) \quad x = 2 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin nx \quad (-\pi < x < \pi).$$

But the differentiated series

$$\sum_{n=1}^{\infty} (-1)^{n+1} \cos nx$$

does not converge since its $n$th term fails to approach zero as $n$ tends to infinity.

Sufficient conditions for differentiability can be stated as follows, where the conditions on $f$ are the same as those in the theorem in Sec. 16, as well as in the lemma in Sec. 15.

**Theorem.** Let $f$ denote a function such that

(i) $f$ is continuous on the interval $-\pi \leq x \leq \pi$;
(ii) $f(-\pi) = f(\pi)$;
(iii) its derivative $f'$ is piecewise continuous on the interval $-\pi < x < \pi$.

The Fourier series representation

$$(2) \quad f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \quad (-\pi \leq x \leq \pi),$$

for $f(x)$ converges uniformly on $-\pi < x < \pi$ to $f(x)$, and the $n$th partial sum $S_n(x)$ converges to $f(x)$ at each point of continuity of $f'$.
where
\[ a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, dx, \quad b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx, \]
is differentiable at each point \( x \) in the interval \(-\pi < x < \pi\) at which the second-order derivative \( f'' \) exists:

\[ f''(x) = \sum_{n=1}^{\infty} (-1)^n n^2 b_n \cos nx. \]

Our proof of this theorem is especially brief. To start, we consider a point \( x \) \((-\pi < x < \pi\) at which \( f'' \) exists; and we note that \( f'' \) is therefore continuous at \( x \). Hence an application of the Fourier theorem in Sec. 12 to the function \( f'' \) shows that

\[ f''(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left( \frac{\alpha_n}{2} \cos nx + \beta_n \sin nx \right), \]

where
\[ \alpha_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f'(x) \cos nx \, dx, \quad \beta_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f'(x) \sin nx \, dx. \]

But since \( f \) and \( f'' \) satisfy all the conditions stated in the lemma in Sec. 15, we know from the proof there that

\[ a_0 = 0, \quad \alpha_n = nb_n, \quad \beta_n = -na_n \quad (n = 1, 2, \ldots). \]

When these substitutions are made, equation (4) takes the form (3); and the proof is complete.

At a point \( x \) where \( f''(x) \) does not exist, but where \( f'' \) has one-sided derivatives, differentiation is still valid in the sense that the series in equation (3) converges to the mean of the values \( f'(x+) \) and \( f'(x-) \). This is also true for the periodic extension of \( f \).

The theorem applies, with obvious changes, to other Fourier series. For instance, if \( f \) is continuous when \( 0 \leq x \leq \pi \) and \( f'' \) is piecewise continuous on the interval \( 0 < x < \pi \), then the Fourier cosine series for \( f \) on \( 0 < x < \pi \) is differentiable at each point \( x \) \((0 < x < \pi)\) where \( f''(x) \) exists.

**EXAMPLE 2.** We know from Problem 5, Sec. 7, and the corollary in Sec. 12 that when \( a \neq 0 \),

\[ \cosh ax = \frac{\sinh ax}{a} \left[ 1 + 2a^2 \sum_{n=1}^{\infty} \frac{(-1)^n}{a^2 + n^2} \cos nx \right] \]
on the closed interval \(-\pi \leq x \leq \pi\). Inasmuch as the hypothesis in the theorem here is satisfied when \( f(x) = \cosh ax \), it follows that

\[ a \sinh ax = \frac{2a \sinh ax}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n}{a^2 + n^2} (-n \sin nx) \]
on the interval \(-\pi < x < \pi\). That is, when \( a \neq 0 \),

\[ \sinh ax = \frac{2 \sinh ax}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{a^2 + n^2} \sin nx \quad (-\pi < x < \pi). \]

Note that equation (6) is, in fact, valid when the condition \( a \neq 0 \) is dropped.

18. INTEGRATION OF FOURIER SERIES

Integration of a Fourier series is possible under much more general conditions than those for differentiation. This is to be expected because an integration of the series in the correspondence

\[ f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \quad (-\pi < x < \pi), \]

where

\[ a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, dx, \quad b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx, \]

introduces a factor \( n \) in the denominator of the general term. In the following theorem, it is not even essential that the original series converge in order that the integrated series converge to the integral of the function.

