Problem 1: ODE ® State Equations

Write the state equations from the following ODE:

$$\ddot{y}(t)+d\cdot\dot{y}^3(t)+a\cdot\sin\big(y(t)\big)=K\cdot u(t)$$

Problem 2: ODE ® Block Diagram

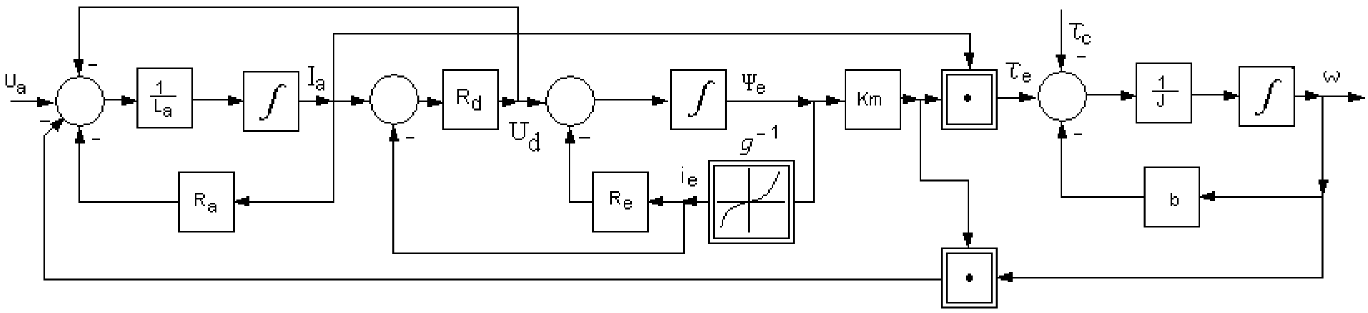
Draw a block diagram from the following ODE:

$$\ddot{z}(t)+a\cdot\dot{z}(t)\cdot\left|\dot{z}(t)\right|+\left\{z^3(t)-z(t)\right\}=g\cdot u(t)+r\cdot\dot{u}(t)$$

(define your own symbol for a non-linear operation if you do not remember a standard one)

Problem 3: Block Diagram ® State Equations

The following is a block diagram of a Series Excited DC-Motor SEDCM with a field-weakening connection (this is **just a comment** for your information; it is **not necessary** to consider it to the aim of this examination !).



The function $\boldsymbol{g}^{-1}$ is the inverse of a nonlinear function $\boldsymbol{g}$ (which represents the magnetic characteristics of the excitation field). In the sequel, use $\boldsymbol{g}^{-1}$ as a symbol for a known function. The other two double-squared blocks are multipliers.

3.1 Identify **all** the state variables: $x_1 =$, $x_2 =$, …

3.2 Write-down **at least one state equation**.

3.3 Write-down **at least one output equation. Output variables** to be considered:

$y_1 = t_e$ (the electromagnetic torque)

$y_2 = U_d$ (the voltage drop on resistor Rd).

Problem 4: State Equations

The following “Volterra system” is an alternative to the “Lotka-Volterrra model”: the time-derivative of each population depends on its present size, on the number of encounters with the other population, and on the past history of the other population (this justifies the terms with the time-integral).

$$\frac{dN_1(t)}{dt}=a.N_1(t)-b.N_1(t).N_2(t)-K_1.N_1(t).\int_0^t N_2(n)dn$$

$$\frac{dN_2(t)}{dt}=-c.N_2(t)+d.N_1(t).N_2(t)+K_2.N_2(t).\int_0^t N_1(n)dn$$

4.1 Identify which variable N_1 or N_2 represents the predator and which the prey population.

4.2 The Volterra Model is an Integro-Differential equation system. Introducing a new pair of variables $H_1(t)$ and $H_2(t)$ the system can be written as a set of four first order differential equations. Define conveniently the new pair of variables $H_1(t)$ and $H_2(t)$ and complete accordingly the differential equations below:

$$\frac{dN_1(t)}{dt} = \quad . \ . \ .$$
$$\frac{dN_2(t)}{dt} = \quad . \ . \ .$$
$$\frac{dH_1(t)}{dt} = \quad . \ . \ .$$
$$\frac{dH_2(t)}{dt} = \quad . \ . \ .$$

Hint: the problem is very easy, but if you don´t realize how to solve it, maybe you can find the way by first drawing a block diagram of the integro-differential system and then writing the state equations from the block diagram.

