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5

Applications to Eigenvalue Problems*

Eigenvalue problems infest applied mathematics. These problems consist of finding nontrivial solutions to a linear differential equation subject to boundary conditions that admit the trivial solution. The differential equation contains an eigenvalue parameter, and nontrivial solutions exist only for special values of this parameter, the eigenvalues. Generally, finding the eigenvalues and the corresponding nontrivial solutions poses a formidable task.

Certain eigenvalue problems can be recast as isoperimetric problems. Indeed, many of the eigenvalue problems have their origin in the calculus of variations. Although the Euler-Lagrange equation is essentially the original differential equation and thus of limited value for deriving solutions, the variational formulation is helpful for extracting results about the distribution of eigenvalues. In this chapter we discuss a few simple applications of the variational approach to Sturm-Liouville problems. The standard reference on this material is Courant and Hilbert [25]. Moiseiwitsch [54] also discusses at length eigenvalue problems in the framework of the calculus of variations. Our brief account is a blend of material from Courant and Hilbert *op. cit.* and Wan [71].

5.1 The Sturm-Liouville Problem

The (regular) Sturm-Liouville problem entails finding nontrivial solutions to differential equations of the form

$$(-p(x)y'(x))' + q(x)y(x) - \lambda r(x)y(x) = 0, \tag{5.1}$$

for the unknown function $y : [x_0, x_1] \to \mathbb{R}$ subject to boundary conditions of the form

$$\begin{aligned} \alpha_0 y(x_0) + \beta_0 y'(x_0) &= 0, \\ \alpha_1 y(x_1) + \beta_1 y'(x_1) &= 0. \end{aligned} \tag{5.2}$$

Here, q and r are functions continuous on the interval $[x_0, x_1]$, and $p \in C^1[x_0, x_1]$. In addition, $p(x) > 0$ and $r(x) > 0$ for all $x \in [x_0, x_1]$. The α_k and β_k in the boundary conditions are constants such that $\alpha_k^2 + \beta_k^2 \neq 0$, and λ is a parameter.

Generically, the only solution to equation (5.1) that satisfies the boundary conditions (5.2) is the trivial solution, $y(x) = 0$ for all $x \in [x_0, x_1]$. There are, however, certain values of λ that lead to nontrivial solutions. These special values are called **eigenvalues** and the corresponding nontrivial solutions are called **eigenfunctions**. The set of all eigenvalues for the problem is called the **spectrum**.

An extensive theory has been developed for the Sturm-Liouville problem. Here, we limit ourselves to citing a few basic results and direct the reader to standard works such as Birkhoff and Rota [9], Coddington and Levinson [24], and Titchmarsh [70] for further details.

The "natural" function space in which to study the Sturm-Liouville problem is the (real) Hilbert space $L^2[x_0, x_1]$, which consists of functions $f : [x_0, x_1] \to \mathbb{R}$ such that

$$\int_{x_0}^{x_1} f^2(x)\,dx < \infty.$$

The inner product on this Hilbert space is defined by

$$\langle f, g\rangle = \int_{x_0}^{x_1} r(x)f(x)g(x)\,dx,$$

for all $f, g \in L^2[x_0, x_1]$.[1] The norm induced by this inner product is defined by

$${}_r\|f\|_2 = \sqrt{\langle f, f\rangle} = \sqrt{\int_{x_0}^{x_1} r(x)f^2(x)\,dx},$$

for all $f \in L^2[x_0, x_1]$. Note that the norm ${}_r\|\cdot\|_2$ is equivalent to the usual norm $\|\cdot\|_2$ defined by

$$\|f\|_2 = \sqrt{\int_{x_0}^{x_1} f^2(x)\,dx},$$

because r is continuous on $[x_0, x_1]$ and positive; hence, r is bounded above and below by positive numbers.[2]

Some notable results from the theory are:

[1] Strictly speaking, the integrals defining the Hilbert space are Lebesgue integrals and the elements of the space are equivalence classes of functions. We deal here with solutions to the Sturm-Liouville problem and these functions are continuous on $[x_0, x_1]$. For such functions the Lebesgue and Riemann integrals are equivalent. Note that $L^2[x_0, x_1]$ also includes much "rougher" functions that are not Riemann integrable.

[2] See Appendix B.1.

(a) There exist an infinite number of eigenvalues. All the eigenvalues are real and isolated. The spectrum can be represented as a monotonic increasing sequence $\{\lambda_n\}$ with $\lim_{n\to\infty} \lambda_n = \infty$. The least element in the spectrum is called the **first eigenvalue**.

