The Fundamental Theorem of Curves and Submanifolds

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Introduction

Introduction

What are the questions/problems?

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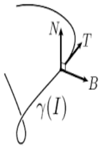
What are the questions/problems?

What answers do we have/know?

The Fundamental Theorem of Curves

The Fundamental Theorem of Curves

We know that if $\gamma : I \to \mathbb{R}^3$ is a curve parametrized by the arc length, then its unit tangent vector field $T = \gamma'$, its unit normal vector field $N = \gamma''/|\gamma''|$, where $\gamma'' \neq 0$, and its binormal vector field $B = T \wedge N$ give us all the geometric details about the trace of γ.

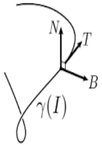


The Fundamental Theorem of Curves

Furthermore, we know that the Frenet frame $\{T, N, B\}$ satisfies the ordinary differential system of equations

$$\begin{aligned} T' &= \kappa N, \\ N' &= -\kappa T - \tau B, \qquad \text{(Frenet equations)} \\ B' &= \tau N, \end{aligned}$$

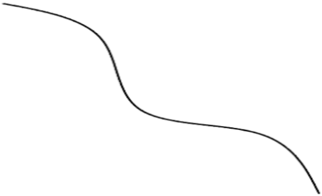
where κ and τ are, respectively, the *curvature* and *torsion* of the curve γ.



The Fundamental Theorem of Curves

The question:

Given the functions $\kappa, \tau : I \to \mathbb{R}$, can we guarantee the existence of a curve $\gamma : I \to \mathbb{R}^3$ with *curvature* κ and *torsion* τ? If so, is that curve unique?



The Fundamental Theorem of Curves

The answer:

Theorem (Fundamental Theorem of curves in $\mathbb{R}^3$)

(1) **Existence:** *Given smooth functions $\kappa, \tau : I \to \mathbb{R}$ so that $\kappa(s) > 0,\ s_0 \in I,\ p_0 \in \mathbb{R}^3$ and (T_0, N_0, B_0) a fixed orthonormal basis of $\mathbb{R}^3$, then there exists a unique curve $\gamma : I \to \mathbb{R}^3$ parametrized by arc length such that $\gamma(s_0) = p_0$, and (T_0, N_0, B_0) is the Frenet frame of γ at $s = s_0$.*

(2) **Uniqueness:** *Suppose that $\gamma, \tilde{\gamma} : I \to \mathbb{R}^3$ are curves parametrized by arc length, and $\gamma, \tilde{\gamma}$ have the same curvature function κ and torsion function τ. Then there exists a rigid motion $f : \mathbb{R}^3 \to \mathbb{R}^3$ such that $\tilde{\gamma} = f(\gamma)$.*

The Fundamental Theorem of Curves

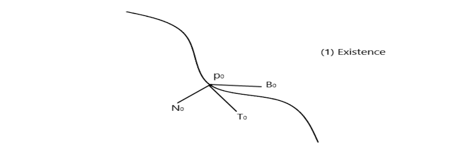
Sketch of the proof.

(1) First, we take the unique solution of the ordinary differential system of equations

$$\begin{aligned} T' &= \kappa N, \\ N' &= -\kappa T - \tau B, \\ B' &= \tau N, \end{aligned} \qquad \text{(Frenet equations)}$$

with initial data $(T, N, B)(s_0) = (T_0, N_0, B_0)$. Then, we define

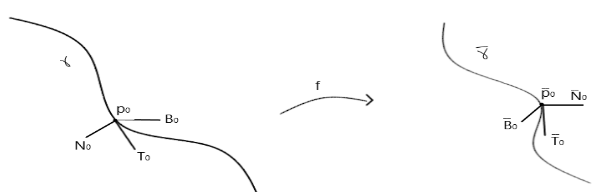
$$\gamma(s) = p_0 + \int_{s_0}^{s} T(t)\,dt, \quad s \in I.$$



The Fundamental Theorem of Curves

Sketch of the proof.

(2) Let $f : \mathbb{R}^3 \to \mathbb{R}^3$ be a rigid motion such that $f(\gamma(s_0)) = \tilde{\gamma}(s_0)$ and f takes the Frenet frame of γ into the Frenet frame of $\tilde{\gamma}$ at s_0.



(2) Uniqueness

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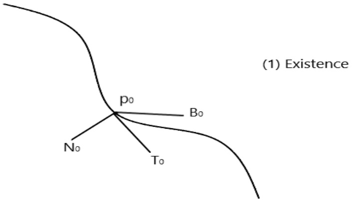
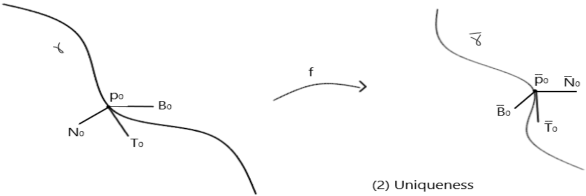
The Fundamental Theorem of Curves

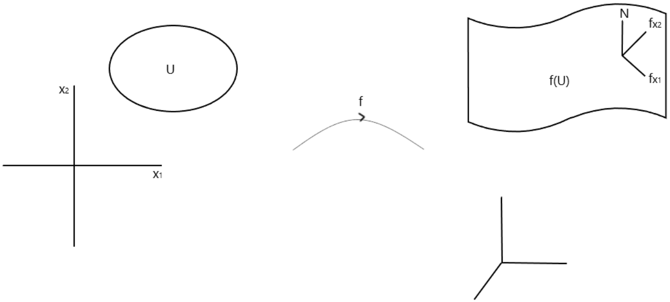




The Fundamental Theorem of Submanifolds: the case of Space Forms

The Fundamental Theorem of Submanifolds: the case of Space Forms

We know that if $f : U \subset \mathbb{R}^2 \to \mathbb{R}^3$ is a parametrized surface in $\mathbb{R}^3$ (i.e., U is an open set and f is an *immersion*), then the frame $\{f_{x_1}, f_{x_2}, N\}$, where $N = \frac{f_{x_1} \wedge f_{x_2}}{||f_{x_1} \wedge f_{x_2}||}$, give us all the geometric details about the submanifold $f(U)$.