**Theorem.** Let \( f \) be a function that is piecewise continuous on the interval \(-\pi < x < \pi\). Regardless of whether series (1) converges or not, the following equation is valid when \(-\pi \leq x \leq \pi\):

\[ \int_{-\pi}^{\pi} f(x) \, dx = \frac{a_0}{2} (x + \pi) + \sum_{n=1}^{\infty} \left[ a_n \sin nx - b_n [\cos nx + (-1)^{n+1}] \right]. \]

Series (3) is, of course, obtained by replacing \( x \) by \( s \) in series (1) and then integrating term by term from \( s = -\pi \) to \( s = x \). Observe that if \( a_0 \neq 0 \), the first term on the right in equation (3) is not of the type encountered in a Fourier series. Hence integrating a Fourier series does not always yield a Fourier series.

Our proof starts with the fact that since \( f \) is piecewise continuous, the function

\[ F(x) = \int_{-\pi}^{x} f(s) \, ds - \frac{a_0}{2} x \quad (-\pi \leq x \leq \pi) \]
is continuous; moreover,

\[ F'(x) = f(x) - \frac{a_0}{2} \quad (-\pi < x < \pi) \]
except at points where \( f \) is discontinuous. Hence \( F' \) is piecewise continuous on the interval \(-\pi < x < \pi\). Since \( F \) is piecewise smooth, then, it follows from Theorem 1 in Sec. 14 that

\[ F(x) = \frac{A_0}{2} + \sum_{n=1}^{\infty} (A_n \cos nx + B_n \sin nx) \quad (-\pi < x < \pi), \]
where

\[ A_n = \frac{1}{\pi} \int_{-\pi}^{\pi} F(x) \cos nx \, dx, \quad B_n = \frac{1}{\pi} \int_{-\pi}^{\pi} F(x) \sin nx \, dx. \]

We note from expression (4) and the first of expressions (2) when \( n = 0 \) that

\[ F(\pi) = \frac{a_0}{2} \pi \quad \text{and} \quad F(-\pi) = \frac{a_0}{2} \pi + \int_{-\pi}^{\pi} f(s) \, ds - \frac{a_0}{2} \pi = a_0 \pi - \frac{a_0}{2} \pi = \frac{a_0}{2} \pi; \]

hence

\[ F(-\pi) = F(\pi). \]

This shows that representation (6) is also valid at the endpoints of the open interval \(-\pi < x < \pi\) (see Sec. 13) and, therefore, at each point of the closed interval \(-\pi \leq x \leq \pi\).

Let us now write the coefficients \( A_n \) and \( B_n \) in terms of \( a_n \) and \( b_n \). When \( n \geq 1 \), we may integrate integrals (7) by parts, using the fact that \( F \) is continuous and \( F' \) is piecewise continuous. Thus

\[ A_n = \frac{1}{\pi} \left\{ \left[ F(x) \frac{\sin nx}{n} \right]_{-\pi}^{\pi} - \int_{-\pi}^{\pi} \frac{\sin nx}{n} F'(x) \, dx \right\} \]

\[ = \frac{1}{n\pi} \int_{-\pi}^{\pi} F'(x) \sin nx \, dx; \]

and, in view of expression (5) for \( F'(x) \), we have

\[ A_n = \frac{1}{n\pi} \int_{-\pi}^{\pi} \left[ f(x) - \frac{a_0}{2} \right] \sin nx \, dx \]

\[ = -\frac{1}{n} \int_{-\pi}^{\pi} f(x) \sin nx \, dx + \frac{a_0}{2n\pi} \int_{-\pi}^{\pi} \sin nx \, dx = -\frac{b_n}{n}. \]

Likewise, keeping relation (9) in mind, we find that

\[ B_n = \frac{1}{n\pi} \int_{-\pi}^{\pi} F'(x) \cos nx \, dx; \]

and, using expression (5) once again, we can see that

\[ B_n = \frac{1}{n} \int_{-\pi}^{\pi} f(x) \cos nx \, dx - \frac{a_0}{2n\pi} \int_{-\pi}^{\pi} \cos nx \, dx = \frac{a_n}{n}. \]

As for \( A_0 \), from the final value for \( F(\pi) \) shown in the second of relations (8) and the fact that representation (6) is valid when \( x = \pi \), we know that

\[ \frac{a_0}{2 \pi} \pi = \frac{a_0}{2} + \sum_{n=1}^{\infty} A_n(-1)^n. \]