(b) The eigenvalues are **simple**. This means that there exists precisely one eigenfunction (apart from multiplicative factors) corresponding to each eigenvalue.

(c) If λ_m and λ_n are distinct eigenvalues with corresponding eigenfunctions ϕ_m and ϕ_n, respectively, the orthogonality relation

$$\langle \phi_m, \phi_n \rangle = 0$$

is satisfied. (Note that $\langle \phi_m, \phi_m \rangle > 0$, since ϕ_m is a nontrivial solution.)

(d) The set of all eigenfunctions $\{\phi_n\}$ forms a basis for the space $L^2[x_0, x_1]$. In other words, for any function $f \in L^2[x_0, x_1]$ there exist constants $\{a_n\}$ such that the series

$$\mathcal{F}(f) = \sum_{n=1}^{\infty} a_n \phi_n$$

converges in the ${}_r\|\cdot\|_2$ norm to f; i.e.,

$$\lim_{k\to\infty} {}_r\|f - \sum_{n=1}^{\infty} a_n \phi_n\|_2 = 0.$$

The series representing f is called an **eigenfunction expansion** or **generalized Fourier series** of f.

The Sturm-Liouville problem can be recast as a variational problem. We do this for the case $\beta_0 = \beta_1 = 0$. The formulation for the general boundary conditions (5.2) can be found in Wan, *op. cit.*, p. 285. Let J be the functional defined by

$$J(y) = \int_{x_0}^{x_1} \left(p y'^2 + q y^2 \right) dx, \tag{5.3}$$

and consider the problem of finding the extremals for J subject to boundary conditions of the form

$$y(x_0) = y(x_1) = 0, \tag{5.4}$$

and the isoperimetric constraint

$$I(y) = \int_{x_0}^{x_1} r(x) y^2(x)\, dx = 1. \tag{5.5}$$

The Euler-Lagrange equation for the functional I is

$$-2r(x)y(x) = 0,$$

which is satisfied only for the trivial solution $y = 0$, because r is positive. No extremals for I can therefore satisfy the isoperimetric condition (5.5). If y is an extremal for the isoperimetric problem, then Theorem 4.2.1 implies that there is a constant λ such that y satisfies the Euler-Lagrange equation

$$\frac{d}{dx}\frac{\partial F}{\partial y'} - \frac{\partial F}{\partial y} = 0, \tag{5.6}$$

for

$$F = py'^2 + qy^2 - \lambda r y^2.$$

But the Euler-Lagrange equation for this choice of F is equivalent to the differential equation (5.1). The isoperimetric problem thus corresponds to the Sturm-Liouville problem augmented by the normalizing condition (5.5), which simply scales the eigenfunctions. Here, the Lagrange multiplier plays the rôle of the eigenvalue parameter.

Example 5.1.1: Let $p(x) = 1$, $q(x) = 0$, $r(x) = 1$, and $[x_0, x_1] = [0, \pi]$. Then the Euler-Lagrange equation reduces to

$$y''(x) + \lambda y(x) = 0, \tag{5.7}$$

and the boundary conditions are

$$y(0) = y(\pi) = 0. \tag{5.8}$$

If $\lambda < 0$, then the general solution to equation (5.7) is

$$y(x) = Ae^{\sqrt{-\lambda}x} + Be^{-\sqrt{-\lambda}x},$$

where A and B are constants. The boundary conditions imply that $A = B = 0$, and therefore there are only trivial solutions if $\lambda < 0$. If $\lambda = 0$, then equation (5.7) has the general solution

$$y(x) = Ax + B.$$

Again the boundary conditions imply that $A = B = 0$, and therefore preclude the possibility of nontrivial solutions. Hence, any eigenvalues for this problem must be positive.

If $\lambda > 0$, then the general solution to equation (5.7) is

$$y(x) = A\cos(\sqrt{\lambda}x) + B\sin(\sqrt{\lambda}x).$$

The condition $y(0) = 0$ implies that $A = 0$; the condition $y(\pi) = 0$ implies that

$$B\sin(\sqrt{\lambda}\pi) = 0. \tag{5.9}$$

Equation (5.9) is satisfied for $B \neq 0$ provided $\sqrt{\lambda}$ is a positive integer, and this leads to the nontrivial solution $y(x) = B\sin(\sqrt{\lambda}x)$. The eigenvalues for this problem are therefore $\lambda_n = n^2$, and the first eigenvalue is $\lambda_1 = 1$. The eigenfunctions corresponding to λ_n are of the form

$$\phi_n(x) = B\sin(nx), \tag{5.10}$$

where B is an arbitrary constant.

