Subspace-based Blind Identification of IIR Systems

Juan Carlos Gómez ¹
<jcgomez@fceia.unr.edu.ar>

Enrique Baeyens ²
<enrbae@eis.uva.es>

¹ Laboratory for System Dynamics and Signal Processing
FCEIA, Universidad Nacional de Rosario, Argentina

² Department of Systems Engineering and Automatic Control
ETSII, Universidad de Valladolid, Spain

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Outline

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Motivation for Blind ID

- **Blind Identification**: based only on output data
- In the digital communication area, blind channel identification avoids the need for transmission of training sequences and, in this way, a more efficient use of the available bandwidth can be achieved.
- **IIR models are more suitable than FIR models for representing systems whose impulse responses have slow decays.**
- **SIMO model structures appear when a source is transmitted through a continuous channel and several measurements are performed at the receiver due to multiple sensors, or oversampling of the received signal to ensure identifiability.**
Contributions

- A new subspace technique for the blind identification of IIR, SIMO systems represented using orthonormal bases with fixed poles is presented.

- Basis coefficients are estimated in closed form by first computing the column space of the output Hankel matrix, using Singular Value Decomposition, and then solving a Least Squares Problem through an Eigenvalue Decomposition.

- The method presents improvements in the estimation accuracy when compared to similar techniques based on FIR or Laguerre models.
Problem Formulation

The LTI, SIMO, IIR system is represented using orthonormal bases as follows

\[ Y(z) = \left\{ \sum_{\ell=0}^{p-1} b_{\ell} B_{\ell}(z) \right\} U(z) \]

where

- \( b_{\ell} = [b_{\ell}^1, b_{\ell}^2, \ldots, b_{\ell}^m]^T \in \mathbb{R}^m \) unknown basis coefficients
- \( B_{\ell}(z) \) rational orthonormal bases on \( H_2(\mathbb{T}) \).
The **Orthonormal Bases with Fixed Poles (OBFP)** described in (Ninness et al., 1997), (Gomez, 1998) are considered. The OBFP are defined as

$$B_{\ell}(z) = \left( \frac{\sqrt{1 - |\xi_{\ell}|^2}}{z - \xi_{\ell}} \right) \prod_{i=0}^{\ell-1} \left( 1 - \frac{\bar{\xi}_i z}{z - \xi_i} \right), \quad \ell \geq 1$$

$$B_0(z) = \frac{\sqrt{1 - |\xi_0|^2}}{z - \xi_0}$$

The OBFP can incorporate an arbitrary number of stable poles, and they have the more common **FIR** and **Laguerre** model structures as special cases.
Fig. 1: IIR Filter Structure using OBFP.
Problem Formulation (cont.)

\[
A_\ell(z) \triangleq \frac{1 - \overline{\xi_\ell}z}{z - \xi_\ell} \quad \text{(all-pass section)},
\]

\[
L_\ell(z) \triangleq \sqrt{1 - |\xi_\ell|^2} \frac{z - \xi_\ell}{z - \xi_\ell} \quad \text{(low-pass section)}
\]

A **state-space realization** of the system can be obtained by giving each all-pass section and each low-pass section a one dimensional state-space realization. It is not difficult to see that the resulting state-space realization will be **non minimal**.
Fig. 2: Input-Output equivalent IIR filter structure.
Problem Formulation (cont.)

\[ Q_0(z) \triangleq \frac{1}{z - \xi_0} \]
\[ Q_\ell(z) \triangleq \frac{1 - \xi_{\ell-1}z}{z - \xi_\ell}, \ \ell \geq 1 \]

A **minimal state-space realization** for the \( Q_\ell(z), (\ell \geq 1) \) blocks is given by

\[
\begin{align*}
    x_{k+1}^\ell &= \left\{ A^\ell \right\} x_k^\ell + \left\{ B^\ell \right\} u_k^\ell \\
    \tilde{y}_k^\ell &= \left\{ C^\ell \right\} x_k^\ell - \left\{ D^\ell \right\} u_k^\ell
\end{align*}
\]

while for the \( Q_0(z) \) blocks is given by

\[
\begin{align*}
    x_{k+1}^0 &= \xi_0 x_k^0 + u_k \\
    \tilde{y}_k^0 &= x_k^0
\end{align*}
\]
Defining

\[ x_k \triangleq \left[ x_0^k, x_1^k, \ldots, x_{p-1}^k \right]^T \]
\[ \tilde{y}_k \triangleq \left[ \tilde{y}_0^k, \tilde{y}_1^k, \ldots, \tilde{y}_{p-1}^k \right]^T \]
\[ y_k \triangleq \left[ y_1^k, y_2^k, \ldots, y_m^k \right]^T \]
\[ b \triangleq [b_0, b_1, \ldots, b_{p-1}] \]