So, by solving for \( A_0 \) here and then using and relation \( A_n = -\frac{b_n}{n} \) found above, we arrive at

\[ A_0 = a_0 \pi - 2 \sum_{n=1}^{\infty} A_n(-1)^n = a_0 \pi - 2 \sum_{n=1}^{\infty} \frac{(-1)^n+1}{n} - b_n. \]
in Example 1, Sec. 17, and the one
\[
2 \sum_{n=1}^{\infty} \frac{\sin(2n-1)x}{2n-1}
\]
appearing in the example in Sec. 16. In each case, describe graphically the function that
is represented by the new series.
6. Let \( p_n \) (\( n = 1, 2, \ldots, N \)) and \( q_n \) (\( n = 1, 2, \ldots, N \)) denote real numbers, where at least
one of the numbers \( p_n \), say \( p_{n_0} \), is nonzero. By writing the quadratic equation
\[
x^2 \sum_{n=1}^{N} p_n^2 + 2x \sum_{n=1}^{N} p_nq_n + \sum_{n=1}^{N} q_n^2 = 0
\]
in the form
\[
\sum_{n=1}^{N} (p_nx + q_n)^2 = 0.
\]
show that the number \( x_0 = -q_{n_0}/p_{n_0} \) is the only possible real root. Conclude that since
there cannot be two distinct real roots, the discriminant
\[
\left( \sum_{n=1}^{N} p_nq_n \right)^2 - 4 \left( \sum_{n=1}^{N} p_n^2 \right) \left( \sum_{n=1}^{N} q_n^2 \right)
\]
of this quadratic equation is negative or zero. Thus derive Cauchy's inequality (Sec. 15)
\[
\left( \sum_{n=1}^{N} p_nq_n \right)^2 \leq \left( \sum_{n=1}^{N} p_n^2 \right) \left( \sum_{n=1}^{N} q_n^2 \right),
\]
which is clearly valid even if all the numbers \( p_n \) are zero.
7. As in the example in Sec. 16, let \( S_0(x) \) denote the partial sum consisting of the sum of
the first \( N \) terms of the Fourier series
\[
2 \sum_{n=1}^{\infty} \frac{\sin(2n-1)x}{2n-1} \quad (-\pi < x < \pi)
\]
for the function \( f \) there.
(a) By writing \( A = x \) and \( B = (2n-1)x \) in the trigonometric identity
\[
2 \sin A \cos B = \sin(A + B) + \sin(A - B)
\]
and then summing each side of the resulting equation from \( n = 1 \) to \( n = N \), derive
the summation formula
\[
2 \sum_{n=1}^{N} \cos(2n-1)x = \frac{\sin 2Nx}{\sin x} \quad (x \neq 0, \pm \pi, \pm 2\pi, \ldots).
\]
Use this formula to write the derivative of \( S_N(x) \) on the interval \( 0 < x < \pi \) as a
simple quotient:
\[
S'_N(x) = \frac{\sin 2Nx}{\sin x} \quad (0 < x < \pi).
\]
(b) With the aid of the expression for the derivative \( S'_N(x) \) in part (a), show that the
first extremum of \( S_N(x) \) in the interval \( 0 < x < \pi \) is a relative maximum occurring
when \( x = \pi/(2N) \).
(c) By integrating each side of the summation formula in part (a) from \( x = 0 \) to
\( x = \pi/(2N) \), show that
\[
S_N \left( \frac{\pi}{2N} \right) = I_1 + I_2
\]
where
\[
I_1 = \int_0^{\pi/(2N)} \frac{x - \sin x}{x \sin x} \sin 2Nx \, dx \quad \text{and} \quad I_2 = \int_0^{\pi/(2N)} \sin 2Nx \, dx.
\]
Verify that the integrands of these two integrals are piecewise continuous on the
interval \( 0 < x < \pi/(2N) \) and hence that the integrals actually exist.
(d) Using the fact that the integrand of the integral \( I_1 \) in part (c) is bounded (see Sec. 1),
show that the value of \( I_1 \) tends to zero as \( N \) tends to infinity. Then conclude that
\[
\lim_{N \to \infty} S_N \left( \frac{\pi}{2N} \right) = \int_0^\pi \frac{\sin t}{t} \, dt.
\]
The value of this last integral is the number \( \sigma \) in the example in Sec. 16.\(^1\)

\(^1\) The integral occurs as a particular value of the sine integral function \( Si(x) \), which is tabulated in, for
instance, the handbook edited by Abramowitz and Stegun (1972, p. 244), listed in the Bibliography. Approximation
methods for evaluating definite integrals can also be used to find \( \sigma \).