In terms of the isoperimetric problem, there are an infinite number of Lagrange multipliers that can be used. Each Lagrange multiplier corresponds to an eigenvalue, and the linearity of the Euler-Lagrange equation implies that any function of the form

$$f(x) = \sum_{n=1}^{\infty} a_n \sin(nx), \tag{5.11}$$

such that the Fourier series is convergent and twice term by term differentiable, is an extremal for the problem, provided f satisfies the isoperimetric condition (5.5). Now,

$$\begin{aligned}
\int_0^{\pi} f^2(x)\,dx &= \int_0^{\pi} \left(\sum_{n=1}^{\infty} a_n \sin(nx) \right)^2 dx \\
&= \sum_{n=1}^{\infty} a_n^2 \int_0^{\pi} \sin^2(nx)\,dx \\
&= \frac{\pi}{2} \sum_{n=1}^{\infty} a_n^2,
\end{aligned}$$

where we have used the orthogonality relation

$$\langle \sin(mx), \sin(nx) \rangle = \begin{cases} 0, & \text{if } m \neq n, \\ \frac{\pi}{2}, & \text{if } m = n. \end{cases}$$

Hence, any eigenfunction expansion of the form (5.11) having the requisite convergence properties and satisfying the condition

$$\sum_{n=1}^{\infty} a_n^2 = \frac{2}{\pi} \tag{5.12}$$

is an extremal for the problem. Any finite combination of the eigenfunctions such as

$$f(x) = a_1 \sin(x) + a_2 \sin(2x) + \cdots + a_m \sin(mx),$$

where

$$a_1^2 + \cdots + a_m^2 = \frac{2}{\pi},$$

for example, is an extremal.

If we are searching among the eigenfunction expansions for extremals that make J a minimum, then the situation changes considerably. Suppose that f is an eigenfunction extremal for the problem. Then

$$y'(x) = \sum_{n=1}^{\infty} na_n \cos(nx),$$

so that

$$\begin{aligned} J(y) = \int_0^{\pi} y'^2(x)\,dx &= \int_0^{\pi} \left(\sum_{n=1}^{\infty} na_n \cos(nx) \right)^2 dx \\ &= \sum_{n=1}^{\infty} n^2 a_n^2 \int_0^{\pi} \cos^2(nx)\,dx \\ &= \frac{\pi}{2} \sum_{n=1}^{\infty} n^2 a_n^2. \end{aligned}$$

Here, we have used the orthogonality relation

$$\langle \cos(mx), \cos(nx) \rangle = \begin{cases} 0, & \text{if } m \neq n, \\ \frac{\pi}{2}, & \text{if } m = n. \end{cases}$$

The eigenfunction extremal for the first eigenvalue is

$$y_1(x) = \sqrt{\frac{2}{\pi}} \sin(x),$$

and for this extremal

$$J(y_1) = 1.$$

In fact, y_1 produces the minimum value for J. To see this, let f be another extremal for the problem. Then the completeness property of the Fourier series implies that f can be expressed as an eigenfunction expansion of the form (5.11), where the coefficients a_n satisfy relation (5.12). If f is distinct from y_1 then there is an integer $m \geq 2$ such that $a_m \neq 0$. Now,

$$\begin{aligned} J(f) &= \frac{\pi}{2} \sum_{n=1}^{\infty} n^2 a_n^2 \\ &\geq \frac{\pi}{2} \left((m^2 - 1) a_m^2 + \sum_{n=1}^{\infty} a_n^2 \right) \\ &> \frac{\pi}{2} \sum_{n=1}^{\infty} a_n^2 = 1, \end{aligned}$$

and hence $J(f) > J(y_1)$.

Exercises 5.1:

1. The **Cauchy-Euler equation** is

$$(xy'(x))' + \frac{\lambda}{x}y(x) = 0.$$

Show that

$$y(x) = c_1\cosh\left(\sqrt{-\lambda}\ln x\right) + c_2\sinh\left(\sqrt{-\lambda}\ln x\right),$$

where c_1 and c_2 are constants, is a general solution to this equation. Given the boundary conditions $y(0) = y(e^{\pi}) = 0$ find the eigenvalues.

