Sistemas y Señales I

Expansión en Fracciones Simples

Temario: Cap. 3: Item 3.1

Caso 1: Raíces reales distintas

$$Y(s)=\frac{(s+2)(s+4)}{s(s+1)(s+3)}=\frac{A}{s}+\frac{B}{s+1}+\frac{C}{s+3}$$

$$A=\lim_{s\to 0}\ sY(s)=\lim_{s\to 0}\ \frac{(s+2)(s+4)}{(s+1)(s+3)}=\frac{8}{3}$$

$$B=\lim_{s\to -1}\ (s+1)Y(s)=\ \lim_{s\to -1}\frac{(s+2)(s+4)}{s(s+3)}=-\frac{3}{2}$$

$$C=\lim_{s\to -3}\ (s+3)Y(s)=\ \lim_{s\to -1}\frac{(s+2)(s+4)}{s(s+1)}=-\frac{1}{6}$$

Luego

$$\boxed{y(t)=\frac{8}{3}\mu(t)-\frac{3}{2}e^{-t}\mu(t)-\frac{1}{6}e^{-3t}\mu(t)}$$

Caso 2: Raíces complejas conjugadas

$$Y(s)=\frac{1}{s(s^2+s+1)}=\frac{A}{s}+\frac{Bs+C}{s^2+s+1}=\frac{As^2+As+A+Bs^2+Cs}{s(s^2+s+1)}$$

Raíces en: $p_1=0\ ;\ p_{2,3}=-\frac{1}{2}\pm j\frac{\sqrt{3}}{2}$

$$\begin{cases}A+B=0\\ A+C=0\\ A=1\end{cases}\Rightarrow\begin{cases}A=1\\ B=-1\\ C=-1\end{cases}$$

$$Y(s)=\frac{1}{s}-\frac{s+1}{s^2+s+1}=\frac{1}{s}-\frac{s+\frac{1}{2}+\frac{1}{2}}{s^2+s+1}=\frac{1}{s}-\frac{s+\frac{1}{2}+\frac{1}{2}}{\left(s+\frac{1}{2}\right)^2+\frac{3}{4}}$$

$\Re e\left\{p_{1,2}\right\}$ $\qquad$ $\Im m\left\{p_{1,2}\right\}^2$

$$Y(s)=\frac{1}{s}-\frac{\left(s+\frac{1}{2}\right)}{\left(s+\frac{1}{2}\right)^{2}+\frac{3}{4}}-\frac{1}{2}\frac{2}{\sqrt{3}}\frac{\frac{\sqrt{3}}{2}}{\left(s+\frac{1}{2}\right)^{2}+\frac{3}{4}}$$

Luego

$$y(t)=\mu(t)-e^{-\frac{t}{2}}\cos\left(\frac{\sqrt{3}}{2}t\right)\mu(t)-\frac{1}{\sqrt{3}}e^{-\frac{t}{2}}sen\left(\frac{\sqrt{3}}{2}t\right)\mu(t)$$

Caso 3: Raíces reales con multiplicidad

$$Y(s)=\frac{s+3}{(s+1)(s+2)^2}=\frac{A}{s+1}+\frac{B}{s+2}+\frac{C}{(s+2)^2}$$

$$A=\lim_{s\to -1}\ (s+1)Y(s)=\lim_{s\to -1}\ \frac{s+3}{(s+2)^2}=2$$

$$B=\lim_{s\to -2}\frac{d}{ds}\left[(s+2)^2\,Y(s)\right]=\lim_{s\to -2}\frac{d}{ds}\left[\frac{s+3}{s+1}\right]=\lim_{s\to -2}\frac{s+1-s-3}{(s+1)^2}=-2$$

$$C=\lim_{s\to -2}\ (s+2)^2\,Y(s)=\lim_{s\to -2}\ \frac{s+3}{s+1}=-1$$

Luego

$$y(t)=2e^{-t}\mu(t)-2e^{-2t}\mu(t)-te^{-2t}\mu(t)$$

Consideremos el caso más general de una raíz real con multiplidad M

$$Y(s) = \frac{1}{(s+a)^M} = \frac{k_1}{s+a} + \frac{k_2}{(s+a)^2} + \cdots + \frac{k_M}{(s+a)^M}$$

$$k_M = \lim_{s\to -a} (s+a)^M Y(s)$$

$$k_j = \lim_{s\to -a} \frac{1}{(M-j)!}\frac{d^{M-j}}{ds^{M-j}}\left[(s+a)^M Y(s)\right] \;\; ; \;\; j = 1, 2, \cdots, M-1$$

Luego

$$y(t) = \left[k_1 e^{-at} + k_2 t e^{-at} + k_3 t^2 e^{-at} + \cdots + k_M t^{M-1} e^{-at}\right]\mu(t)$$