A **minimal state-space realization of the IIR model** is given by

\[
x_{k+1} = Ax_k + Bu_k \\
y_k = Cx_k + Du_k
\]
**Problem Formulation (cont.)**

\[
A = \begin{bmatrix}
A^0 & 0 & \cdots & 0 \\
B_1^1 C_0^0 & A^1 & \cdots & 0 \\
B_2^2 D_1^1 C_0^0 & B_2^2 C^1 & \cdots & 0 \\
B_3^3 D_2^2 D_1^1 C_0^0 & B_3^3 D_2^2 C^1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
B_{p-1}^{p-1} D_{p-2}^p \cdots D_1^1 C_0^0 & B_{p-1}^{p-1} D_{p-2}^p \cdots D_2^2 C^1 & \cdots & A_{p-1}^{p-1}
\end{bmatrix},
\]

\[
B = [1, 0, \cdots, 0]^T, \quad C = b\Lambda\tilde{C}, \quad D = 0
\]

\[
\tilde{C} = \begin{bmatrix}
C^0 & 0 & \cdots & 0 \\
D_1^1 C^0 & C^1 & \cdots & 0 \\
D_2^2 D_1^1 C^0 & D_2^2 C^1 & \cdots & 0 \\
D_3^3 D_2^2 D_1^1 C^0 & D_3^3 D_2^2 C^1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
D_{p-1}^{p-1} D_{p-2}^p \cdots D_1^1 C_0^0 & D_{p-1}^{p-1} D_{p-2}^p \cdots D_2^2 C^1 & \cdots & C_{p-1}^{p-1}
\end{bmatrix},
\]

\[
\Lambda \triangleq \begin{bmatrix}
\sqrt{1-\frac{\xi^2_0}{\lambda}} & 0 & \cdots & 0 \\
0 & \sqrt{1-\frac{\xi^2_1}{\lambda}} & \cdots & 0 \\
0 & 0 & \ddots & 0 \\
0 & 0 & \cdots & \sqrt{1-\frac{\xi^2_{p-1}}{\lambda}}
\end{bmatrix},
\]
Considering that there are \( N + \alpha - 1 \) measurements available, the system's input-output equation can be written in matrix form as

\[
Y_\alpha = \Gamma_\alpha X + H_\alpha U_\alpha = \begin{bmatrix} \Gamma_\alpha & H_\alpha \end{bmatrix} \begin{bmatrix} X \\ U_\alpha \end{bmatrix}
\]

where

\[
U_\alpha \triangleq \begin{bmatrix} u_1 & u_2 & \cdots & u_N \\ u_2 & u_3 & \cdots & u_{N+1} \\ \vdots & \vdots & \ddots & \vdots \\ u_\alpha & u_{\alpha+1} & \cdots & u_{N+\alpha-1} \end{bmatrix}, \quad Y_\alpha \triangleq \begin{bmatrix} y_1 & y_2 & \cdots & y_N \\ y_2 & y_3 & \cdots & y_{N+1} \\ \vdots & \vdots & \ddots & \vdots \\ y_\alpha & y_{\alpha+1} & \cdots & y_{N+\alpha-1} \end{bmatrix}
\]

\[
\Gamma_\alpha \triangleq \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{\alpha-1} \end{bmatrix}, \quad H_\alpha \triangleq \begin{bmatrix} D \\ CB \\ CAB \\ \vdots \\ D \end{bmatrix}, \quad X \triangleq \begin{bmatrix} x_1 & x_2 & \cdots & x_N \end{bmatrix}
\]
Considering that $C = b\Lambda\tilde{C}$, the system equation becomes

$$Y_\alpha = (\mathbb{I}_\alpha \otimes b) \begin{bmatrix} X \\ U_\alpha \end{bmatrix}$$