2. Reformulate the differential equation

$$y^{(iv)}(x) + (\lambda + \rho(x))y(x) = 0$$

along with the boundary values $y(0) = y'(0) = 0$, $y(1) = y'(1) = 0$ as an isoperimetric problem.

5.2 The First Eigenvalue

The first eigenvalue in Example 5.1.1 has the notable property that the corresponding eigenfunction produced the minimum value for J. If fact, this relationship persists for the general Sturm-Liouville problem.

Theorem 5.2.1 *Let λ_1 be the first eigenvalue for the Sturm-Liouville problem* (5.1) *with boundary conditions* (5.4)*, and let y_1 be the corresponding eigenfunction normalized to satisfy the isoperimetric constraint* (5.5)*. Then, among functions in $C^2[x_0, x_1]$ that satisfy the boundary conditions* (5.4) *and the isoperimetric condition* (5.5)*, the functional J defined by equation* (5.3) *is minimum at $y = y_1$. Moreover,*

$$J(y_1) = \lambda_1.$$

Proof: Suppose that J has a minimum at y. Then y is an extremal and thus satisfies equation (5.1) and conditions (5.4) and (5.5). Multiplying equation (5.1) by y and integrating from x_0 to x_1 gives

$$-pyy'\Big|_{x_0}^{x_1} + \int_{x_0}^{x_1} \left(py'^2 + qy^2\right)\,dx = \lambda\int_{x_0}^{x_1} ry^2\,dx.$$

The first term on the left-hand side of the above expression is zero since $y(x_0) = y(x_1) = 0$; the integral on the left-hand side of the equation is one by the isoperimetric condition. Hence we have

$$J(y) = \lambda.$$

Any extremal to the problem must be a nontrivial solution to equation (5.1) because of the isoperimetric condition; consequently, λ must be an eigenvalue. By property (a) there must be a least element in the spectrum, the first eigenvalue λ_1, and a corresponding eigenfunction y_1 normalized to meet the isoperimetric condition. Hence the minimum value for J is λ_1 and $J(y_1) = \lambda_1$.

□

Eigenvalues for the Sturm-Liouville problem signal a bifurcation: in a deleted neighbourhood of an eigenvalue there is only the trivial solution available; at an eigenvalue there are nontrivial solutions (multiples of the eigenfunction) available in addition to the trivial solution. In applications such as those involving the stability of elastic bodies, eigenvalues indicate potential abrupt changes. Often the most vital piece of information in a model is the location of the first eigenvalue. For example, an engineer may wish to design a column so that the first eigenvalue in the problem modelling the deflection of the column is sufficiently high that it will not be attained under normal loadings.[3]

Theorem 5.2.1 suggests a characterization of the first eigenvalue in terms of the functionals J and I. Let R be the functional defined by

$$R(y) = \frac{J(y)}{I(y)}. \tag{5.13}$$

The functional R is called the **Rayleigh quotient** for the Sturm-Liouville problem. If $I(y) = 1$, then for any nontrivial solution y we have

$$\lambda = R(y). \tag{5.14}$$

We can, however, drop this normalization restriction on I because both J and I are homogeneous quadratic functions in y and y' so that any normalization factors cancel out in the quotient. Relation (5.14) is thus valid for any nontrivial solution, and we can make use of this observation to characterize the first eigenvalue as the minimum of the Rayleigh quotient.

Theorem 5.2.2 *Let S' denote the set of all functions in $C^2[x_0, x_1]$ that satisfy the boundary conditions* (5.4) *except the trivial solution $y \equiv 0$. The minimum of the Rayleigh quotient R for the Sturm-Liouville problem* (5.1), (5.4) *over all functions in S' is the first eigenvalue; i.e.,*

$$\min_{y \in S'} R(y) = \lambda_1. \tag{5.15}$$

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[3] The governing differential equation for this model is in fact of fourth order, but similar comments apply. The variational formulation of this model is discussed in Courant and Hilbert, *op. cit.*, p. 272.

