FINITE ELEMENT APPROXIMATIONS IN A NON-LIPSCHITZ DOMAIN

GABRIEL ACOSTA, MARÍA G. ARMENTANO, RICARDO G. DURÁN, AND ARIEL L. LOMBARDI

Abstract. In this paper we analyze the approximation by standard piecewise linear finite elements of a non homogeneous Neumann problem in a cuspidal domain.

Since the domain is not Lipschitz, many of the results on Sobolev spaces which are fundamental in the usual error analysis do not apply. Therefore, we need to work with weighted Sobolev spaces and to develop some new theorems on traces and extensions.

We show that, in the domain considered here, suboptimal order can be obtained with quasi-uniform meshes even when the exact solution is in H^2, and we prove that the optimal order with respect to the number of nodes can be recovered by using appropriate graded meshes.

1. Introduction

The finite element method has been widely analyzed in its different forms for all kind of partial differential equations. However, as far as we know, all analyses are restricted to the case of polygonal or smooth domains and no results have been obtained for the case in which the domain is non Lipschitz, with the exception of the well known fracture problems.

The goal of this paper is to start the analysis of finite element approximations in non-Lipschitz domains. As a first step in this direction we consider a model problem in a plane domain with an external cusp.

Several difficulties arise in this problem because many of the results on Sobolev spaces, which are fundamental in the analysis of partial differential equations in variational form, do not apply. For example, the standard trace theorems do not hold in this case and this fact makes the analysis of non homogeneous Neumann problems harder.

Given $\alpha > 1$, let $\Omega \subset \mathbb{R}^2$ be the domain defined by

$$\Omega = \{(x, y) : 0 < x < 1,\, 0 < y < x^\alpha\},$$

and $\Gamma = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$ its boundary, with

$$\Gamma_1 = \{0 \le x \le 1, y = 0\}, \quad \Gamma_2 = \{x = 1, 0 \le y \le 1\} \quad \text{and} \quad \Gamma_3 = \{0 \le x \le 1, y = x^\alpha\}$$

(see Figure 1).

Some of our arguments require that $\alpha < 3$ and so our main result will be valid under this restriction.

Our model problem is

1991 *Mathematics Subject Classification.* 65N30,46E35.

Key words and phrases. Cuspidal domains, Neumann problem, Finite elements, Graded meshes.

Supported by ANPCyT under grant PICT 03-13719, Fundación Antorchas and by Universidad de Buenos Aires under grant X052. The second and third authors are members of CONICET, Argentina. The first author is partially supported by PICT 03-10724.

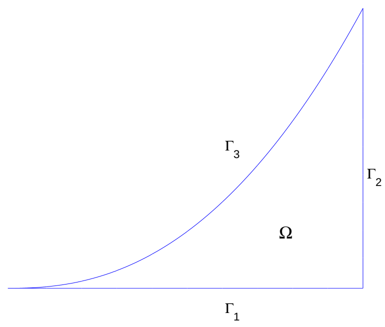

FIGURE 1. Cuspidal domain

$$\begin{cases} -\Delta u = f\,, & \text{in } \Omega \\ \dfrac{\partial u}{\partial \nu} = g\,, & \text{on } \Gamma_3 \\ \dfrac{\partial u}{\partial \nu} = 0\,, & \text{on } \Gamma_1 \\ u = 0\,, & \text{on } \Gamma_2 \end{cases} \tag{1.1}$$

where ν denotes the outside normal.

A natural way to approximate the solution of problem (1.1) is to replace Ω by a polygonal domain and to use the standard linear finite element method. It is known that, under appropriate conditions on the data, the solution of this problem is in $H^2(\Omega)$ (see [1]). Therefore, based on the experience and theory for smooth domains, one would expect that the optimal order of convergence could be obtained by using quasi-uniform meshes. However, numerical examples show that this is not the case (see Section 2). The reason for this behavior seems to be the fact that the solution can not be extended to an H^2 function on the polygonal domain approximating the original domain. Indeed, it is known that the standard extension theorems in Sobolev spaces do not apply for our domain (see for example [12]).

We will show that the optimal order with respect to the number of nodes in the H^1 norm can be recovered by using appropriate graded meshes. To obtain this result, we will first prove an extension theorem for the domain Ω which shows that the solution of problem (1.1) can be extended to a function in a weighted H^2 space, the weight being a power of the distance to the cuspidal point.

The rest of the paper is organized as follows. In Section 2 we introduce the finite element approximation of our problem and show that the use of quasi-uniform meshes can give bad results. Section 3 deals with some extension and trace theorems in weighted Sobolev spaces that we need for our error analysis. Finally, in Section 4 we prove that optimal order approximations are obtained by using appropriate graded meshes.

2. Finite element approximations

In this section we introduce the finite element approximation of our model problem and show that, if the meshes are quasi-uniform, the approximation may be of suboptimal order even when the exact solution is in $H^2(\Omega)$.

Introducing the space

$$V = \{v \in H^1(\Omega) : v|_{\Gamma_2} = 0\},$$

the weak form of Problem (1.1) is to find $u \in V$ such that

$$\int_\Omega \nabla u \cdot \nabla v = \int_\Omega fv + \int_{\Gamma_3} gv \qquad \forall v \in V. \tag{2.1}$$

The following existence and regularity results have been proved in [1]: Define $z(t) := g(t, t^\alpha)$. If $f \in L^2(\Omega)$ and $z\, t^{-\frac{\alpha}{2}} \in L^2(0,1)$ this problem has a unique solution. If in addition we assume that $z'\, t^{1-\frac{\alpha}{2}} \in L^2(0,1)$, the solution is in $H^2(\Omega)$ and there exists a constant C such that

$$\|u\|_{H^2(\Omega)} \leq C \left\{ \|f\|_{L^2(\Omega)} + \|z\, t^{-\frac{\alpha}{2}}\|_{L^2(0,1)} + \|z'\, t^{1-\frac{\alpha}{2}}\|_{L^2(0,1)} \right\}. \tag{2.2}$$

To approximate the solution of (1.1) we replace Ω by a polygonal domain Ω_h and use the standard linear finite element method. We will construct Ω_h in such a way that $\Omega \subset \Omega_h$ and the nodes on Γ_h, the boundary of Ω_h, are also on Γ.

Let $\{\mathcal{T}_h\}$ be a family of triangulations of Ω_h satisfying the maximum angle condition. Associated with $\{\mathcal{T}_h\}$ we have the finite element space

$$V_h = \{v \in H^1(\Omega_h) : v|_{\Gamma_2} = 0 \,\text{and}\; v|_T \in \mathcal{P}_1 \quad \forall T \in \mathcal{T}_h\}$$

where $\mathcal{P}_1$ denotes the space of linear polynomials.

Denote with $\Gamma_{3,h}$ the part of Γ_h approximating Γ_3 and with I_h the piecewise linear interpolation at the endpoints of the segments which lie on $\Gamma_{3,h}$.

Then, our discrete problem is to find $u_h \in V_h$ such that

$$\int_{\Omega_h} \nabla u_h \cdot \nabla v = \int_\Omega fv + \int_{\Gamma_{3,h}} I_h(gv) \quad \forall v \in V_h. \tag{2.3}$$

Observe that the discrete problem corresponds to a boundary problem on Ω_h if we consider f as being extended by zero outside Ω.

One could think that, when the solution is in $H^2(\Omega)$, the numerical approximation obtained with quasi-uniform meshes would be of optimal order. However, the following example shows that this is not the case.

Example 2.1. *Consider*

$$f(x,y) = s(s-1)(1+y^2/2)x^{s-2} + x^s - 1$$

and

$$z(t) = g(t, t^\alpha) = \frac{-s\alpha t^{\alpha+s-2}(1+t^{2\alpha}/2) + (1-t^s)t^\alpha}{\sqrt{1+\alpha^2 t^{2(\alpha-1)}}}.$$

Then, the solution of (1.1) is

$$u(x,y) = (1-x^s)(1+y^2/2)$$

and an easy calculation shows that $u \in H^2(\Omega)$ whenever $s > \frac{3-\alpha}{2}$.

We take $\alpha = 2$, $\frac{1}{2} < s < 1$ and solve Problem (2.3) by using quasi-uniform meshes. The numerical results are presented in Table 1.