The Fundamental Theorem of Submanifolds: the case of Space Forms

Furthermore, we know that the frame $\{f_{x_1}, f_{x_2}, N\}$ satisfies the partial differential system of equations

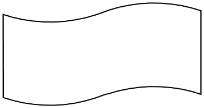
$$\begin{aligned} f_{x_i x_j} &= \Gamma_{ij}^k f_{x_k} + b_{ij} N, \\ N_{x_j} &= a_{ij} f_{x_i}, \end{aligned}$$

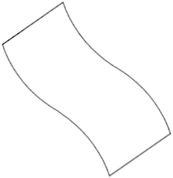
for $1 \leq i, j \leq 2$, where $\Gamma_{ij}^k = \frac{1}{2}\left(\frac{\partial}{\partial x_i} g_{jl} + \frac{\partial}{\partial x_j} g_{il} - \frac{\partial}{\partial x_l} g_{ij}\right) g^{lk}$ are the *Christoffel symbols*, $I = g_{ij} dx_i\, dx_j = \langle f_{x_i}, f_{x_j}\rangle dx_i\, dx_j$ is the *first fundamental form*, $II = b_{ij} dx_i\, dx_j = \langle f_{x_i x_j}, N\rangle dx_i\, dx_j$ is the *second fundamental form*, and $a_{ij} = -b_{ik} g^{kj}$ is the matrix of the *shape operator* associated to the submanifold $f(U)$.

The Fundamental Theorem of Submanifolds: the case of Space Forms

The question:

Given the functions $g_{ij}, b_{ij} : U \subset \mathbb{R}^2 \to \mathbb{R}$, for $1 \le i, j \le 2$, with $g_{ij} = g_{ji} > 0$, $g_{11}g_{22} - g_{12}^2 > 0$ and $b_{ij} = b_{ji}$, can we guarantee the existence of a parametrized surface $f : U \to \mathbb{R}^3$ such that the *first* and *second fundamental forms* of the submanifold $f(U)$ are given by (g_{ij}) and (b_{ij}) respectively? If so, is that submanifold unique?





The Fundamental Theorem of Submanifolds: the case of Space Forms

The answer: Not yet!

The Fundamental Theorem of Submanifolds: the case of Space Forms

The answer: Not yet!

We need to keep in mind that in order to solve the partial differential system of equations

$$\begin{cases} f_{x_i x_j} & = & \Gamma_{ij}^k f_{x_k} + b_{ij} N, \\ N_{x_j} & = & a_{ij} f_{x_i}, \end{cases}$$

for $1 \leq i, j \leq 2$, and get a surface, we must "obey" the *Frobenius Theorem*, which states:

Theorem (Frobenius Theorem in $\mathbb{R}^3$)

Let $U \subset \mathbb{R}^2$ *and* $V \subset \mathbb{R}^3$ *be open subsets,* $A = (A_1, A_2, A_3),\ B = (B_1, B_2, B_3) : U \times V \to \mathbb{R}^3$ *smooth maps,* $u_0 \in U$*, and* $v_0 \in V$*. Then the following first order system*

$$\begin{cases} \phi_{x_1} = A(x_1, x_2, \phi(x_1, x_2)) \\ \phi_{x_2} = B(x_1, x_2, \phi(x_1, x_2)) \\ \phi(u_0) = v_0, \end{cases}$$

has a (unique) smooth solution ϕ *in a neighborhood of* u_0 *for all possible* $u_0 \in U$ *and* $v_0 \in V$ *(fixed) if and only if*

$$(A_i)_{x_2} + \frac{\partial A_i}{\partial \phi_j} B_j = (B_i)_{x_1} + \frac{\partial B_i}{\partial \phi_j} A_j, \quad 1 \leq i \leq 3,$$

hold identically on $U \times V$*.*

The Fundamental Theorem of Submanifolds: the case of Space Forms

So if we write the system of equations

$$\begin{aligned} f_{x_i x_j} &= \Gamma_{ij}^k f_{x_k} + b_{ij} N, \\ N_{x_j} &= a_{ij} f_{x_i}, \end{aligned}$$

for $1 \leq i, j \leq 2$, as

$$\begin{aligned} (f_{x_1}, f_{x_2}, N)_{x_1} &= (f_{x_1}, f_{x_2}, N) P, \\ (f_{x_1}, f_{x_2}, N)_{x_2} &= (f_{x_1}, f_{x_2}, N) Q, \end{aligned}$$

where P, Q are $M_{3\times 3}(\mathbb{R})-$value maps given in terms of g_{ij} and b_{ij}, then this system has solution if and only if

$$P_{x_2} - Q_{x_1} = PQ - QP := [P, Q].$$

This last equation is called the *Gauss-Codazzi equation*.

The Fundamental Theorem of Submanifolds: the case of Space Forms

Now we have the answer:

Theorem (Bonnet)

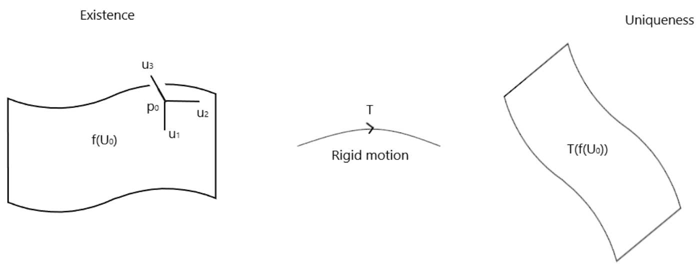
Let $g_{ij}, b_{ij} : U \subset \mathbb{R}^2 \to \mathbb{R}$*, for* $1 \leq i, j \leq 2$*, with* $g_{ij} = g_{ji} > 0,\ g_{11}g_{22} - g_{12}^2 > 0$ *and* $b_{ij} = b_{ji}$*, be smooth maps satisfying the Gauss-Codazzi equation. Let* $(x_1^0, x_2^0) \in U,\ p_0 \in \mathbb{R}^3$ *and* $\{u_1, u_2, u_3\}$ *a basis for* $\mathbb{R}^3$ *be given such that* $u_i \cdot u_j = g_{ij}(x_1^0, x_2^0)$ *and* $u_i \cdot u_3 = 0$ *for* $1 \leq i \leq 2$*. Then there exists a neighborhood* $U_0 \subset U$ *of* (x_1^0, x_2^0) *and a unique immersion* $f : U_0 \to \mathbb{R}^3$ *so that* f *maps* U_0 *homeomorphically to* $f(U_0)$ *such that*