Fundamental equation

where

$$H \triangleq \begin{bmatrix} \tilde{\Gamma}_\alpha & \tilde{H}_\alpha \end{bmatrix}$$

$$\tilde{\Gamma}_\alpha \triangleq \begin{bmatrix} \Lambda\tilde{C} \\
\Lambda\tilde{C}A \\
\vdots \\
\Lambda\tilde{C}A^{\alpha-1} \end{bmatrix}, \quad \tilde{H}_\alpha \triangleq \begin{bmatrix}
0 & 0 & 0 & \cdots & 0 \\
0 & 0 & \cdots & \cdots & 0 \\
\Lambda\tilde{C}B & 0 & \cdots & \cdots & 0 \\
\Lambda\tilde{C}AB & \Lambda\tilde{C}B & 0 & \cdots & 0 \\
\Lambda\tilde{C}A^{\alpha-2}B & \Lambda\tilde{C}A^{\alpha-3}B & \cdots & \cdots & 0
\end{bmatrix}$$
Blind ID Problem: given the (possibly noisy) output measurements $Y_\alpha$ and the basis poles, find an estimate of the parameter matrix $b$, and possibly of the structured matrices $X$ and $U_\alpha$.

$$Y_\alpha = (\mathbb{I}_\alpha \otimes b) H \begin{bmatrix} X \\ U_\alpha \end{bmatrix}$$

Fundamental equation

Provided that $\alpha m > p + \alpha$, $N > p + \alpha$, and that matrices $(\mathbb{I}_\alpha \otimes b) H$ and $\begin{bmatrix} X \\ U_\alpha \end{bmatrix}$ have full column rank, then the following subspace relationships hold

$$\text{col} (Y_\alpha) = \text{col} ((\mathbb{I}_\alpha \otimes b) H), \quad \text{row} (Y_\alpha) = \text{row} \left( \begin{bmatrix} X \\ U_\alpha \end{bmatrix} \right)$$
The parameter matrix $b$ can then be estimated by computing the column space of the output Hankel matrix $Y_\alpha$ and exploiting the Kronecker structure of the system. Assuming $Y_\alpha$ has rank $(p + \alpha)$, and performing its SVD yields

$$Y_\alpha = \Phi \Sigma \Psi^T = \underbrace{\begin{bmatrix} \Phi_1 & \Phi_2 \end{bmatrix}}_{\Phi} \underbrace{\begin{bmatrix} \Sigma_1 & 0 \\ 0 & \Sigma_2 \end{bmatrix}}_{\Sigma} \underbrace{\begin{bmatrix} \Psi_1^T \\ \Psi_2^T \end{bmatrix}}_{\Psi^T} \approx \Phi_1 \Sigma_1 \Psi_1^T,$$

Then, the following subspace relationship holds

$$\text{col} (Y_\alpha) = \text{col} \left( (I_\alpha \otimes b) H \right) = \text{col} \left( \Phi_1 \right)$$
There must exist a nonsingular matrix $T$ such that the columns of $\Phi_1$ can be converted into the columns of matrix $(I_\alpha \otimes b)H$. Matrices $b$ and $T$ can then be estimated by solving the **Least Squares Problem**

$$\begin{align*}
(\hat{b}, \hat{T}) &= \arg \min_{T, \|b\|_2=1} \| (I_\alpha \otimes b)H - \Phi_1 T \|^2_F \\
\text{or equivalently}
\hat{b} &= \arg \min_{\|b\|_2=1} \| (I - \Phi_1 (\Phi_1^T \Phi_1)^{-1} \Phi_1^T) (I_\alpha \otimes b)H \|^2_F \\
\hat{T} &= (\Phi_1^T \Phi_1)^{-1} \Phi_1^T (I_\alpha \otimes \hat{b})H
\end{align*}$$
Remark: The constraint \( \|b\|_2 = 1 \) has been imposed to avoid the trivial solution \((b = 0)\) inherent to this least squares problem. There is no loss of generality since the parameter matrix \(b\) is identifiable only up to scalar factor.