The reason for this behavior seems to be the fact that the solution can not be extended to an H^2 function on Ω_h. Indeed, it is well known that the standard extension theorems in Sobolev spaces do not apply for our domain (see for example [12]).

value of s ($\alpha = 2$)	in number of nodes	in h
0.55	0.35251875520376	0.67178134623590
0.6	0.38975451668143	0.74274009553459
0.65	0.42721035098256	0.81411822909419
0.7	0.46467723338678	0.88551741659622
0.75	0.50173913962734	0.95614485669926
0.8	0.53758303749553	1.02445118539463
0.85	0.57069966903139	1.08756026820953
0.9	0.59856739468917	1.14066671427110
0.95	0.61772738036031	1.17717915730572

TABLE 1. H^1 order using quasi-uniform meshes

3. EXTENSION AND TRACE THEOREMS

The standard results on extensions and restrictions in Sobolev spaces do not apply for domains with external cusps. In this section we prove some weaker results using weighted norms.

First, we develop an extension theorem in a weighted Sobolev space for $H^2(\Omega)$ functions with vanishing normal derivative on Γ_1. In particular, our theorem applies to solutions of (1.1) which, in view of (2.2), are in $H^2(\Omega)$ under appropriate assumptions on the data.

Second, we prove a trace theorem for functions in $H^1(\Omega)$ which will be useful to estimate the error due to the approximation of the non homogeneous Neumann type boundary condition.

Given a domain $D \subset \mathbb{R}^2$ we introduce the weighted Sobolev space

$$H^2_\alpha(D) = \left\{ v : r^{\frac{\alpha-1}{2}} D^\gamma v \in L^2(D) \text{ for any } |\gamma| \le 2 \right\}$$

where $r = \sqrt{x^2+y^2}$, and its natural norm

$$\|v\|^2_{H^2_\alpha(D)} = \sum_{|\gamma|\le 2} = \|r^{\frac{\alpha-1}{2}} D^\gamma v\|^2_{L^2(D)}.$$

Our argument proceeds in two steps. First, we extend the given function to the Lipschitz domain

$$D := \{(x,y) \in \mathbb{R}^2 : \quad -x < y < x^\alpha , \quad 0 < x < 1\}.$$

(see Figure 2) in such a way that the extension belongs to $H^2_\alpha(D)$. Then, we apply known theorems for weighted Sobolev spaces on Lipschitz domains to obtain an extension which belongs to $H^2_\alpha(\mathbb{R}^2)$.

We call W the subspace of $H^2(\Omega)$ defined by

$$W = \{u \in H^2(\Omega) : \quad \frac{\partial u}{\partial \nu} = 0 \text{ on } \Gamma_1\}$$

Lemma 3.1. *Given $u \in W$ there exists a function $\tilde{u} \in H^2_\alpha(D)$ such that $\tilde{u}|_\Omega = u$ and*

$$\|\tilde{u}\|_{H^2_\alpha(D)} \le C\|u\|_{H^2(\Omega)}.$$

Proof. We extend u by a reflection in the following way. Given $(x,y) \in D$ with $y \le 0$, let $\eta = -x^{\alpha-1}y \in \Omega$. Observe that $(x,\eta) \in \Omega$ and therefore we can define

$$\begin{cases} \tilde{u}(x,y) = u(x,y), & \text{for } \quad (x,y) \in \Omega \\ \tilde{u}(x,y) = u(x,\eta), & \text{for } \quad (x,y) \in D \setminus \Omega \end{cases}$$

To simplify notation define $T_L := D \setminus \overline{\Omega}$.

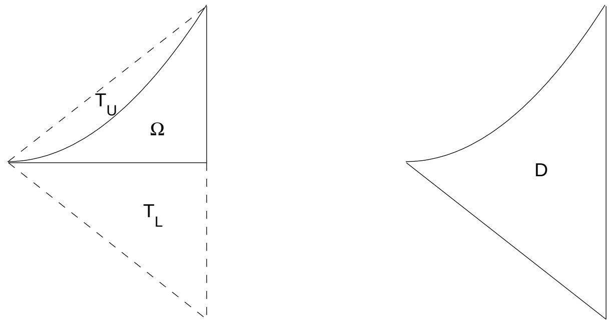

FIGURE 2

We claim that $\tilde{u} \in H^2_\alpha(T_L)$. Observe first that for $(x,y) \in T_L$ we have $x \sim r$ and therefore we can replace the weight $r^{\alpha-1}$ by $x^{\alpha-1}$ in our estimates.

By a change of variables we obtain

$$\int_{T_L} \tilde{u}^2(x,y)x^{\alpha-1}dxdy = \int_\Omega u^2(x,\eta)dxd\eta = \|u\|^2_{L^2(\Omega)}.$$

Now, for $(x,y) \in T_L$ we have

$$\frac{\partial \tilde{u}}{\partial x}(x,y) = \frac{\partial u}{\partial x}(x,\eta) - \frac{\partial u}{\partial \eta}(x,\eta)(\alpha-1)x^{\alpha-2}y$$

and

$$\frac{\partial \tilde{u}}{\partial y}(x,y) = -\frac{\partial u}{\partial \eta}(x,\eta)x^{\alpha-1}.$$

Then, recalling that $\eta = -x^{\alpha-1}y$, we obtain

$$\int_{T_L} \left(\frac{\partial \tilde{u}}{\partial x}\right)^2 x^{\alpha-1}dxdy \le C\left\{\int_\Omega \left(\frac{\partial u}{\partial x}\right)^2 dxd\eta + \int_\Omega \left(\frac{\partial u}{\partial \eta}\right)^2 \left(\frac{\eta}{x}\right)^2 dxd\eta\right\}$$

but, since $(x,\eta) \in \Omega$, we have $\frac{\eta}{x} \le x^{\alpha-1} \le 1$ and then

$$\int_{T_L} \left(\frac{\partial \tilde{u}}{\partial x}\right)^2 x^{\alpha-1}dxdy \le C\|\nabla u\|^2_{L^2(\Omega)}.$$

Analogously we get

$$\int_{T_L} \left(\frac{\partial \tilde{u}}{\partial y}\right)^2 x^{\alpha-1}dxdy \le C\|\nabla u\|^2_{L^2(\Omega)}.$$

Bounds for the second derivatives of $\tilde{u}$ follow similarly. For instance, we have

$$\begin{aligned}\frac{\partial^2 \tilde{u}}{\partial x^2}(x,y) &= \frac{\partial^2 u}{\partial x^2}(x,\eta) - 2(\alpha-1)\frac{\partial^2 u}{\partial \eta \partial x}(x,\eta)x^{\alpha-2}y \\ &- \frac{\partial^2 u}{\partial \eta^2}(x,\eta)(\alpha-1)^2x^{2(\alpha-2)}y^2 - \frac{\partial u}{\partial \eta}(x,\eta)(\alpha-2)(\alpha-1)x^{\alpha-3}y\end{aligned}$$

hence,

$$\begin{aligned}\int_{T_L}\left(\frac{\partial^2\tilde{u}}{\partial x^2}\right)^2 x^{\alpha-1}dxdy \quad &\leq \quad C\left\{\int_\Omega\left(\frac{\partial^2 u}{\partial x^2}\right)^2 dxd\eta+\int_\Omega\left(\frac{\partial^2 u}{\partial\eta\partial x}\right)^2\left(\frac{\eta}{x}\right)^2 dxd\eta\right.\\ &\quad\left.+\int_\Omega\left(\frac{\partial^2 u}{\partial\eta^2}\right)^2\left(\frac{\eta}{x}\right)^4 dxd\eta+\int_\Omega\left(\frac{\partial u}{\partial\eta}\right)^2\left(\frac{\eta}{x^2}\right)^2 dxd\eta\right\}.\end{aligned}$$

Now, the first three terms on the right hand side can be bounded using again that $\frac{\eta}{x}\leq 1$. For the last term we have

$$\int_\Omega\left(\frac{\partial u}{\partial\eta}\right)^2\left(\frac{\eta}{x^2}\right)^2 dxdy\leq\int_0^1\int_0^{x^\alpha}\left(\frac{\partial u}{\partial\eta}\right)^2\frac{1}{\eta^2}d\eta dx\leq C\int_0^1\int_0^{x^\alpha}\left(\frac{\partial^2 u}{\partial\eta^2}\right)^2 d\eta dx$$

where the last inequality follows from the Hardy inequality [10] and the fact that $\frac{\partial u}{\partial\eta}(x,0)=0$. Hence,

$$\int_{T_L}\left(\frac{\partial^2\tilde{u}}{\partial x^2}\right)^2 x^{\alpha-1}dxdy\leq C\|D^2u\|^2_{L^2(\Omega)}.$$

In a similar way we can show that

$$\int_{T_L}\left(\frac{\partial^2\tilde{u}}{\partial y\partial x}\right)^2 x^{\alpha-1}dxdy\leq C\|D^2u\|^2_{L^2(\Omega)}$$

and

$$\int_{T_L}\left(\frac{\partial^2\tilde{u}}{\partial y^2}\right)^2 x^{\alpha-1}dxdy\leq C\|D^2u\|^2_{L^2(\Omega)}.$$

Therefore, we have proved that $\tilde{u}\in H^2_\alpha(T_L)$ and that

$$\|\tilde{u}\|_{H^2_\alpha(T_L)}\leq C\|u\|_{H^2(\Omega)}.$$

On the other hand, using that $\frac{\partial u}{\partial\nu}=0$ on Γ_1, it is easy to see that $\tilde{u}\in H^2_\alpha(D)$, thus concluding the proof. □

Now, using known extension theorems for weighted Sobolev spaces on Lipschitz domains due to Chua [6], we can extend functions in W to $H^2_\alpha(\mathbb{R}^2)$.