(1) *the first and second fundamental forms of the submanifold* $f(U_0)$ *are given by* (g_{ij}) *and* (b_{ij}) *respectively,*

(2) $f(x_1^0, x_2^0) = p_0$*, and* $f_{x_i}(x_1^0, x_2^0) = u_i$ *for* $i = 1, 2$*.*

The Fundamental Theorem of Submanifolds: the case of Space Forms

Sketch of the proof.



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The Fundamental Theorem of Submanifolds: the case of Space Forms

Remark

The Gauss-Codazzi equation (Frobenius condition)

$$P_{x_2} - Q_{x_1} = [P, Q]$$

usually appears as **Gauss equation**

$$(\Gamma^2_{12})_{x_1} - (\Gamma^2_{11})_{x_2} + \Gamma^1_{12}\Gamma^2_{11} + \Gamma^2_{12}\Gamma^2_{12} - \Gamma^2_{11}\Gamma^2_{22} - \Gamma^1_{11}\Gamma^2_{12} = -g_{11}K,$$

where $K = \frac{b_{11}b_{22}-b_{12}^2}{g_{11}g_{22}-g_{12}^2}$ *is called the Gauss curvature of the submanifold, and* **Codazzi equations**

$$\begin{aligned} (b_{11})_{x_2} - (b_{12})_{x_1} &= g_{11}\Gamma^1_{12} + g_{12}(\Gamma^2_{12} - \Gamma^1_{11}) - b_{22}\Gamma^2_{11} \\ (b_{12})_{x_2} - (b_{22})_{x_1} &= g_{11}\Gamma^1_{22} + g_{12}(\Gamma^2_{22} - \Gamma^1_{12}) - b_{22}\Gamma^2_{12}. \end{aligned}$$

The Fundamental Theorem of Submanifolds: the case of Space Forms

The main question:

What is the most general situation?

The Fundamental Theorem of Submanifolds: the case of Space Forms

We know that if $f : M^n \to \overline{M}^{n+k}$ is an *isometric immersion* between Riemannian manifolds, X, Y, Z, W are tangent to the immersion, η, ζ are normal, and $\overline{\nabla}$ is the Riemannian connection of $\overline{M}$, then the *Gauss equation*

$$\begin{aligned}\langle \overline{R}(X,Y)Z,W\rangle &= \langle R(X,Y)Z,W\rangle \\ &\quad -\langle B(Y,W),B(X,Z)\rangle + \langle B(X,W),B(Y,Z)\rangle,\end{aligned}$$

the *Codazzi equation*

$$\langle \overline{R}(X,Y)Z,\eta\rangle = (\overline{\nabla}_Y B)(X,Z,\eta) - (\overline{\nabla}_X B)(Y,Z,\eta),$$

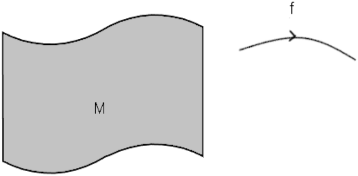
and the *Ricci equation*

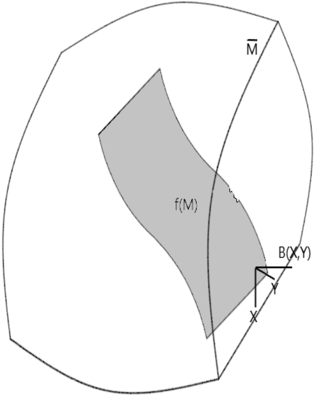
$$\langle \overline{R}(X,Y)\eta,\zeta\rangle = \langle R^{\perp}(X,Y)\eta,\zeta\rangle + \langle [A_\eta, A_\zeta]X,Y\rangle,$$

hold.

The Fundamental Theorem of Submanifolds: the case of Space Forms

R (resp. $\bar{R}$) = Riemannian tensor curvature of M (resp. $\bar{M}$)
$R^{\perp}$ = normal curvature tensor
A = shape operator
B = second fundamental form
$\langle , \rangle$ = Riemannian metric (first fundamental form)





The Fundamental Theorem of Submanifolds: the case of Space Forms

Remark

In the case of a hypersurface $f : M^n \to \overline{M}^{n+1}$, the Ricci equation

$$\langle \overline{R}(X, Y)\eta, \zeta \rangle = \langle R^{\perp}(X, Y)\eta, \zeta \rangle + \langle [A_\eta, A_\zeta]X, Y \rangle$$

disappears and we only have the Gauss and Codazzi equations, as we saw in the case of surfaces in $\mathbb{R}^3$.

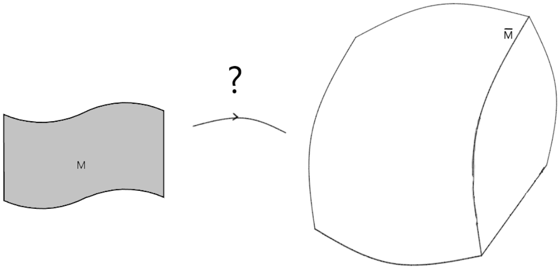
The Fundamental Theorem of Submanifolds: the case of Space Forms

Are Gauss, Codazzi and Ricci equations sufficient to guarantee the existence and uniqueness of an isometric immersion $f : M^n \to \overline{M}^{n+k}$?