The estimate of matrix \(b\) can be written as

\[
\hat{b} = \arg \min_{\|b\|_2 = 1} \text{Tr} \left\{ H^T (I_\alpha \otimes b^T) \Phi_1^\perp (I_\alpha \otimes b) H \right\}
\]

Or equivalently

\[
\hat{b} = \arg \min_{\|b\|_2 = 1} b^T Gb \quad \text{Quadratic Form}
\]
where \( b \triangleq \text{vec}(b) \), and the entries of matrix \( G \) are given by

\[
G_{ij} = \mathrm{Tr}\left\{ H^T (I_\alpha \otimes e_j^T) \Phi_1^\perp (I_\alpha \otimes e_i) H \right\}
\]

where \( e_i \triangleq \text{vec}^{-1}(e_i) \), with \( e_i \) being a vector of zeros, except for a one in the \( i \)-th position.

**Solution:** By invoking Rayleigh-Ritz theorem, the estimate of vector \( b \) is given by the unit length eigenvector \((v_{\min})\) corresponding to the smallest eigenvalue \((\lambda_{\min})\) of matrix \( G \).
Blind Identification Algorithm

- **Step 1:** Compute the output Hankel matrix $Y_\alpha$ from output measurements.
- **Step 2:** Perform the SVD of $Y_\alpha$.
- **Step 3:** Compute matrix $G$.
- **Step 4:** Compute the EVD of matrix $G$.
- **Step 5:** Compute the estimate $\hat{b}$ as
  $$\hat{b} = \text{vec}^{-1}(v_{\text{min}})$$
**Remark:** The EVD in *Step 4* of the Blind ID Algorithm can actually be solved by performing the SVD of the square root of matrix $G$, which is guaranteed to exist since $G$ is a symmetric positive definite matrix.
Simulation results

$m = 3$ outputs
$p = 5$ internal states
Poles at:
$(0.1, 0.3, 0.6, 0.8, 0.9)$

Fig. 3: True system.

\[
F_1(z) = \frac{0.3365z^4 - 0.8025z^3 + 0.8448z^2 - 0.5488z + 0.1832}{z^5 - 2.700z^4 + 2.6900z^3 - 1.1970z^2 + 0.2250z - 0.0130}
\]

\[
F_2(z) = \frac{0.1581z^4 - 0.2805z^3 + 0.2140z^2 - 0.1530z + 0.0780}{z^5 - 2.700z^4 + 2.6900z^3 - 1.1970z^2 + 0.2250z - 0.0130}
\]

\[
F_3(z) = \frac{0.3026z^4 - 0.7236z^3 + 0.7939z^2 - 0.5348z + 0.1776}{z^5 - 2.700z^4 + 2.6900z^3 - 1.1970z^2 + 0.2250z - 0.0130}
\]
Simulation results (cont.)

- The outputs were corrupted with zero mean, colored, independent noise signals obtained by filtering Gaussian white noise with variance $\sigma^2$. The resulting noise power spectral density is given by

$$\Phi(\omega) = \frac{0.64\sigma^2}{1.04 - 0.4 \cos(\omega)}$$

- 4000 samples were collected
- $\alpha = 20$ (number of rows in Hankel matrices)
- $SNR \triangleq \frac{\sum_{k=1}^{N+\alpha-1} ||y_k||^2}{m\sigma^2(N+\alpha-1)}$
- $RMSE \triangleq \sqrt{\sum_{i=1}^{m} ||M_i - \hat{M}_i||^2}$, where $M_i$ is the vector of numerator polynomial coefficients of the transfer function between the input and the $i$-th output, and $\hat{M}_i$ is its corresponding estimate.
Location of bases poles:

- FIR bases: poles at $z = 0$
- Laguerre bases: poles at $z = 0.5$
- OBFP-exact: poles at $z = (0.1, 0.3, 0.6, 0.8, 0.9)$
- OBFP-approx.: poles at $z = (0.15, 0.35, 0.65, 0.85, 0.95)$
Simulation results (cont.)

Fig. 4: RMSE vs. SNR for different locations of the basis poles.
Fig. 5: Measured (solid blue line) and estimated (dashed green line) outputs (SNR = 35.5646 dB, FIT = 87.62 %).
A new subspace technique for the blind identification of IIR, SIMO systems represented using orthonormal bases with fixed poles has been presented.

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The method presents improvements in the estimation accuracy when compared to similar techniques based on FIR or Laguerre models.
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