Theorem 3.1. *If $\alpha<3$ and $u\in W$, there exists a function $\tilde{u}\in H^2_\alpha(\mathbb{R}^2)$ such that $\tilde{u}|_\Omega=u$, and*

$$\|\tilde{u}\|_{H^2_\alpha(\mathbb{R}^2)}\leq C\|u\|_{H^2(\Omega)}.$$

Proof. In view of Lemma 3.1 we only have to show that for $v\in H^2_\alpha(D)$ there exists an extension $\tilde{v}\in H^2_\alpha(\mathbb{R}^2)$ such that

$$\|\tilde{v}\|_{H^2_\alpha(\mathbb{R}^2)}\leq C\|v\|_{H^2_\alpha(D)}.$$

But this follows immediately from the results in [6] because, for $1<\alpha<3$, our weight belongs to the class considered in that paper (the Muckenhoupt class A_2) [7, page 145]. □

In the rest of this section we prove a trace theorem for functions in $H^1(\Omega)$. In [1] it was proved that

$$\|u\|_{L^2(\Gamma)}\leq C(\|ux^{-\frac{\alpha}{2}}\|_{L^2(\Omega)}+\|\nabla ux^{\frac{\alpha}{2}}\|_{L^2(\Omega)}). \tag{3.1}$$

Our trace theorem is a consequence of this result and the known imbedding theorem

$$H^1(\Omega)\subset L^r(\Omega)\quad\text{for}\quad 2\leq r\leq\frac{2(\alpha+1)}{\alpha-1} \tag{3.2}$$

which is a particular case of the results given in [2].

Theorem 3.2. *Let* $u \in H^1(\Omega)$*,*

(1) *If* $\alpha < 2$ *then* $u \in L^2(\Gamma)$ *and* $\|u\|_{L^2(\Gamma)} \leq C\|u\|_{H^1(\Omega)}$
(2) *If* $\alpha \geq 2$ *then* $x^\beta u \in L^2(\Gamma)$ *and* $\|x^\beta u\|_{L^2(\Gamma)} \leq C\|u\|_{H^1(\Omega)}$ *,* $\forall \beta > \alpha/2 - 1$.

Proof. Part (1) was proved in [1]. Therefore, we will only prove here (2).

Using (3.1) for the function $x^\beta u$ we have

$$\|x^\beta u\|_{L^2(\Gamma)} \leq C(\|x^\beta u x^{-\frac{\alpha}{2}}\|_{L^2(\Omega)} + \|\nabla(x^\beta u)x^{\frac{\alpha}{2}}\|_{L^2(\Omega)}).$$

It is easy to see that the second term on the right hand side is bounded by $\|u\|_{H^1(\Omega)}$ because $\alpha \geq 2$ and $\beta > \alpha/2 - 1$. Then, it is enough to show that

$$\|x^\beta u x^{-\frac{\alpha}{2}}\|_{L^2(\Omega)} \leq \|u\|_{H^1(\Omega)}. \tag{3.3}$$

Using the Hölder inequality we have

$$\int_\Omega u^2 x^{2\beta-\alpha} \leq \Big(\int_\Omega u^{2q}\Big)^{\frac{1}{q}} \Big(\int_\Omega x^{(2\beta-\alpha)\frac{q}{q-1}}\Big)^{\frac{q-1}{q}}.$$

Choosing $q = r/2$ with $r = 2(\alpha+1)/(\alpha-1)$ and using the imbedding theorem (3.2) we obtain

$$\|x^\beta u x^{-\frac{\alpha}{2}}\|_{L^2(\Omega)} \leq \Big(\int_\Omega x^{(2\beta-\alpha)\frac{q}{q-1}}\Big)^{\frac{q-1}{2q}} \|u\|_{H^1(\Omega)}.$$

But

$$\int_\Omega x^{(2\beta-\alpha)\frac{q}{q-1}} = \int_\Omega x^{(2\beta-\alpha)\frac{\alpha+1}{2}} < \infty$$

because $\beta > \alpha/2 - 1$ and therefore (3.3) holds. □

4. Optimal approximations using graded meshes

In this section we obtain error estimates in H^1 of quasi-optimal order (i.e., optimal up to a logarithmic factor) with respect to the number of nodes by using appropriate graded meshes.

Finite element methods using graded meshes of the type considered here have been analyzed for problems with corner type singularities in [3, 4, 9]. In [4, 9] the error estimates were obtained under the classic regularity condition on the meshes (the minimum angle condition). This hypothesis has been relaxed in [3], where the author obtained error estimates under the maximum angle condition. This generalization is very important for our problem because we can not avoid small angles in those elements which are near the cusp.

Consider $1 < \alpha < 3$ and define $\gamma = (\alpha-1)/2$. Let Ω_h be an approximating polygon and $\mathcal{T}_h$ a triangulation of it, where $h > 0$ is a parameter that goes to 0. For each $T \in \mathcal{T}_h$ we denote by h_T its diameter and by β_T its maximum angle. We assume that there exist positive constants σ and $\beta < \pi$, independent of h, such that

(1) $\beta_T < \beta$, $\forall T \in \mathcal{T}_h$ *(the maximal angle condition).*
(2) $h_T \sim \sigma\, h^{\frac{1}{1-\gamma}}$, if $(0,0) \in T$.
(3) $h_T \leq \sigma\, h\, \inf_T x^\gamma$, if $(0,0) \notin T$.

Since we know that the solution of our problem has an extension $\tilde{u} \in H^2_\alpha(\Omega_h)$, we are interested in interpolation error estimates for functions in this space. We call $\Pi v \in V_h$ the piecewise linear Lagrange interpolation of v.

Theorem 4.1. *If* $v \in H^2_\alpha(\Omega_h)$ *and the family of triangulations satisfies conditions (1), (2) and (3), there exists a constant* C *depending only on* β*,* σ *and* α *such that*

$$\|v - \Pi v\|_{H^1(\Omega_h)} \leq Ch\|v\|_{H^2_\alpha(\Omega_h)}.$$

Proof. It follows as in [9, page 392] but using the error estimates obtained by Apel under the maximum angle condition (see Theorem 6.9 in [3, page 40]). ☐

Now we introduce some notation which will be used in the rest of this section. We denote by $\Gamma^j_{3,h}$, $1 \leq j \leq n$ the edges on the boundary of Ω_h, by $(x_{j-1}, x^\alpha_{j-1})$ and (x_j, x^α_j) their endpoints and by Γ^j_3 the part on Γ_3 with the same endpoints. Let Ω^j_h be the region bounded by Γ^j_3 and $\Gamma^j_{3,h}$.

In addition to the assumptions (1), (2) and (3) we will need for our error analysis the following hypothesis on the meshes:

(H) For $1 \leq j \leq n$ the region Ω^j_h is contained in only one triangle.

We denote by T_j the triangle containing Ω^j_h and by h_j its diameter (see Figure 3).

It can be seen from our hypotheses that there exists a constant C, independent of h, such that, for $2 \leq j \leq n$,

$$x_j \leq C x_{j-1}. \tag{4.1}$$

In fact, from (H) we have $x_j - x_{j-1} \leq C|\Gamma^j_{3,h}|$ for some constant C depending only on α. Then, $x_j - x_{j-1} \leq Ch_j$, and therefore from assumption (3) we have

$$x_j \leq x_{j-1}\left(1 + Chx^{\gamma-1}_{j-1}\right)$$

and, since $j \geq 2$, $x_{j-1} \geq x_1 \sim h^{1/(1-\gamma)}$ by assumption (2), we obtain (4.1).