Proof: Suppose that R has minimum value Λ at $y \in S'$, and let

$$\hat{y} = y + \epsilon\eta,$$

where ϵ is small and η is a smooth function such that $\eta(x_0) = \eta(x_1) = 0$, to ensure that $\hat{y} \in S'$. Now,

$$I(y + \epsilon\eta) = I(y) + 2\epsilon \int_{x_0}^{x_1} \eta r y \, dx + O(\epsilon^2),$$

so that

$$\frac{1}{I(\hat{y})} = \frac{1}{I(y)} + O(\epsilon),$$

where $I(y) \neq 0$, and

$$\begin{aligned} J(\hat{y}) &= J(y) + 2\epsilon \int_{x_0}^{x_1} \eta \left((-py')' + qy\right) \, dx + O(\epsilon^2) \\ &= \Lambda I(y) + 2\epsilon \int_{x_0}^{x_1} \eta \left((-py')' + qy\right) \, dx + O(\epsilon^2), \end{aligned}$$

so that

$$\begin{aligned} J(\hat{y}) - \Lambda I(y) = 2\epsilon \int_{x_0}^{x_1} \eta \left((-py')' + qy - \Lambda r y\right) \, dx \\ + O(\epsilon^2). \end{aligned}$$

We thus have

$$\begin{aligned} R(\hat{y}) - R(y) &= \frac{J(\hat{y})}{I(\hat{y})} - \frac{J(y)}{I(y)} = \frac{J(\hat{y}) - \Lambda I(\hat{y})}{I(\hat{y})} \\ &= 2\epsilon \frac{\int_{x_0}^{x_1} \eta \left((-py')' + qy - \Lambda r y\right) \, dx}{I(y)} + O(\epsilon^2), \end{aligned}$$

and since R is minimum at y, the terms of order ϵ must vanish in the above expression for arbitrary η. We can apply Lemma 2.2.2 to the numerator of the order ϵ term and deduce that y must satisfy equation (5.1). Since $y \in S'$, the constant Λ must be an eigenvalue. Any extremal for R must therefore be a nontrivial solution to the Sturm-Liouville problem.

If λ_m is an eigenvalue for the problem and y_m is a corresponding eigenfunction, then the calculation in the proof of Theorem 5.2.1 can be used to show that

$$R(y_m) = \frac{J(y_m)}{I(y_m)} = \lambda_m.$$

Since R is minimum at y and Λ is the corresponding eigenvalue we have that

$$\lambda_m = R(y_m) \geq \Lambda$$

for all eigenvalues. Therefore we have $\Lambda = \lambda_1$. □

Generally, the eigenvalues (and hence eigenfunctions) for a Sturm-Liouville problem cannot be determined explicitly. Bounds for the first eigenvalue, however, can be obtained using the Rayleigh quotient. Upper bounds for λ_1 can be readily obtained since λ_1 is a minimum value: for any function $\phi \in S'$ we have

$$R(\phi) \geq \lambda_1, \tag{5.16}$$

so that an upper bound can be derived by using any function in S'. Lower bounds require a bit more work.

To get a lower bound, the strategy is to construct a comparison problem that can be solved explicitly, the first eigenvalue $\bar{\lambda}_1$ of which is guaranteed to be no greater than λ_1. To construct a comparison problem, we make the following simple observations.

(a) Let $\bar{p} \in C^1[x_0, x_1]$ be any function such that $p(x) \geq \bar{p}(x) > 0$ for all $x \in [x_0, x_1]$, and let $\bar{q} \in C^0[x_0, x_1]$ be any function such that $q(x) \geq \bar{q}(x)$ for all $x \in [x_0, x_1]$. Then, for

$$\bar{J}(y) = \int_{x_0}^{x_1} \left(\bar{p}y'^2 + \bar{q}y^2\right)\, dx,$$

we have

$$\bar{J}(y) \leq J(y).$$

(b) Let $\bar{r} \in C^0[x_0, x_1]$ be any function such that $\bar{r}(x) \geq r(x) > 0$ for all $x \in [x_0, x_1]$. Then, for

$$\bar{I}(y) = \int_{x_0}^{x_1} \bar{r}y^2\, dx$$

we have

$$\bar{I}(y) \geq I(y).$$

If we choose $\bar{p}$, $\bar{q}$, and $\bar{r}$ as above, then

$$\bar{R}(y) = \frac{\bar{J}(y)}{\bar{I}(y)} \leq \frac{J(y)}{I(y)} = R(y),$$

and hence

$$\bar{\lambda}_1 \leq \lambda_1. \tag{5.17}$$

Inequality (5.17) is useful only if we can determine $\bar{\lambda}_1$ explicitly. We have considerable freedom, however, in our choices for $\bar{p}$, $\bar{q}$, and $\bar{r}$, and the simplest choice is when these functions are constants; i.e.,

$$\begin{aligned}
\bar{p}(x) &= \min_{x \in [x_0, x_1]} p(x) \equiv p_m, \\
\bar{q}(x) &= \min_{x \in [x_0, x_1]} q(x) \equiv q_m, \\
\bar{r}(x) &= \max_{x \in [x_0, x_1]} r(x) \equiv r_M.
\end{aligned}$$