The Fundamental Theorem of Submanifolds: the case of Space Forms

Are Gauss, Codazzi and Ricci equations sufficient to guarantee the existence and uniqueness of an isometric immersion $f : M^n \to \overline{M}^{n+k}$?

In fact, if we give a Riemannian manifold M, how could we express Gauss, Codazzi and Ricci equations for a target manifold $\overline{M}$?



The Fundamental Theorem of Submanifolds: the case of Space Forms

When $\overline{M}^{n+k} = \overline{M}_c^{n+k}$ is a Riemannian manifold with *constant secional curvature* c, then the Gauss equation becomes

$$R(X,Y)Z = c\left(\langle Y,Z\rangle X - \langle X,Z\rangle Y\right) + A_{B(Y,Z)}X - A_{B(X,Z)}Y,$$

the Codazzi has the version

$$(\nabla^{\perp}_X B)(Y,Z) = (\nabla^{\perp}_Y B)(X,Z),$$

whereas the Ricci equation reduces to

$$R^{\perp}(X,Y)\eta = B(X, A_\eta Y) - B(A_\eta X, Y).$$

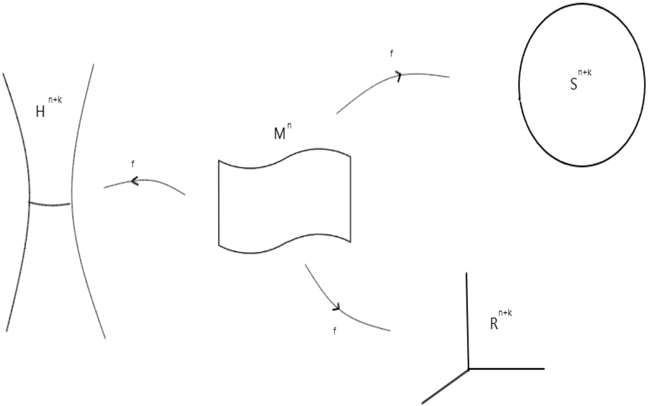
The Fundamental Theorem of Submanifolds: the case of Space Forms

In particular, the equations

$$\begin{aligned} R(X,Y)Z &= c\left(\langle Y,Z\rangle X-\langle X,Z\rangle Y\right)+A_{B(Y,Z)}X-A_{B(X,Z)}Y,\\ (\nabla^{\perp}_{X}B)(Y,Z) &= (\nabla^{\perp}_{Y}B)(X,Z),\\ R^{\perp}(X,Y)\eta &= B(X,A_{\eta}Y)-B(A_{\eta}X,Y), \end{aligned}$$

hold when $\overline{M}_c^{n+k}$ is one of the simply connected complete space forms $\mathbb{Q}_c^{n+k}$, i.e., Euclidean space $\mathbb{R}^{n+k}$, the sphere $\mathbb{S}_c^{n+k}$ or the hyperbolic space $\mathbb{H}_c^{n+k}$, according as $c=0$, $c>0$ or $c<0$, respectively.

The Fundamental Theorem of Submanifolds: the case of Space Forms



Theorem (Isometric immersions into space forms)

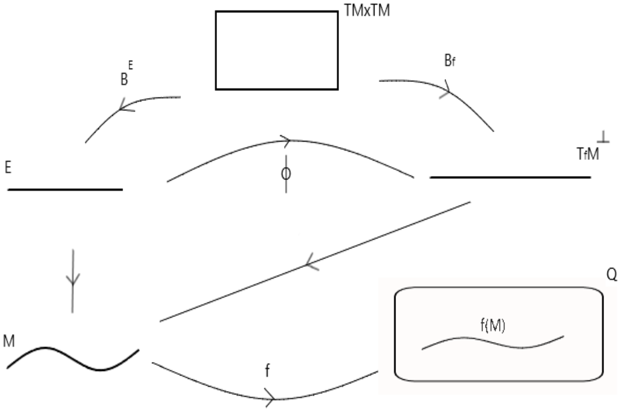
(1) **Existence:** *Let M^n be a simply connected Riemannian manifold, let $\mathcal{E}$ be a Riemannian vector bundle of rank k over M^n with compatible connection $\nabla^{\mathcal{E}}$ and curvature tensor $R^{\mathcal{E}}$ and let $B^{\mathcal{E}}$ be a symmetric section of $Hom(TM \times TM, \mathcal{E})$. For each $\eta \in \Gamma(\mathcal{E})$, define $A^{\mathcal{E}}_{\eta} \in \Gamma(Hom(TM, TM))$ by $\langle A^{\mathcal{E}}_{\eta} X, Y \rangle = \langle B^{\mathcal{E}}(X, Y), \eta \rangle$. Assume that $(\nabla^{\mathcal{E}}, B^{\mathcal{E}}, A^{\mathcal{E}}, R^{\mathcal{E}})$ satisfies Gauss, Ricci and Codazzi equations for a fixed space form $\mathbb{Q}^{n+k}_c$. Then, there exist an isometric immersion $f : M^n \to \mathbb{Q}^{n+k}_c$ and a vector bundle isometry $\phi : \mathcal{E} \to T_f M^{\perp}$ such that $B_f = \phi \circ B^{\mathcal{E}}$ and $\nabla^{\perp}\phi = \phi\nabla^{\mathcal{E}}$.*

(2) **Uniqueness:** *Let $f, g : M^n \to \mathbb{Q}^{n+k}_c$ be isometric immersions. Assume that there exists a vector bundle isometry $\phi : T_f M^{\perp} \to T_g M^{\perp}$ such that*

$$\phi \circ B_f = B_g \quad \text{and} \quad \phi\, {}^f\nabla^{\perp} = {}^g\nabla^{\perp}\phi.$$

Then, there exists an isometry $\tau : \mathbb{Q}^{n+k}_c \to \mathbb{Q}^{n+k}_c$ such that $\tau \circ f = g$ and $\tau_*|_{T_f M^{\perp}} = \phi$.

The Fundamental Theorem of Submanifolds: the case of Space Forms



The Fundamental Theorem of Submanifolds: the case of Space Forms

What is behind the proof of this theorem?