We will show below that meshes satisfying all our assumptions can indeed be constructed.

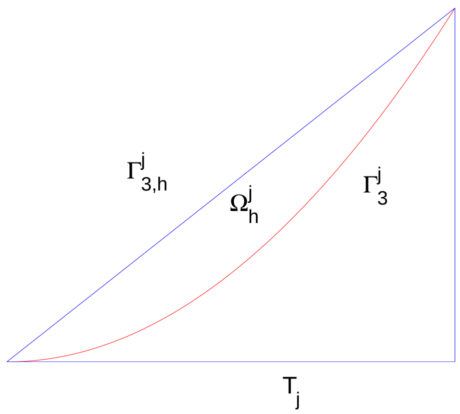

FIGURE 3

The next lemma deals with the error arising from the approximation of the domain by polygonal domains. We will work with an extension $\tilde{u}$ of the solution u of (1.1). Since $u \in W$ we know from Theorem 3.1 that there exists $\tilde{u} \in H^2_\alpha(\mathbb{R}^2)$ such that $\tilde{u}|_\Omega = u$ and

$$\|\tilde{u}\|_{H^2_\alpha(\mathbb{R}^2)} \leq C\|u\|_{H^2(\Omega)}. \tag{4.2}$$

We will make use of the well known imbedding $H^1(D) \subset L^p(D)$ for planar Lipschitz domains and $1 \leq p < \infty$, and of the explicit dependence on p of the constant in the continuity of this inclusion (see for example [8]), namely,

$$\|v\|_{L^p(D)} \leq C\sqrt{p}\,\|v\|_{H^1(D)}. \tag{4.3}$$

Lemma 4.1. *If $1 < \alpha < 3$, then there exists a constant C, which depends only on α, β and σ, such that*

$$\|\nabla \tilde{u}\|_{L^2(\Omega_h \setminus \Omega)} \leq Ch\sqrt{\log(1/h)}\, \|u\|_{H^2(\Omega)}.$$

Proof. Clearly, for every h, the polygonal domain Ω_h is contained in the triangle

$$T_U = \{0 \leq x \leq 1,\, 0 \leq y \leq x\}$$

(see Figure 2). Writing

$$\int_{T_U} |v|^p = \int_{T_U} |v|^p x^{p(\frac{\alpha-1}{2})} x^{-p(\frac{\alpha-1}{2})}$$

and applying the Hölder inequality with $2/p$ and its dual exponent we obtain

$$\|v\|_{L^p(T_U)} \leq C\|v x^{\frac{\alpha-1}{2}}\|_{L^2(T_U)}$$

for any function v and $1 \leq p < \frac{4}{\alpha+1}$. Therefore, using (4.2) we conclude that $\tilde{u} \in W^{2,p}(T_U)$ and that

$$\|\tilde{u}\|_{W^{2,p}(T_U)} \leq C\|u\|_{H^2(\Omega)}. \tag{4.4}$$

As a consequence, we obtain that, for $\beta > \frac{\alpha-1}{2}$, $\nabla \tilde{u}\, x^\beta \in H^1(T_U)$ and

$$\|\nabla \tilde{u}\, x^\beta\|_{H^1(T_U)} \leq C\|u\|_{H^2(\Omega)}. \tag{4.5}$$

Indeed, since $\tilde{u} \in H^2_\alpha(\mathbb{R}^2)$, we already know that $\nabla \tilde{u}\, x^\beta \in L^2(T_U)$ and so, we only have to see that the first derivatives of $\nabla \tilde{u}\, x^\beta$ belong to $L^2(T_U)$. But, taking the derivative of $\nabla \tilde{u}\, x^\beta$ and using again that $\tilde{u} \in H^2_\alpha(\mathbb{R}^2)$, we see that it only remains to prove that $\nabla \tilde{u}\, x^{\beta-1} \in L^2(T_U)$.

Now, from (4.4) and a well known Sobolev imbedding theorem we obtain that $\nabla \tilde{u} \in L^{p^*}(T_U)$ for $1 \leq p < \frac{4}{\alpha+1}$ and $p^* = \frac{2p}{2-p}$, moreover,

$$\|\nabla \tilde{u}\|_{L^{p^*}(T_U)} \leq C\|u\|_{H^2(\Omega)}.$$

Therefore, applying the Hölder inequality with $p^*/2$ and its dual exponent q we have

$$\int_{T_U} |\nabla \tilde{u}|^2 x^{2(\beta-1)} \leq \|\nabla \tilde{u}\|^2_{L^{p^*}(T_U)} \|x^{2(\beta-1)}\|_{L^q(T_U)}$$

but, since $\beta > \frac{\alpha-1}{2}$, it is possible to choose $p < \frac{4}{\alpha+1}$ such that $\|x^{2(\beta-1)}\|_{L^q(T_U)}$ is finite, thus concluding the proof of (4.5).

Now, let $\beta > \frac{\alpha-1}{2}$ and $2 \leq p < \infty$ to be chosen below. Applying the Hölder inequality for $p/2$ and its dual exponent q we have

$$\int_{\Omega_h \setminus \Omega} |\nabla \tilde{u}|^2 \leq \Big(\int_{\Omega_h \setminus \Omega} |\nabla \tilde{u}|^p\, x^{\beta p}\Big)^{\frac{2}{p}} \Big(\int_{\Omega_h \setminus \Omega} x^{-2\beta q}\Big)^{\frac{1}{q}}, \tag{4.6}$$

and therefore, from the Sobolev imbedding (4.3) and (4.5) we obtain

$$\int_{\Omega_h \setminus \Omega} |\nabla \tilde{u}|^2 \leq \frac{C}{q-1}\, \|u\|^2_{H^2(\Omega)} \Big(\int_{\Omega_h \setminus \Omega} x^{-2\beta q}\Big)^{\frac{1}{q}} \tag{4.7}$$

for $q \to 1$. Then, we have to estimate

$$\int_{\Omega_h \setminus \Omega} x^{-2\beta q} = \sum_{j=1}^{N} \int_{\Omega_h^j} x^{-2\beta q}. \tag{4.8}$$

Since $\gamma = \frac{\alpha-1}{2}$ and $1 < \alpha < 3$ we can choose β and $q > 1$ such that

$$\gamma < \beta < \min\{2\gamma, 1\} \qquad \text{and} \qquad \beta q < \min\{2\gamma, 1\}.$$

Let us estimate each term in the right hand side of (4.8). Since $\Omega_h^1 \subset T_1$ we have

$$\int_{\Omega_h^1} x^{-2\beta q} \le \int_{T_1} x^{-2\beta q}.$$

Hence, using now that $h_1 \le \sigma\, h^{\frac{1}{1-\gamma}}$, we obtain

$$\int_{T_1} x^{-2\beta q} \le C h_1^{2(\gamma+1-\beta q)} \le C h^{2\frac{\gamma+1-\beta q}{1-\gamma}}$$

and therefore

$$\int_{T_1} x^{-2\beta q} \le C h^2$$

because $q\beta < 2\gamma$.

On the other hand, we have

$$\sum_{j>1} \int_{\Omega_h^j} x^{-2\beta q} \le \sum_{j>1} x_{j-1}^{-2\beta q} |\Omega_h^j|,$$

but, by using the well known error formula for the trapezoidal rule we obtain

$$|\Omega_h^j| \le C h_j^3 x_{j-1}^{\alpha-2} = C h_j^3 x_{j-1}^{2\gamma-1}$$

where in the case $\alpha > 2$ we have used (4.1). Therefore, since $h_j \le \sigma h\, x_{j-1}^{\gamma}$, we have

$$\begin{aligned}
\sum_{j>1} \int_{\Omega_h^j} x^{-2\beta q} &\le C \sum_{j>1} x_{j-1}^{-2\beta q+2\gamma-1} h_j^3 \le C h^2 \sum_{j>1} x_{j-1}^{-2\beta q+4\gamma-1} h_j \\
&\le C h^2 \int_0^1 x^{-2\beta q+4\gamma-1}
\end{aligned}$$

where we have used again (4.1). But the last integral is finite because $\beta q < 2\gamma$. Moreover, it is bounded by a constant which remains bounded when $q \to 1$.