For this choice, the differential equation is

$$(-p_m y')' + q_m y - \bar{\lambda} r_M y = 0;$$

i.e.,

$$y'' + \frac{1}{p_m}\left(\bar{\lambda} r_M - q_m\right) y = 0. \tag{5.18}$$

The solution of equation (5.18) subject to boundary conditions (5.4) follows essentially along the same lines as that given in Example 5.1.1. The eigenvalues for this problem are

$$\bar{\lambda}_n = \frac{1}{r_M}\left(\frac{p_m n^2 \pi^2}{(x_0 - x_1)^2} + q_m\right).$$

We thus get the lower bound

$$\bar{\lambda}_1 = \frac{1}{r_M}\left(\frac{p_m \pi^2}{(x_0 - x_1)^2} + q_m\right) \leq \lambda_1. \tag{5.19}$$

Example 5.2.1: Mathieu's Equation
Let $p(x) = r(x) = 1$, and $q(x) = 2\theta \cos(2x)$, where $\theta \in \mathbb{R}$ is a constant. Let $x_0 = 0$ and $x_1 = \pi$. For this choice of functions equation (5.1) is equivalent to

$$y'' + (\lambda - 2\theta \cos(2x))\, y = 0, \tag{5.20}$$

and the boundary conditions are

$$y(0) = y(\pi) = 0. \tag{5.21}$$

The expression (5.20) is called **Mathieu's equation**, and its solutions have been investigated in depth (cf. McLachlan [52] and Whittaker and Watson [74]). If $\theta = 0$, then the problem reduces to that studied in Example 5.1.1. If $\theta \neq 0$, then the nontrivial solutions to this problem cannot be expressed in closed form in terms of elementary functions. Indeed, this problem defines a new class of functions $\{se_n\}$ called **Mathieu functions**,[4] that correspond to the eigenfunctions of the problem. The determination of the eigenvalues for this problem is a more complicated affair compared to the simple problem of Example 5.1.1. Briefly, it can be shown that the first eigenvalue λ_1 and the corresponding eigenfunction se_1 are given asymptotically by

$$\lambda_1 = 1 - \theta - \frac{1}{8}\theta^2 + \frac{1}{64}\theta^3 - \frac{1}{1536}\theta^4 - \frac{11}{36864}\theta^5 + O(\theta^6), \tag{5.22}$$

and

[4] The notation se_n is an abbreviation for "sine-elliptic." There are also "cosine-elliptic" Mathieu functions ce_n.

$$
\begin{aligned}
se_1(x) = \sin(x) - \frac{1}{8}\theta\sin(3x) + \frac{1}{64}\theta^2\left(\sin(3x) + \frac{1}{3}\sin(5x)\right)\\
- \frac{1}{512}\theta^3\left(\frac{1}{3}\sin(3x) + \frac{4}{9}\sin(5x) + \frac{1}{18}\sin(7x)\right)\\
+ O(\theta^4),
\end{aligned}
$$

for $|\theta|$ small (cf. McLachlan, *op. cit.* p. 10–14).

In contrast, a rough lower bound for λ_1 can be readily gleaned from inequality (5.19). Suppose that $\theta \geq 0$, and let $p_m = r_M = 1$, $q_m = -2\theta \leq 2\theta\cos(2x)$. Inequality (5.19) then implies

$$1 - 2\theta \leq \lambda_1. \tag{5.23}$$

Given the asymptotic expression (5.22), if $\theta \geq 0$ is small then the lower bound (5.23) can be verified directly. But inequality (5.23) is also valid for θ large, and this is not so obvious.

Note that if $\theta < 0$, we cannot use $q_m = 2\theta$ in our comparison problem since $-2\theta \geq 2\theta\cos(2x)$ for $x \in [x_0, x_1]$. For this case we can use $q_m = 2\theta$ and thus get the lower bound

$$1 + 2\theta \leq \lambda_1.$$

Exercises 5.2:

1. Mathieu's equation (5.20) can have a first eigenvalue λ_1 that is negative depending on the constant θ. Write out the Rayleigh quotient for Mathieu's equation. Now, $\phi = \sin(x)$ is in the space S'. Use this function and inequality (5.16) to get an upper bound for λ_1, and show that $\lambda_1 < 0$ whenever $\theta > 1$. Compare this with expression (5.22). (For the choice $\theta = 5$ the value of λ_1 is given in table 5.1 at the end of Section 5.3.)
2. **Halm's equation** is

$$(1 + x^2)^2 y''(x) + \lambda y(x) = 0.$$

Under the boundary conditions $y(0) = y(\pi) = 0$, find a lower bound for λ_1.
3. The **Titchmarsh equation** is

$$y''(x) + (\lambda - x^{2n})y(x) = 0,$$

where n is a nonnegative integer. Under the boundary conditions $y(0) = y(1) = 0$ show that the first eigenvalue λ_1 satisfies $\pi^2 < \lambda_1 < 11$. (The function $\phi = x(x - 1)$ can be used to get the upper bound.)

5.3 Higher Eigenvalues

The Rayleigh quotient can be used to frame a variational characterization of higher eigenvalues. The eigenfunctions for the Sturm-Liouville problem are mutually orthogonal, and this relationship can be exploited to give such a characterization. For example, it can be shown that the eigenvalue λ_2 corresponds to the minimum of R among functions in $y \in S'$ that also satisfy the orthogonality condition

$$\langle y, y_1 \rangle = 0,$$

where y_1 is an eigenfunction corresponding to λ_1. More generally, we have the following result the proof of which we omit.

Theorem 5.3.1 *Let y_k denote the eigenfunction associated with the eigenvalue λ_k, and let S'_n be the set of functions $y \in S'$ such that*

$$\langle y, y_k \rangle = 0 \tag{5.24}$$

for $k = 1, 2, \ldots, n-1$. Then

$$\lambda_n = \min_{y \in S'_n} R(y). \tag{5.25}$$

The above theorem is of limited practical value because, in general, the eigenvalues $\lambda_1, \ldots, \lambda_{n-1}$ and corresponding eigenfunction $y_1, \ldots, y_{n-1}$ are not known explicitly. Constraints such as (5.24) require precise knowledge of the eigenfunctions as opposed to approximations. Fortunately, we can characterize higher eigenvalues with a "max-min" type principle involving the Rayleigh quotient, and circumvent the problem of finding eigenfunctions. The next results we state without proof. Some details can be found in Wan *op. cit.*, p. 284, and in Courant and Hilbert *op. cit.*, p. 406.

Lemma 5.3.2 *Let $z_1, \ldots, z_{n-1}$ be any functions in S' and let $\bar{\lambda}_n$ be the minimum of R subject to the $n-1$ constraints*

$$\langle y, z_k \rangle = 0,$$

where $k = 1, \ldots n-1$. Then

$$\bar{\lambda}_n \leq \lambda_n.$$

Lemma 5.3.2 is a key result used to establish the following "max-min" principle for higher eigenvalues.

Theorem 5.3.3 *Let Ω_{n-1} be the set of all functions $\mathbf{z} = (z_1, \ldots, z_{n-1})$ such that $z_k \in S'$ for $k = 1, \ldots, n-1$. Then*

$$\lambda_n = \max_{\mathbf{z} \in \Omega_{n-1}} \{\bar{\lambda}_n(\mathbf{z})\},$$

where

$$\bar{\lambda}_n(\mathbf{z}) = \min_{y \in S'} \{R(y) : \langle y, z_k \rangle = 0, \quad k = 1, \ldots, n-1\}.$$

The "max-min" property of eigenvalues can be exploited to get a simple asymptotic estimate of the eigenvalues λ_n as $n \to \infty$. Note that the problem

$$(-py')' + qy - \lambda r y = 0,$$

$$y(0) = y(\pi) = 0,$$

can be converted into the problem

$$\phi''(t) - f(t)\phi(t) + \lambda\phi(t) = 0, \tag{5.26}$$

$$\phi(0) = \phi(\ell) = 0, \tag{5.27}$$

by the transformation

$$\phi = \sqrt[4]{rp}\,y, \quad t = \int_0^x \sqrt{\frac{r(\xi)}{p(\xi)}}\, d\xi, \quad \ell = \int_0^\pi \sqrt{\frac{r(x)}{p(x)}}\, dx.$$