The Fundamental Theorem of Submanifolds: the case of Space Forms

What is behind the proof of this theorem?

When we consider an orthonormal frame $\{E_a\}_{a=1}^{n+k}$ defined in an open set $\overline{U}$ of a Riemannian manifold $\overline{M}^{n+k}$, then its curvature tensor can be described by the 2-forms $\{\Theta_a^b\}_{a,b=1}^{n+k}$ given by

$$d\theta_a^b + \sum_{c=1}^{n+k} \theta_c^b \wedge \theta_a^c = \Theta_a^b,$$

where $\{\theta^a\}_{a=1}^{n+k}$ denotes the co-frame dual to $\{E_a\}_{a=1}^{n+k}$, and $\{\theta_a^b\}_{a,b=1}^{n+k}$ are the corresponding connection forms characterized by

$$d\theta^a + \sum_{c=1}^{n+k} \theta_c^a \wedge \theta^c = 0, \quad \theta_a^b = -\theta_b^a.$$

The Fundamental Theorem of Submanifolds: the case of Space Forms

If we consider another orthonormal frame $\{e_a\}_{a=1}^{n+k}$ with corresponding co-frame $\{\omega^a\}_{a=1}^{n+k}$, connection forms $\{\omega_a^b\}_{a,b=1}^{n}$ and curvature forms $\{\Omega_a^b\}_{a,b=1}^{n+k}$, then those differential forms are related by

$$\omega = P^{-1}dP + P^{-1}\theta P,$$

$$\Omega = P^{-1}\Theta P,$$

where $P : \overline{U} \subset \overline{M}^{n+k} \to \mathrm{SO}_{n+k}$ is the map

$$e_a = \sum_{b=1}^{n+k} P_a^b E_b.$$

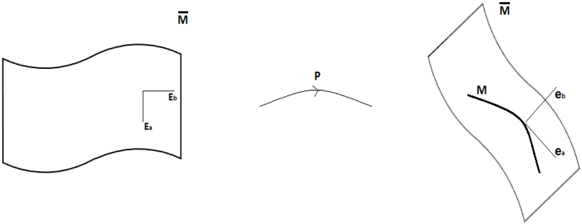
The Fundamental Theorem of Submanifolds: the case of Space Forms

In particular, when we have an isometric immersion $f : M^n \to \overline{M}^{n+k}$ and $\{e_a\}_{a=1}^{n+k}$ is chosen to be adapted to the immersion, that is, in such a way that, along points of M, the first n fields in this frame are tangent to M and the other k ones are local sections of the normal bundle $T_f M^{\perp}$, then

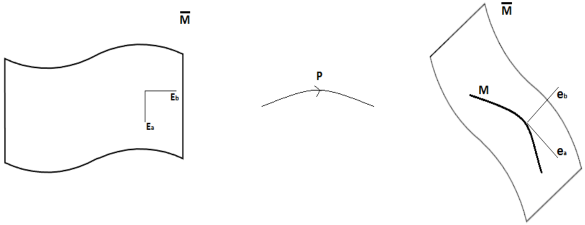
$$d\omega_b^a + \sum_{c=1}^{n+k} \omega_c^a \wedge \omega_b^c = \Omega_b^a = (P^{-1}\Theta P)_b^a$$

corresponds to Gauss, Codazzi and Ricci equations, considering suitable ranges of indices a and b.

The Fundamental Theorem of Submanifolds: the case of Space Forms

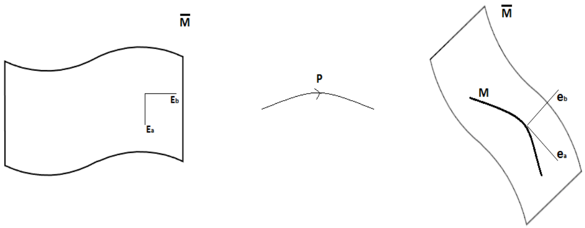


The Fundamental Theorem of Submanifolds: the case of Space Forms



But what if we don't have the immersion $f : M^n \to \overline{M}^{n+k}$?

The Fundamental Theorem of Submanifolds: the case of Space Forms



But what if we don't have the immersion $f : M^n \to \overline{M}^{n+k}$?
Go to the realm of vector bundles!

The Fundamental Theorem of Submanifolds: the case of Space Forms

The idea is the following:

Given a frame $\{E_a\}_{a=1}^{n+k}$ in $TM \oplus \mathcal{E}$ (where $\mathcal{E}$ is a vector bundle over M), we try to build a smooth map

$$P : U \subset M \to \mathrm{SO}_{n+k},$$

(which will play the role of producing an adapted frame) such that

$$P^{-1}dP = \omega - \lambda,$$

where λ has to be expressed (if possible) in terms of a given data (remember that $\lambda = P^{-1}\theta P$ when the immersion is given).

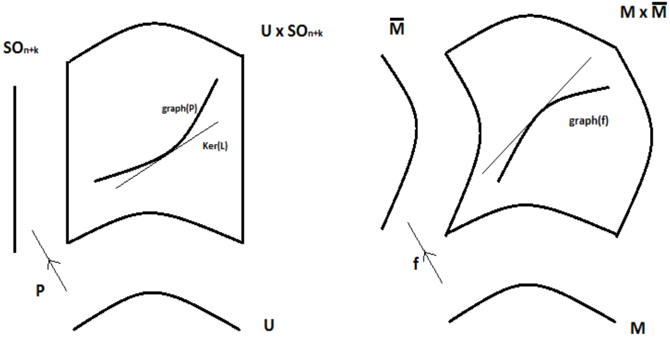
The Fundamental Theorem of Submanifolds: the case of Space Forms

For solving the equation

$$\begin{cases} P^{-1}dP &= \omega - \lambda \\ P(x_0) &= Id \end{cases}$$

we consider on $U \times SO_{n+k}$ the distribution $\ker L_{(x,Z)}$, where $L_{(x,Z)} = \omega - \lambda - Z^{-1}dZ$, and use (in the presence of Gauss, Codazzi and Ricci equations) the Frobenius Theorem (in the context of differential forms) to get an integral submanifold in $U \times SO_{n+k}$, which is the graph of a map $P : U \to SO_{n+k}$. Finally, the idea is to use the map $P : U \to SO_{n+k}$ to build the isometric immersion $f : U \subset M^n \to f(U) \subset \overline{M}^{n+k}$ (for example, the graph of $f : U \subset M^n \to f(U) \subset \overline{M}^{n+k}$ is an integral submanifold in $M \times \overline{M}$ obtained from a suitable distribution on $M \times \overline{M}$).