Therefore, summing up the estimates obtained and replacing in (4.7) we have

$$\|\nabla \tilde{u}\|_{L^2(\Omega_h \setminus \Omega)} \le \frac{C}{\sqrt{q-1}} \|u\|_{H^2(\Omega)} h^{\frac{1}{q}}$$

with a constant C which does not blow up when $q \to 1$.

The proof concludes with a standard extrapolation argument taking $q = \frac{2\log(1/h)}{2\log(1/h)-1}$. □

Now, we want to estimate the error arising in the numerical integration of the boundary term. With this goal we introduce an extension $\tilde{g}$ of the function g to $\Gamma_{3,h}$. Calling $\phi(t) = (t, t^\alpha)$ we define $\tilde{g}$ on each $\Gamma_{3,h}^j$ as follows,

$$\tilde{g}(\psi_j(t)) := g(\phi(t)) = z(t), \qquad x_{j-1} \le t \le x_j$$

where

$$\psi_j(t) = (t, t^\alpha + \delta_j(t))$$

with

$$\delta_j(t) = \frac{x_j^\alpha - x_{j-1}^\alpha}{x_j - x_{j-1}}(t - x_{j-1}) + x_{j-1}^\alpha - t^\alpha.$$

The following lemma gives some estimates for the functions δ_j and their derivatives that will be useful in our error analysis.

Lemma 4.2. *There exists a constant C, which depends only on α, such that*

i) $|\delta_1(t)| \le 2h_1^\alpha$ *and* $|\delta_1'(t)| \le C h_1^{\alpha-1}$.

ii) $|\delta_j(t)| \le C h_j^2 x_j^{\alpha-2}$ *and* $|\delta_j'(t)| \le C h_j x_j^{\alpha-2}$, $2 \le j \le n$.

Proof. The estimates in i) follow immediately from $\delta_1(t) = x_1^{\alpha-1}t - t^\alpha$, $0 \le t \le x_1$ and $x_1 \le h_1$. Consider now $2 \le j \le n$. Since $\delta_j(x_{j-1}) = \delta_j(x_j) = 0$, δ_j' vanishes at some point in the interval (x_{j-1}, x_j) and therefore

$$|\delta_j'(t)| \le C(x_j - x_{j-1})x_j^{\alpha-2}$$

where we have used (4.1) to bound δ'' in the case $\alpha < 2$. So, the second part of ii) follows from $x_j - x_{j-1} \le h_j$. Finally, the bound for δ_j follows immediately from the bound for its derivative using again that $\delta_j(x_{j-1}) = 0$ and $x_j - x_{j-1} \le h_j$. □

Observe that if we apply a standard trace result in the polygonal domain Ω_h, the constant depends on h. However, since $\Gamma_{3,h}$ approximates Γ_3, a trace theorem with a constant independent of h can be derived from Theorem 3.2. This is the object of the next lemma.

Lemma 4.3. *There exists a constant C independent of h such that, for all $v \in V_h$,*

$$\|x^r v\|_{L^2(\Gamma_{3,h})} \le C\,\|v\|_{H^1(\Omega_h)}$$

for $r > \alpha/2 - 1$ if $\alpha \ge 2$ and $r = 0$ if $\alpha < 2$.

Proof. Since $h_j \le C x_j$, it follows from ii) of Lemma 4.2 that

$$|\delta_j(t)| \le C x_j^{\alpha-1} h_j.$$

Then, since v is linear in each triangle

$$\begin{aligned}\int_{\Gamma_{3,h}^j} v^2 x^{2r} &= \int_{x_{j-1}}^{x_j} \left| v(\phi(t)) + \delta_j(t)\frac{\partial v}{\partial y}(\phi(t))\right|^2 t^{2r}|\psi_j'(t)| \\ &\le C\int_{x_{j-1}}^{x_j} |v(\phi(t))|^2 t^{2r}|\psi_j'(t)| + C\int_{x_{j-1}}^{x_j}\left|\frac{\partial v}{\partial y}|_{\Gamma_3^j}\right|^2 |\delta_j(t)|^2 t^{2r}|\psi_j'(t)|.\end{aligned}$$

Since $\psi_j(t) = \left(t, \frac{x_j^\alpha - x_{j-1}^\alpha}{x_j - x_{j-1}}(t - x_{j-1}) + x_{j-1}^\alpha\right)$, it follows that $|\psi_j'(t)| \sim |\phi'(t)| \sim C$, thus

$$\int_{\Gamma_{3,h}^j} v^2 x^{2r} \le C\|x^r v\|_{0,\Gamma_3^j}^2 + C h_j^3 x_j^{2\alpha-2+2r}\left|\frac{\partial v}{\partial y}|_{\Gamma_3^j}\right|^2. \tag{4.9}$$

If $j = 1$ we have $\left|\frac{\partial v}{\partial y}|_{\Gamma_3^1}\right|^2 \sim \left\|\frac{\partial v}{\partial y}\right\|_{L^2(T_1)}^2 h_1^{-1-\alpha}$ and using that $h_1 \sim x_1$ we obtain

$$\|x^r v\|_{L^2(\Gamma_{3,h}^1)}^2 \le C\|x^r v\|_{L^2(\Gamma_3^1)}^2 + C h_1^{\alpha+2r}\left\|\frac{\partial v}{\partial y}\right\|_{L^2(T_1)}^2$$

while if $j > 1$ we have $\left|\frac{\partial v}{\partial y}|_{\Gamma_3^j}\right|^2 \sim \left\|\frac{\partial v}{\partial y}\right\|_{0,T_j}^2 h_j^{-2} x_j^{1-\alpha}$ and then

$$\|x^r v\|_{L^2(\Gamma_{3,h}^j)}^2 \le C\|x^r v\|_{L^2(\Gamma_3^j)}^2 + C h_j x_j^{\alpha-1+2r}\left\|\frac{\partial v}{\partial y}\right\|_{L^2(T_j)}^2.$$

Therefore, for every j we have

$$\|x^r v\|_{L^2(\Gamma_{3,h}^j)}^2 \le C\left(\|x^r v\|_{L^2(\Gamma_3^j)}^2 + \left\|\frac{\partial v}{\partial y}\right\|_{L^2(T_j)}^2\right), \qquad j = 1\ldots, n,$$

and the lemma follows by summing up the previous inequalities for $j = 1, \ldots, n$ and using Theorem 3.2. □

Lemma 4.4. *There exists a constant C independent of h such that, for all $v \in V_h$*

i) *If* $\alpha < 2$ *and* $z' \in L^2(0,1)$

$$\left| \int_{\Gamma_3} gv - \int_{\Gamma_{3,h}} I_h(gv) \right| \leq Ch \|z'\|_{L^2(0,1)} \|v\|_{H^1(\Omega_h)}.$$

ii) *If* $2 \leq \alpha < 3$, $\beta > \alpha/2 - 1$ *and* $z' t^{-\beta} \in L^2(0,1)$

$$\left| \int_{\Gamma_3} gv - \int_{\Gamma_{3,h}} I_h(gv) \right| \leq Ch \|z'\, t^{-\beta}\|_{L^2(0,1)} \|v\|_{H^1(\Omega_h)}.$$

Proof. First, we observe that since g and $\tilde{g}$ agree at the nodes on $\Gamma_3 \cap \Gamma_{3,h}$ we have

$$\begin{aligned} \left| \int_{\Gamma_3} gv - \int_{\Gamma_{3,h}} I_h(gv) \right| &= \left| \int_{\Gamma_3} gv - \int_{\Gamma_{3,h}} \tilde{g}v + \int_{\Gamma_{3,h}} (\tilde{g}v - I_h(\tilde{g}v)) \right| \\ &\leq \sum_{j=1}^{n} \left| \int_{\Gamma_3^j} gv - \int_{\Gamma_{3,h}^j} \tilde{g}v \right| + \sum_{j=1}^{n} \int_{\Gamma_{3,h}^j} |\tilde{g}v - I_h(\tilde{g}v)| \\ &=: I + II. \end{aligned} \tag{4.10}$$