Here, the function f is given by

$$f = \frac{g''}{g} + \frac{q}{r},$$

where $g = \sqrt[4]{rp}$. We can thus restrict our attention to the problem (5.26), (5.27).[5] The Rayleigh quotient for this problem is

$$R(\phi) = \frac{J(\phi)}{I(\phi)},$$

where

$$J(\phi) = \int_0^\ell \left(\phi'^2 + f(t)\phi^2\right)\, dt$$

and

$$I(\phi) = \int_0^\ell \phi^2\, dt.$$

Let

$$k = \max_{t \in [0,\ell]} |f(t)|, \tag{5.28}$$

and

[5] This formulation is called the **Liouville normal form** of the problem. Details on this transformation and extensions to more general intervals can be found in Birkhoff and Rota [9], p. 320.

$$J^+(\phi) = \int_0^\ell \left(\phi'^2 + k\phi^2\right) dt,$$
$$R^+(\phi) = \frac{J^+(\phi)}{I(\phi)},$$
$$J^-(\phi) = \int_0^\ell \left(\phi'^2 - k\phi^2\right) dt,$$
$$R^-(\phi) = \frac{J^-(\phi)}{I(\phi)}.$$

Then,

$$R^+(\phi) = \frac{\int_0^\ell \phi'^2 \, dt}{I(\phi)} + k,$$
$$R^-(\phi) = \frac{\int_0^\ell \phi'^2 \, dt}{I(\phi)} - k,$$

and, since $J^+(\phi) \geq J(\phi) \geq J^-(\phi)$,

$$R^+(\phi) \geq R(\phi) \geq R^-(\phi);$$

i.e.,

$$|\bar{R}(\phi) - R(\phi)| \leq k, \tag{5.29}$$

where

$$\bar{R}(\phi) = \frac{\int_0^\ell \phi'^2 \, dt}{I(\phi)}. \tag{5.30}$$

The Rayleigh quotient defined by equation (5.30) is associated with the Sturm-Liouville problem

$$\phi'' + \bar{\lambda}\phi = 0, \tag{5.31}$$
$$\phi(0) = \phi(\ell) = 0, \tag{5.32}$$

and the eigenvalues for this problem are given by

$$\bar{\lambda}_n = \frac{n^2\pi^2}{\ell^2}. \tag{5.33}$$

Inequality (5.29) indicates that $R(\phi)$ can differ from $\bar{R}(\phi)$ by no more than $\pm k$. By the "max-min" principle for higher eigenvalues we see that λ_n and $\bar{\lambda}_n$ can differ by no more than $\pm k$ and thus deduce the asymptotic relation

$$\lambda_n = \frac{n^2\pi^2}{\ell^2} + O(1), \tag{5.34}$$

as $n \to \infty$. The function f influences only the $O(1)$ term (a term that is bounded as $n \to \infty$); λ_n is approximately $n^2\pi^2/\ell^2$ for large values of n. If we return back to the original problem, the relation (5.34) can be recast as

n	n^2	λ_n
1	1	−5.790
2	4	2.099
3	9	9.236
4	16	16.648
5	25	25.511
6	36	36.359

Table 5.1. Eigenvalues for Mathieu's equation, $\theta = 5$

$$\lambda_n = n^2\pi^2 \left\{ \int_0^\pi \sqrt{\frac{r(x)}{p(x)}}\, dx \right\}^{-2} + O(1); \tag{5.35}$$

i.e.,

$$\lim_{n\to\infty} \frac{n^2}{\lambda_n} = \frac{1}{\pi^2} \left\{ \int_0^\pi \sqrt{\frac{r(x)}{p(x)}}\, dx \right\}^{2}. \tag{5.36}$$

Note that q does not influence the leading order behaviour for the asymptotic distribution of eigenvalues.

Equation (5.35), for example, predicts that the higher eigenvalues for Mathieu's equation (Example 5.2.1) are

$$\lambda_n = n^2 + O(1), \tag{5.37}$$

as $n \to \infty$. In fact, the approximation is not "too bad" for θ small even with n small (cf. Table 5.1).

In closing, we note that the results of this chapter can be extended for the general Sturm-Liouville boundary conditions (5.2). Some extensions can also be made to cope with singular Sturm-Liouville problems. The reader is directed to Courant and Hilbert, *op. cit.* Chapter 5, for a fuller discussion and a wealth of examples from mathematical physics.

Exercises 5.3:

1. For Mathieu's equation (5.20) show that $|\lambda_n - n^2| \le 2\theta$ for all n.
2. Determine a constant Λ such that for Halm's equation (Exercise 5.2-2)

$$\lambda_n = \Lambda n^2 + O(1).$$

Derive a number M such that $|\lambda_n - \Lambda n^2| \le M$ for all n.