The Fundamental Theorem of Submanifolds: the case of Space Forms



The Fundamental Theorem of Submanifolds: the case of Space Forms

Remark

In the case of Space Forms, the differential form $\lambda = P^{-1}\theta P$, *which originally appears in the change of coordinates* $\omega = P^{-1}dP + \lambda$, *can be expressed in terms of a given data, and the equation*

$$\begin{cases} P^{-1}dP & = & \omega - \lambda \\ P(x_0) & = & Id \end{cases}$$

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What about other cases?

Some other cases

Some other cases

In **2007**, Benoit Daniel showed that the Gauss and Codazzi equations are not sufficient to guarantee the existence of isometric immersions into 3-dimensional homogeneous manifold with 4-dimensional isometry group, which includes the spaces $\mathbb{Q}^2_c \times \mathbb{R}$ (for $c \neq 0$) and the Heisenberg Lie group Nil_3. In **2009**, he obtained a similar result for isometric immersion into the produt space $\mathbb{Q}^n_c \times \mathbb{R}$.

He proved the following results:

Theorem (B. Daniel, 2007)

Let M^2 be a simply connected oriented Riemannian manifold, ds^2 its metric, ∇ its Riemannian connection and J the rotation of angle $\frac{\pi}{2}$ on TM. Let A be a field of symmetric operators $A_y : T_yM \to T_yM$, T a vector field on M and ν a smooth function on M such that $||T||^2 + \nu^2 = 1$. Let $\mathbb{E}$ be a 3-dimensional homogeneous manifold with 4-dimensional isometry group and ξ its vertical vector field. Let κ be its base curvature and τ its bundle curvature. Then, there exists an isometric immersion $f : M \to \mathbb{E}$ such that the shape operator with respect to the normal N associated to f is $df \circ A \circ df^{-1}$ and such that $\xi = df(T) + N$ if and only if (ds^2, A, T, ν) satisfies the Gauss and Codazzi equations for $\mathbb{E}$ and, for all vector fields X on M, the following equations:

$$\nabla_X T = \nu(AX - \tau JX), \qquad d\nu(X) + \langle AX - \tau JX, T\rangle = 0.$$

In this case, the immersion is unique up to a global isometry of $\mathbb{E}$ preserving the orientations of both the fibers and the base of the fibration.

Theorem (B. Daniel, 2009)

Let M^n be simply connected oriented Riemannian manifold, ds^2 its metric and ∇ its Riemannian connection. Let A be a field of symmetric operators $A_y : T_yM \to T_yM$, T a vector field on M and ν a smooth function on M such that $||T||^2 + \nu^2 = 1$. Assume that (ds^2, A, T, ν) satisfies the Gauss and Codazzi equations for $\mathbb{Q}^n_c \times \mathbb{R}$ (with $c \neq 0$) and the following equations:

$$\nabla_X T = \nu AX, \qquad d\nu(X) = -\langle SX, T\rangle.$$

Then, there exists an isometric immersion $f : M^n \to \mathbb{Q}^n_c \times \mathbb{R}$ such that the shape operator with respect to the normal N associated to f is given by $df \circ A \circ df^{-1}$ and such that $\frac{\partial}{\partial t} = df(T) + N$.
Moreover, the immersion is unique up to a global isometry of $\mathbb{Q}^n_c \times \mathbb{R}$ preserving the orientations of both $\mathbb{Q}^n_c$ and $\mathbb{R}$.

Some other cases

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In **2010**, J.H. Lira, R. Tojero and F. Vitório extended to semi-Riemannian product of space forms $\mathbb{Q}^{n_1}_{\kappa_1,\mu_1} \times \mathbb{Q}^{n_2}_{\kappa_2,\mu_2}$ the isometric immersion result obtained by B. Daniel.

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How does the case of Lie groups work?

Some other cases

For the case of a two-step nilpotent Lie group N, we use the decomposition of its Lie algebra $\mathfrak{n} = \mathfrak{v} \oplus \mathfrak{z}$ with the Lie bracket relations

$$[\mathfrak{v}, \mathfrak{v}] \subset \mathfrak{z}, \quad [\mathfrak{z}, \mathfrak{n}] = \{0\}$$

to write the co-frame, connection forms and curvature forms associated to a fixed orthonormal left-invariant frame field

$$E_1, \cdots, E_n, E_{n+1}, \cdots, E_{n+n'}$$

so that the first n vector are in $\mathfrak{v}$ and the next n' ones are in $\mathfrak{z}$. The additional conditions are given by the tensor $J_Z : \mathfrak{n} \to \mathfrak{n},\ Z \in \mathfrak{z}$, given by

$$J_Z = -2\bar{\nabla} Z.$$

Some other cases

The work is done when we denote $J_k = J_{E_{n+k}},\ 1 \leq k \leq n'$, obtain

$$\langle J_k V, W\rangle = \sum_{l,r=1}^{n+n'} \langle V, E_l\rangle\langle W, E_r\rangle\sigma_{lr}^{n+k},$$

where $[E_l, E_r] = \sum_{k=1}^{n+n'} \sigma_{lr}^{k} E_k$ are the structure constants of N, and write the form λ (which is related to a future adapted frame) and the curvature form associated do the frame $\{E_a\}_{a=1}^{n+n'}$ in terms of the tensors $J_k,\ 1 \leq k \leq n'$.