For any $v \in V_h$, we have

$$\begin{aligned} I = \sum_{j=1}^{n} \left| \int_{\Gamma_3^j} gv - \int_{\Gamma_{3,h}^j} \tilde{g}v \right| &\leq \sum_{j=1}^{n} \int_{x_{j-1}}^{x_j} |z(t)| \left| v(\phi(t))|\phi'(t)| - v(\psi_j(t))|\psi_j'(t)| \right| \\ &\leq \sum_{j=1}^{n} \int_{x_{j-1}}^{x_j} |z(t)| \, |v(\phi(t)) - v(\psi_j(t))| \, |\phi'(t)| \\ &\quad + \sum_{j=1}^{n} \int_{x_{j-1}}^{x_j} |z(t)| |v(\psi_j(t))| \left| |\phi'(t)| - |\psi_j'(t)| \right| \\ &\leq \sum_{j=1}^{n} \int_{x_{j-1}}^{x_j} |z(t)| \left| \frac{\partial v}{\partial y}|_{\Gamma_3^j} \right| |\delta_j(t)| \, |\phi'(t)| + C \int_{x_{j-1}}^{x_j} |z(t)| |v(\psi_j(t))| \left| \delta_j'(t) \right| \\ &=: \sum_{j=1}^{n} A_j + B_j \end{aligned}$$

and

$$II = \sum_{j=1}^{n} \int_{\Gamma_{3,h}^j} |\tilde{g}v - I_h(\tilde{g}v)| \leq \sum_{j=1}^{n} \int_{x_{j-1}}^{x_j} |z(t)v(\psi_j(t)) - I_h(z(v \circ \psi_j))(t)| \, |\psi_j'(t)|.$$

Let

$$w_j(t) = (z(t) - \bar{z}_j) v(\psi_j(t)), \qquad t \in I_j, j = 1, \ldots, n$$

where $\bar{z}_j, j = 1, \ldots, n$ are constants to be chosen below. It follows that

$$II \leq C \sum_{j=1}^{n} \int_{x_{j-1}}^{x_j} |w_j(t) - I_h w_j(t)| \leq C \sum_{j=1}^{n} h_j \int_{x_{j-1}}^{x_j} |w_j'(t)|,$$

where we have used a standard L^1 interpolation error estimate. Since, for $t \in I_j$,

$$\begin{aligned} |w_j'(t)| &\leq |z'(t)||v(\psi_j(t))| + |z(t) - \bar{z}_j| \left| \frac{\partial v}{\partial x}(\psi_j(t)) + \frac{\partial v}{\partial y}(\psi_j(t))\psi_j'(t) \right| \\ &\leq |z'(t)||v(\psi_j(t))| + C|z(t) - \bar{z}_j||\nabla v(\psi_j(t))| \end{aligned}$$

thus,

$$II \leq C\sum_{j=1}^{n} h_j \left(\int_{x_{j-1}}^{x_j} |z'(t)||v(\psi_j(t))| + \int_{x_{j-1}}^{x_j} |z(t)-\bar{z_j}||\nabla v(\psi_j(t))| \right).$$

Clearly, since $h_j \leq h$ for any $j = 1, \cdots, n$ it follows that

$$\begin{aligned} II &\leq Ch\sum_{j=1}^{n} \left(\int_{x_{j-1}}^{x_j} |z'(t)||v(\psi_j(t))| + \int_{x_{j-1}}^{x_j} |z(t)-\bar{z_j}||\nabla v(\psi_j(t))| \right) \\ &=: Ch\sum_{j=1}^{n} C_j + D_j. \end{aligned}$$

Now, we consider the case $\alpha < 2$ and we prove the result given in i).

For $j = 1$, using i) of Lemma 4.2 and that

$$|\nabla v(\phi(t))| \sim \|\nabla v\|_{L^2(T_1)} h_1^{-\frac{\alpha+1}{2}}, \qquad t \in I_1, v \in V_h \tag{4.11}$$

we obtain, for $\beta = \max\{0, \frac{3}{2} - \alpha\}$

$$\begin{aligned} A_1 &\leq Ch_1^{\alpha+\frac{1}{2}+\beta} \|zt^{-\beta}\|_{L^2(I_1)} |\nabla v|_{T_1}| \leq Ch_1^{\frac{\alpha}{2}+\beta} \|zt^{-\beta}\|_{L^2(I_1)} \|\nabla v\|_{L^2(T_1)} \\ &\leq Ch^{\left(\frac{\alpha}{2}+\beta\right)\frac{2}{3-\alpha}} \|zt^{-\beta}\|_{L^2(I_1)} \|\nabla v\|_{L^2(T_1)}. \end{aligned}$$

Hence, since $\alpha < 2$ then $\left(\frac{\alpha}{2}+\beta\right)\frac{2}{3-\alpha} \geq 1$ and therefore,

$$A_1 \leq Ch\|zt^{-\beta}\|_{L^2(I_1)} \|\nabla v\|_{L^2(T_1)}. \tag{4.12}$$

Similary, for $\beta = \max\{0, \frac{5}{2} - \frac{3}{2}\alpha\}$ we have

$$\begin{aligned} B_1 &\leq Ch_1^{\alpha+\beta-1} \|zt^{-\beta}\|_{L^2(I_1)} \|v\|_{L^2(\Gamma^1_{3,h})} \leq h^{(\alpha+\beta-1)\frac{2}{3-\alpha}} \|zt^{-\beta}\|_{L^2(I_1)} \|v\|_{L^2(\Gamma^1_{3,h})} \\ &\leq Ch\|zt^{-\beta}\|_{L^2(I_1)} \|v\|_{L^2(\Gamma^1_{3,h})}. \end{aligned}$$

For $j > 1$, using ii) of Lemma 4.2 and that

$$|\nabla v(\phi(t))| \sim h_j^{-1} x_j^{\frac{1-\alpha}{2}} \|\nabla v\|_{L^2(T_j)}, \qquad t \in I_j \tag{4.13}$$

since $h_j \leq Chx_j^{\frac{\alpha-1}{2}}$, we have, for $\beta = \max\{0, \frac{3}{2} - \alpha\}$,

$$\begin{aligned} A_j &\leq Ch_j^2 x_j^{\alpha+\beta-\frac{3}{2}} \|zt^{-\beta}\|_{L^2(I_j)} |\nabla v|_{T_j}| = Ch_j x_j^{\beta+\frac{\alpha}{2}-1} \|zt^{-\beta}\|_{L^2(I_j)} \|\nabla v\|_{L^2(T_j)} \\ &\leq Chx_j^{\beta+\alpha-\frac{3}{2}} \|zt^{-\beta}\|_{L^2(I_j)} \|\nabla v\|_{L^2(T_j)} \\ &\leq Ch\|zt^{-\beta}\|_{L^2(I_j)} \|\nabla v\|_{L^2(T_j)}. \end{aligned} \tag{4.14}$$

Similarly, for $j > 1$ and $\beta = \max\{0, \frac{5}{2} - \frac{3}{2}\alpha\}$, applying the Cauchy-Schwartz inequality we have that

$$\begin{aligned} B_j &\leq Ch_j x_j^{\alpha-2+\beta} \|zt^{-\beta}\|_{L^2(I_j)} \|v\|_{L^2(\Gamma^j_{3,h})} \leq Chx_j^{\frac{3}{2}\alpha-\frac{5}{2}+\beta} \|zt^{-\beta}\|_{L^2(I_j)} \|v\|_{L^2(\Gamma^j_{3,h})} \\ &\leq Ch\|zt^{-\beta}\|_{L^2(I_j)} \|v\|_{L^2(\Gamma^j_{3,h})}. \end{aligned} \tag{4.15}$$

So, since $\frac{3}{2} - \alpha < \frac{5}{2} - \frac{3}{2}\alpha$, if we take $\beta = \max\{0, \frac{5}{2} - \frac{3}{2}\alpha\}$, we obtain for any j

$$\left|\int_{\Gamma_3^j} gv - \int_{\Gamma_{3,h}^j} \tilde{g}v\right| \leq Ch\|zt^{-\beta}\|_{L^2(I_j)}\left(\|\nabla v\|_{L^2(T_j)} + \|v\|_{L^2(\Gamma_{3,h}^j)}\right) \tag{4.16}$$

and adding for $j = 1, \dots, n$, we have

$$I \leq Ch\|zt^{-\beta}\|_{L^2(0,1)}\left(\|\nabla v\|_{L^2(\Omega_h)} + \|v\|_{L^2(\Gamma_{3,h})}\right).$$

Now, since $z(0) = 0$ and $\beta < 1$, by using the Hardy inequality

$$\|zt^{-\beta}\|_{L^2(0,1)} \leq C\|z'\|_{L^2(0,1)} \tag{4.17}$$

and Lemma 4.3 for the case $\alpha < 2$, we conclude that

$$I \leq Ch\|z'\|_{L^2(0,1)}\|v\|_{H^1(\Omega_h)}. \tag{4.18}$$

On the other hand, we have that for all $j \geq 1$

$$C_j = \int_{x_{j-1}}^{x_j} |z'(t)||v(\psi_j(t))| \leq \left\|z'\right\|_{L^2(I_j)} \|v\|_{L^2(\Gamma_{3,h}^j)}. \tag{4.19}$$

Taking $\bar{z_j} = \frac{1}{x_j - x_{j-1}} \int_{x_{j-1}}^{x_j} z$, it follows from the Poincaré inequality that

$$\|z - \bar{z_j}\|_{L^2(I_j)} \leq Ch_j\|z'\|_{L^2(I_j)}.$$

Then, using (4.11) for $j = 1$ and (4.13) for $j > 1$ we obtain

$$\begin{aligned} D_1 &\leq Ch_1^{1-\frac{\alpha}{2}}\|z'\|_{L^2(I_1)}\|\nabla v\|_{L^2(T_1)} \leq Ch^{\frac{2-\alpha}{3-\alpha}}\|z'\|_{L^2(I_1)}\|\nabla v\|_{L^2(T_1)} \\ D_j &\leq Ch_j^{\frac{1}{2}}x_j^{\frac{1-\alpha}{2}}\|z'\|_{L^2(I_j)}\|\nabla v\|_{L^2(T_j)} \leq Ch^{\frac{1}{2}}\|z'\|_{L^2(I_j)}\|\nabla v\|_{L^2(T_j)}, \qquad j > 1 \end{aligned}$$

and therefore

$$D_j \leq C\|z'\|_{L^2(I_j)}\|\nabla v\|_{L^2(T_j)}, \qquad \forall j \geq 1. \tag{4.20}$$

So, adding inequalities (4.19) and (4.20) for $j = 1, \cdots, n$ we have that

$$II \leq Ch\left(\|z'\|_{L^2(0,1)}\|v\|_{L^2(\Gamma_{3,h})} + \|z'\|_{L^2(0,1)}\|\nabla v\|_{L^2(\Omega_h)}\right)$$

and using Lemma 4.3 again we conclude that

$$II \leq Ch\|z'\|_{L^2(0,1)}\|v\|_{H^1(\Omega_h)}. \tag{4.21}$$

From this inequality, (4.18) and (4.10) the proof of i) concludes.

Now, consider the case $\alpha \geq 2$. By the same arguments used in the previous case we have

$$\begin{aligned} A_1 &\leq Ch_1^{\alpha+\frac{1}{2}}\|z\|_{L^2(I_1)}|\nabla v|_{T_1}| \leq Ch_1^{\frac{\alpha}{2}}\|z\|_{L^2(I_1)}\|\nabla v\|_{L^2(T_1)} \\ &\leq Ch^{\frac{\alpha}{3-\alpha}}\|z\|_{L^2(I_1)}\|\nabla v\|_{L^2(T_1)} \end{aligned}$$

but $\frac{\alpha}{3-\alpha} \geq 1$ so,

$$A_1 \leq Ch\|z\|_{L^2(I_1)}\|\nabla v\|_{L^2(T_1)}. \tag{4.22}$$

For $j > 1$,

$$\begin{aligned} A_j &\leq Ch_j^2 x_j^{\alpha-\frac{3}{2}}\|z\|_{L^2(I_j)}|\nabla v|_{T_j}| \leq Ch_j x_j^{\frac{\alpha}{2}-1}\|z\|_{L^2(I_j)}\|\nabla v\|_{L^2(T_j)} \\ &\leq Chx_j^{\alpha-\frac{3}{2}}\|z\|_{L^2(I_j)}\|\nabla v\|_{L^2(T_j)} \leq Ch\|z\|_{L^2(I_j)}\|\nabla v\|_{L^2(T_j)}. \end{aligned} \tag{4.23}$$

Similarly, for any $\beta \geq 0$ we have

$$\begin{aligned} B_1 &\leq Ch_1^{\alpha-1}\|zt^{-\beta}\|_{L^2(I_1)}\|vx^\beta\|_{L^2(\Gamma^1_{3,h})} \leq h^{(\alpha-1)\frac{2}{3-\alpha}}\|zt^{-\beta}\|_{L^2(I_1)}\|vx^\beta\|_{L^2(\Gamma^1_{3,h})} \\ &\leq Ch\|zt^{-\beta}\|_{L^2(I_1)}\|vx^\beta\|_{L^2(\Gamma^1_{3,h})} \end{aligned} \tag{4.24}$$

and

$$\begin{aligned} B_j &\leq Ch_j x_j^{\alpha-2}\|zt^{-\beta}\|_{L^2(I_j)}\|vx^\beta\|_{L^2(\Gamma^j_{3,h})} \leq Chx_j^{\frac{3}{2}\alpha-\frac{5}{2}}\|zt^{-\beta}\|_{L^2(I_j)}\|vx^\beta\|_{L^2(\Gamma^j_{3,h})} \\ &\leq Ch\|zt^{-\beta}\|_{L^2(I_j)}\|vx^\beta\|_{L^2(\Gamma^j_{3,h})}. \end{aligned} \tag{4.25}$$

and so, adding inequalities (4.22), (4.23), (4.24) and (4.25) for $j = 1, \cdots, n$ we conclude that for any $\beta \geq 0$,

$$I \leq Ch\left(\|z\|_{L^2(0,1)}\|\nabla v\|_{L^2(\Omega_h)} + \|zt^{-\beta}\|_{L^2(0,1)}\|vx^\beta\|_{L^2(\Gamma_{3,h})}\right).$$

Taking $\frac{\alpha}{2} - 1 < \beta < 1$, using the Hardy inequality (4.17) and our trace result for the case $2 \leq \alpha < 3$, we obtain

$$I \leq Ch\|z'\|_{L^2(0,1)}\|v\|_{H^1(\Omega_h)}. \tag{4.26}$$

On the other hand, for any j and $\frac{\alpha}{2} - 1 < \beta < 1$ it follows that

$$C_j \leq \|z't^{-\beta}\|_{L^2(I_j)}\|vx^\beta\|_{L^2(\Gamma^j_{3,h})}$$

and by using (4.11) for $j = 1$ and (4.13) for $j > 1$ we get

$$D_j \quad \leq \quad C\|z'\|_{L^2(I_j)}\|\nabla v\|_{L^2(T_j)}, \qquad j \geq 1.$$

Therefore, we conclude that for $\frac{\alpha}{2} - 1 < \beta < 1$,

$$II \leq Ch\left(\|z't^{-\beta}\|_{L^2(0,1)}\|vx^\beta\|_{L^2(\Gamma_{3,h})} + \|z'\|_{L^2(0,1)}\|\nabla v\|_{L^2(\Omega_h)}\right).$$

Hence, using Lemma 4.3 again we obtain

$$II \leq Ch\|z't^{-\beta}\|_{L^2(0,1)}\|v\|_{H^1(\Omega_h)} \tag{4.27}$$

and thus, using (4.26) and (4.27) in (4.10) we conclude the proof of ii). □

We can now prove our main theorem which gives quasi-optimal error estimates in H^1 for the piecewise linear approximation on appropriate graded meshes.

Theorem 4.2. *Let u be the solution of (2.1) and $u_h \in V_h$ be its finite element approximation using the mesh $\mathcal{T}_h$. Assume $\alpha < 3$, $f \in L^2(\Omega)$, $z\,t^{-\frac{\alpha}{2}} \in L^2(0,1)$ and $z't^{-r} \in L^2(0,1)$, with $r = 0$ when $\alpha < 2$ and $r > \alpha/2 - 1$ when $\alpha \geq 2$.*

If the family of meshes satisfies (1), (2), (3) and (H), then there exists a constant C depending only on α, β and σ such that

$$\|u - u_h\|_{H^1(\Omega)} \leq Ch\sqrt{\log(1/h)}\left\{\|f\|_{L^2(\Omega)} + \|z\,t^{-\frac{\alpha}{2}}\|_{L^2(0,1)} + \|z'\,t^{-r}\|_{L^2(0,1)}\right\}.$$

Proof. In view of (2.2) and since $r > \alpha/2 - 1$, it is enough to prove

$$\|u - u_h\|_{H^1(\Omega)} \leq Ch\sqrt{\log(1/h)}\left\{\|u\|_{H^2(\Omega)} + \|z'\,t^{-r}\|_{L^2(0,1)}\right\}.$$