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where $[E_l, E_r] = \sum_{k=1}^{n+n'} \sigma_{lr}^k E_k$ are the structure constants of N, and write the form λ (which is related to a future adapted frame) and the curvature form associated do the frame $\{E_a\}_{a=1}^{n+n'}$ in terms of the tensors $J_k,\ 1 \leq k \leq n'$. Precisely, the form λ comes from the tensor field in N

$$\begin{aligned} L(X, Y, V) &= -\frac{1}{2} \sum_{k=1}^{n'} \langle J_k V, X \rangle \langle Y, E_{n+k} \rangle + \frac{1}{2} \sum_{k=1}^{n'} \langle J_k Y, X \rangle \langle V, E_{n+k} \rangle \\ &+ \frac{1}{2} \sum_{k=1}^{n'} \langle J_k Y, V \rangle \langle X, E_{n+k} \rangle, \quad X, Y, V \in \Gamma(TN), \end{aligned}$$

Some other cases

and the curvature tensor is the $(0,4)$ covariant tensor Q in N given by

$$Q(X,Y,V,W) = Q_1(X,Y,V,W) + Q_2(X,Y,V,W), \quad X,Y,V,W \in \Gamma(TN),$$

where Q_1 and Q_2 are the $(0,4)-$tensor fields given by

$$\begin{aligned} Q_1(X,Y,V,W) & \\ = \; & \frac{1}{4}\langle J_k X, W\rangle\langle J_k V, Y\rangle + \frac{1}{2}\langle J_k Y, X\rangle\langle J_k W, V\rangle - \frac{1}{4}\langle J_k Y, W\rangle\langle J_k V, X\rangle \\ & -\frac{1}{2}\langle W, E_{n+k}\rangle\langle(\bar{\nabla}_X J_k)V, Y\rangle + \frac{1}{2}\langle V, E_{n+k}\rangle\langle(\bar{\nabla}_X J_k)W, Y\rangle \\ & +\frac{1}{2}\langle Y, E_{n+k}\rangle\langle(\bar{\nabla}_X J_k)W, V\rangle + \frac{1}{2}\langle W, E_{n+k}\rangle\langle(\bar{\nabla}_Y J_k)V, X\rangle \\ & -\frac{1}{2}\langle V, E_{n+k}\rangle\langle(\bar{\nabla}_Y J_k)W, X\rangle - \frac{1}{2}\langle X, E_{n+k}\rangle\langle(\bar{\nabla}_Y J_k)W, V\rangle \end{aligned}$$

Some other cases

and

$$Q_2(X, Y, V, W) =$$

$$-\frac{1}{4}\sum_{k,l=1}^{n'}\langle E_{n+k}, W\rangle\langle E_{n+l}, V\rangle\langle J_k Y, J_l X\rangle - \frac{1}{4}\sum_{k,l=1}^{n'}\langle E_{n+k}, W\rangle\langle E_{n+l}, X\rangle\langle J_k Y, J_l V\rangle$$

$$-\frac{1}{4}\sum_{k,l=1}^{n'}\langle E_{n+k}, Y\rangle\langle E_{n+l}, V\rangle\langle J_k W, J_l X\rangle - \frac{1}{4}\sum_{k,l=1}^{n'}\langle E_{n+k}, Y\rangle\langle E_{n+l}, X\rangle\langle J_k W, J_l V\rangle$$

$$+\frac{1}{4}\sum_{k,l=1}^{n'}\langle E_{n+k}, W\rangle\langle E_{n+l}, V\rangle\langle J_k X, J_l Y\rangle + \frac{1}{4}\sum_{k,l=1}^{n'}\langle E_{n+k}, W\rangle\langle E_{n+l}, Y\rangle\langle J_k X, J_l V\rangle$$

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Some other cases

The theorem is the following:

Theorem (J.H. Lira, - , 2012)

Let M^m be a Riemannian simply connected manifold and let $\mathcal{E}$ be a real Riemannian vector bundle with rank $m' = n + n' - m$ so that $\mathcal{S} = TM \oplus \mathcal{E}$ is a trivial vector bundle. Let $\hat{\nabla}$ and $\hat{R}$ be respectively the compatible connection and curvature tensor in $\mathcal{S}$ and ∇ and $\nabla^{\mathcal{E}}$ the compatible connections induced in TM and $\mathcal{E}$, respectively. We fix a global orthonormal frame $\{\hat{E}_k\}_{k=1}^{n+n'}$ in $\mathcal{S}$. Define tensors $\hat{J}_k$ and $\hat{Q}$ (as those defined in a fixed two-step nilpotent Lie group N), and assume that these fields satisfy the Gauss, Codazzi and Ricci equations

$$\hat{R} = \hat{Q}$$

and the additional equations

$$\hat{\nabla}\hat{E}_{n+k} = -1/2\hat{J}_k, \quad k = 1, \cdots, n'.$$

Thus, there exists an isometric immersion $f : M \to N$ covered by a bundle isomorphism $\Phi : \mathcal{E} \to T_f M^{\perp}$, where $T_f M^{\perp}$ is the normal bundle along f, so that Φ is an isometry when restrited to the fibers and satisfies

Theorem (J.H. Lira, - , 2012)

$$\Phi\nabla^{\mathcal{E}}_X V = \nabla^{\perp}_X \Phi V, \quad X \in \Gamma(TM), \quad V \in \Gamma(\mathcal{E}),$$

$$\Phi B^{\mathcal{E}}(X, Y) = \bar{\nabla}_{f_* X} f_* Y - f_*(\nabla_X Y) =: B_f(X, Y), \quad X, Y \in \Gamma(TM),$$

where $\bar{\nabla}$ *and* $\nabla^{\perp}$ *denote, respectively, the connections in* N *and* $T_f M^{\perp}$ *and the covariant symmetric tensor* $B^{\mathcal{E}} \in \Gamma(T^*M \otimes T^*M \otimes \mathcal{E})$ *is defined by*

$$\hat{\nabla}_X Y = \nabla_X Y + B^{\mathcal{E}}(X, Y), \quad X, Y \in \Gamma(TM).$$