Since $\Omega \subset \Omega_h$, we have

$$\|u - u_h\|_{H^1(\Omega)} \leq \|\tilde{u} - u_h\|_{H^1(\Omega_h)},$$

and therefore it is enough to prove that

$$\|\tilde{u} - u_h\|_{H^1(\Omega_h)} \leq Ch\sqrt{\log(1/h)}\, \left\{\|u\|_{H^2(\Omega)} + \|z'\, t^{-r}\|_{L^2(0,1)}\right\}. \tag{4.28}$$

Using the Poincaré inequality we have,

$$\|\tilde{u}-u_h\|^2_{H^1(\Omega_h)} \leq C|\tilde{u}-u_h|^2_{H^1(\Omega_h)} = \int_{\Omega_h} \nabla(\tilde{u}-u_h)\cdot\nabla(\tilde{u}-\Pi\tilde{u}) + \int_{\Omega_h} \nabla(\tilde{u}-u_h)\cdot\nabla(\Pi\tilde{u}-u_h), \tag{4.29}$$

but we know from (4.2) and Theorem 4.1 that

$$|\tilde{u} - \Pi\tilde{u}|_{H^1(\Omega_h)} \leq Ch\|\tilde{u}\|_{H^2_\alpha(\Omega_h)} \leq Ch\|u\|_{H^2(\Omega)}.$$

Thus,

$$\int_{\Omega_h} \nabla(\tilde{u} - u_h)\cdot\nabla(\tilde{u} - \Pi\tilde{u}) \leq Ch|\tilde{u} - u_h|_{H^1(\Omega_h)}\|u\|_{H^2(\Omega)}$$

and therefore, using the Young inequality, we obtain

$$\int_{\Omega_h} \nabla(\tilde{u} - u_h)\cdot\nabla(\tilde{u} - \Pi\tilde{u}) \leq C_\varepsilon h^2|\tilde{u} - u_h|^2_{H^1(\Omega_h)} + \varepsilon\|u\|^2_{H^2(\Omega)} \tag{4.30}$$

with ε to be chosen below.

Then, we only have to estimate the second term of (4.29). To simplify notation we introduce $w_h := \Pi\tilde{u} - u_h$. From (2.1) and (2.3) we have

$$\begin{aligned}
\int_{\Omega_h} \nabla(\tilde{u} - u_h)\cdot\nabla w_h &= \int_{\Omega} \nabla(\tilde{u} - u_h)\cdot\nabla w_h + \int_{\Omega_h\setminus\Omega} \nabla(\tilde{u} - u_h)\cdot\nabla w_h \\
&= \int_{\Omega} \nabla u\cdot\nabla w_h + \int_{\Omega_h\setminus\Omega} \nabla\tilde{u}\cdot\nabla w_h - \int_{\Omega_h} \nabla u_h\cdot\nabla w_h \\
&= \int_{\Omega_h\setminus\Omega} \nabla\tilde{u}\cdot\nabla w_h + \int_{\Gamma_3} g w_h - \int_{\Gamma_{3,h}} I_h(g w_h).
\end{aligned}$$

Then, from Lemmas 4.1 and 4.4, using (4.2), Theorem 4.1, and again the Young inequality we obtain

$$\int_{\Omega_h} \nabla(\tilde{u} - u_h)\cdot\nabla w_h \leq C_\varepsilon h^2\log 1/h \left\{\|u\|^2_{H^2(\Omega)} + \|z'\, t^{-r}\|^2_{L^2(0,1)}\right\} + \varepsilon\|w_h\|^2_{H^1(\Omega_h)}.$$

Therefore, (4.28) follows from (4.29), (4.30) by choosing an appropriate small ε. □

Now we show that meshes satisfying the hypotheses (1)-(3) and (H) can be constructed. To define the mesh $\mathcal{T}_h$, with $h = 1/n$ we use the following method given in [9, 11].

(1) Introduce the partition of the interval $(0,1)$ given by

$$x_j = \left(\frac{j}{n}\right)^{\frac{2}{3-\alpha}} \qquad 0 \leq j \leq n.$$

(2) Take the points $(x_j, 0)$ in Γ_1, (x_j, x_j^α) in Γ_3 and divide the vertical lines $x = x_j$, for $j > 1$, in a uniform way.

Figure 4 shows an example of one of these meshes.

If N is the number of nodes in the partition $\mathcal{T}_h$, it can be proved that $h^2 \sim 1/N$ [9, 11]. Therefore, using these meshes we have the following error estimate in terms of the number of nodes,

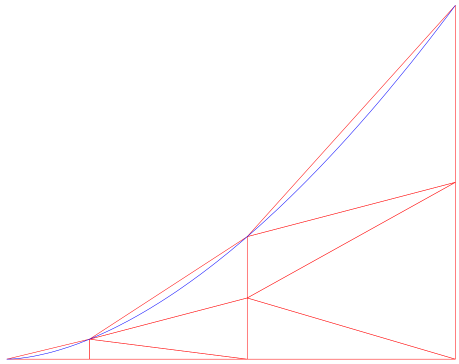

FIGURE 4

value of s $(\alpha = 2)$	in number of nodes	in h
0.55	0.58821838532072	1.05409143915600
0.6	0.58550066423886	1.04922126406817
0.65	0.58412286609630	1.04675224021035
0.7	0.58376172979933	1.04610508145349
0.75	0.58410183082038	1.04671454484955
0.8	0.58483732089851	1.04803254818689
0.85	0.58568475211108	1.04955115081601
0.9	0.58640674380833	1.05084496496090
0.95	0.58683593873302	1.05161408525217

TABLE 2. H^1 order using graded meshes

$$\|u - u_h\|_{H^1(\Omega)} \leq C\sqrt{\frac{\log N}{N}}\left\{\|f\|_{L^2(\Omega)} + \|z\,t^{-\frac{\alpha}{2}}\|_{L^2(0,1)} + \|z'\,t^{-r}\|_{L^2(0,1)}\right\}.$$

Observe that this estimate is quasi-optimal. Indeed, up to the logarithmic factor, the order with respect to the number of nodes is the same as that obtained for a smooth problem using quasi-uniform meshes.

Table 2 shows the numerical results obtained with these graded meshes for the example (2.1).

REFERENCES

[1] G. ACOSTA, M. G. ARMENTANO, R. G. DURÁN AND A. L. LOMBARDI, Nonhomogeneous Neumann problem for the Poisson equation in domains with an external cusp, Journal of Mathematical Analysis and Applications **310**(2), pp. 397–411, 2005.

[2] R. A. ADAMS, Sobolev Spaces, Academic Press, New York, 1975.

[3] T. APEL, Anisotropic finite elements: local estimates and applications, Advances in Numerical Mathematics, 1999.

[4] I. BABUŠKA, R. B. KELLOG AND J. PITKARANTA, Direct and inverse error estimates for finite elements with mesh refinements, Numer. Math. **33**(4), pp. 447–471, 1979.

[5] J. H. BRAMBLE AND J. T. KING, A robust finite element method for nonhomogeneous Dirichlet problems in domains with curved boundaries, Math. Comp. **63**(207), pp. 1-17, 1994.

[6] S. K. CHUA, Extension theorems on weighted Sobolev Spaces, Indiana Math. J. **41**(4), pp. 1027-1076,1992.

[7] J. DUOANDIKOETXEA ZUAZO, Análisis de Fourier, Addison-Wesley Iberoamericana, 1995.

[8] D. GILBARG, N. S. TRUDINGER, Elliptic Partial Differential Equations of Second Order, Springer Verlag, Berlin, 1983.

[9] P. Grisvard, Elliptic Problems in Nonsmooth Domains, Pitman, Boston, 1985.
[10] G. Hardy, J. E. Littlewood and G. Polya, Inequalities, Cambridge UP, 1952.
[11] G. Raugel, Résolution numérique par une méthode d'éléments finis du problème Dirichlet pour le laplacien dans un polygone, C. R. Acad. Sci., Paris, Ser. A, **286**(18), pp. A791-A794, 1978.
[12] E. M. Stein, Singular Integrals and Differentiability Properties of Functions, Princeton University Press, 1970.

Instituto de Ciencias, Universidad Nacional de General Sarmiento, J.M. Gutierrez 1150, Los Polvorines, B1613GSX Provincia de Buenos Aires, Argentina
E-mail address: gacosta@ungs.edu.ar

Departamento de Matemática, Facultad de Ciencias Exactas y Naturales, Universidad de Buenos Aires, 1428 Buenos Aires, Argentina.
E-mail address: garmenta@dm.uba.ar
E-mail address: rduran@dm.uba.ar
E-mail address: aldoc7@dm.uba.ar