Moreover, if $f, g : M \to N$ *are isometric immersions such that* $\Phi B_f = B_g$ *and* $\Phi^f\nabla^{\perp} = {}^g\nabla^{\perp}\Phi$, *for an isometry* $\Phi : T_f M^{\perp} \to T_g M^{\perp}$ *such that*

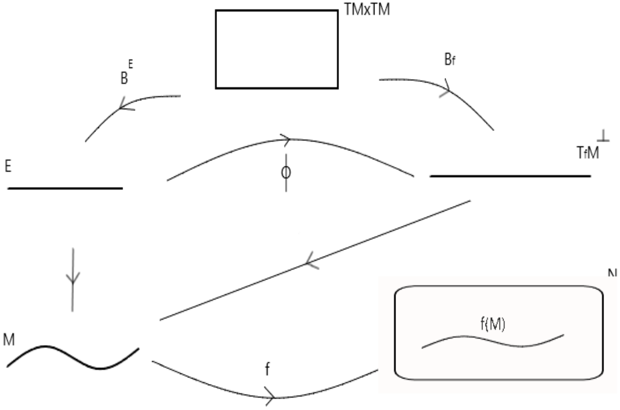
$$\langle f_* X, E_{n+k}\rangle = \langle g_* X, E_{n+k}\rangle, \quad X \in \Gamma(TM), \; k = 1, \cdots, n'$$

and

$$\langle V, E_{n+k}\rangle = \langle \Phi(V), E_{n+k}\rangle, \quad V \in T_f M^{\perp}, \; k = 1, \cdots, n',$$

for a fixed left-invariant frame $\{E_k\}_{k=1}^{n+n'}$ *in* N, *then there exists an isometry* $\tau : N \to N$ *such that* $\tau \circ f = g$ *and* $\tau_*|_{T_f M^{\perp}} = \Phi$.

Some other cases



Some other cases

Remark

The same can be done for a three-step nilpotent Lie group S whose Lie algebra may be decomposed as

$$\mathfrak{s} = \mathfrak{z} \oplus \mathfrak{v} \oplus \mathfrak{a},$$

where $\mathfrak{a} = \mathbb{R}H$ is a one-dimensional factor, with

$$[\mathfrak{v}, \mathfrak{v}] \subset \mathfrak{z}, \quad [\mathfrak{z}, \mathfrak{v}] = \{0\}, \quad [\mathfrak{z}, \mathfrak{z}] = \{0\},$$

and the Lie bracket extended to $\mathfrak{a}$ by the relations

$$[H, E] = \frac{1}{2}E, \quad [H, Z] = Z,$$

for $E \in \mathfrak{v}$ and $Z \in \mathfrak{z}$.

Some other cases

Recently, we found necessary and sufficient conditions for a non-degenerate arbitrary signature manifold M^n to be realized as a submanifold in the large class of warped product manifolds $\varepsilon I \times_a \mathbb{Q}_c^N$, where $\varepsilon = \pm 1$ and $a : I \subset \mathbb{R} \to \mathbb{R}^+$ is the scale factor.
We have proved the following result:

Theorem (C.A.D. Ribeiro, - ,2017)

Let M^n be a semi-Riemannian manifold of index p and E a semi-Riemannian vector bundle of index q and rank $m = N + 1 - n$ over M with compatible connection ∇^E and curvature operator R^E. Let us algo give B^E a symmetric section in $\Gamma(T^*M \otimes T^*M \otimes E)$, ξ a section in $\Gamma(E)$, real numbers $c, \varepsilon \in \{-1, 1\}$ and smooth functions $a : I \to \mathbb{R}^+$ and $\pi : M \to I$. We define the vector field $T \in TM$ by $T = \varepsilon \cdot \mathrm{grad}(\pi)$ and, for each $\eta \in \Gamma(E)$, we define a section $A^E_\eta \in \Gamma(T^*M \otimes TM)$ by $\langle B^E(X, Y), \eta\rangle = \langle A^E_\eta X, Y\rangle$. Assume that $(\nabla^E, B^E, A^E, R^E)$ satisfies Gauss, Codazzi and Ricci equations for $\varepsilon I \times_a \mathbb{Q}^N_c$, and the additional equations

$$\langle T, T\rangle + \langle \xi, \xi\rangle = \varepsilon, \quad \nabla_X T = \frac{a'}{a}(X - \varepsilon\langle X, T\rangle) + A^E_\eta X$$

$$\nabla^E_X \xi = \frac{-\varepsilon a'}{a}\langle X, T\rangle\xi - B^E(T, X).$$

Theorem (C.A.D. Ribeiro, - ,2017)

Then, there exists an isometric immersion $f : M^n \to \varepsilon I \times_a \mathbb{Q}_c^N$ *and a vector bundle isometry* $\Phi : E \to T_f M^\perp$*, such that:*

1. $\partial t = df(T) + \Phi(\xi)$,
2. $\pi = \pi_I \circ f$*, where* $\pi_I : \varepsilon I \times_a \mathbb{Q}_c^N \to I$ *is the projection,*
3. $B_f = \Phi \circ B^E \circ df^{-1}$*, where* B_f *is the second fundamental form of* f*,*
4. $\nabla^\perp \Phi = \Phi \nabla^E$.

Moreover, if $f, g : M \to \varepsilon I \times_a \mathbb{Q}_c^N$ *are isometric immersions such that* $\Phi B_f = B_g$ *and* $\Phi^f \nabla^\perp = {}^g\nabla^\perp \Phi$*, for an isometry* $\Phi : T_f M^\perp \to T_g M^\perp$*, then there exists an isometry* $\tau : \varepsilon I \times_a \mathbb{Q}_c^N \to \varepsilon I \times_a \mathbb{Q}_c^N$ *such that* $\tau \circ f = g$ *and* $\tau_*|_{T_f M^\perp} = \Phi$.

Muito obrigado!

Thank you very much!

Muchas gracias!