Problem 5: State Equations ® Matrix Form

Put the following sets of state equations into the matrix form $\dot{x}(t) = A \cdot x(t) + B \cdot u(t)$

$$\dot{w}(t) \quad = \frac{K_m}{J} I_a(t) - \frac{k_T}{J} q(t) - \frac{b}{J} w(t)$$
$$\dot{I}_a(t) = -\frac{R_a}{L_a} I_a(t) + \frac{1}{L_a} U_a(t) - \frac{K_m}{L_a} w(t)$$
$$\dot{q}(t) = w(t)$$

Problem 6: Model Classification – Operational Notation – Transfer Functions

6.1 Classify the following models in **Linear** (**L**) / **Non-Linear** (**NL**); **Time-Invariant** (**TI**) / **Time-Variant** (**TV**):

No indication of time dependence means a constant (in the following examples, T_1 and K are constants).

6.1.0.i	*Example i:*	$T_1 \ \dot{y}(t) + y(t) = K \ u(t)$	**LTI**
6.1.0.ii	*Example ii:*	$\ddot{y}(t) + t \cdot y(t) = K \ u(t)$	**L<u>TV</u>(*<u>t</u>* x *y*)**
6.1.0.iii	*Example iii:*	$\ddot{y}(t) + t \cdot \dot{y}^2(t) = K \cdot y(t) \cdot u(t)$	**NL(** *y-dot-<u>square</u>; <u>y x u</u>***)** **TV(** *<u>t</u>* x *y-dot-square***)**

In each case justify your classification in terms of the model particularities. For instance, *Example iii* is **NL** because of the presence of the square of *y-dot* ($\dot{y}^2(t)$) in the first member and of the product *y-times-u* ($y(t) \cdot u(t)$) in the second member, and is **TV** because of the coefficient “*t*” ($\underline{t} \cdot \dot{y}^2$) in the second term of the left-hand side.

Problem 6.1

	Problem	Solution
6.1.1	$\ddot{y}(t) + a_1\ \dot{y}(t) + a_2\ y(t) = b_0\ u(t) + b_1\ \dot{u}(t)$	
6.1.2	$\ddot{y}(t) + a_1\ (1 - y(t)) \cdot \dot{y}(t) + a_2\ y(t) = b_0\ u(t)$	
6.1.3	$\ddot{y}(t) + a_1\ (1 - \sin(w \cdot t)) \cdot \dot{y}(t) + a_2\ y(t) = b_0\ u(t)$	
6.1.4	$\ddot{y}(t) + e^{-t} \cdot \dot{y}(t) + a_2\ \sin(y(t)) = b_0\ u(t)$	
6.1.5	$\ddot{y}(t) + a_1\ \dot{y}(t) + a_2\ y(t) + a_3 \int y(t)\, dt = K \cdot u(t)$	
6.1.6	$\ddot{y}(t) + a_1\ \dot{y}(t) + a_2\ y(t) = 0$	

6.2 Using the operational notations $\boxed{s \equiv \frac{d}{dt}}$ and $\boxed{\frac{1}{s} \equiv \int\ dt}$ and their natural extensions to higher order derivatives and time-integrals, **write the transfer functions G(s) corresponding to the models in the previous problem, when the TFs do exist!** For each of the six models, clearly explain (in terms of the models particularities) why it is possibly / non-possibly to write the corresponding transfer function.

<u>Problem 7: Numerical Approximations</u>

Recall that for the general system $\boxed{\frac{d\ x(t)}{d\ t} = f\ (\ x(t), u(t), t)}$

the one-step approximation to the exact solution delivered by the forward-Euler technique is

$\boxed{x_{k+1} = x_k\ + h\ .\ f\ (x_k\ , u_k\ , t_k)}$

and that the one-step approximation to the exact solution delivered by the backward-Euler technique is

$\boxed{x_{k+1} = x_k\ + h\ .\ f\ (x_{k+1}\ , u_{k+1}\ , t_{k+1})}$

(in both cases a *constant integration step h* was assumed)

7.1 Consider the free (no input) system $T_1\ \dot{y}(t) + y(t) = 0$ with the initial condition $y(0) = y_0$. The exact solution to this Cauchy problem is known to be $y(t) = y_0 \cdot e^{-t/T_1}$, which exponentially converges to zero when time goes to infinity.

7.1.1 ***i)*** Obtain the one-step approximation applying forward-Euler. ***ii)*** Then, obtain the expression for y_k as function (not of the previous value but) of y_0. ***iii)*** Using this latter expression, find the set of values of the step $h > 0$ which assure the stability of the approximation (convergence to zero of the approximating sequence $\{y_k, k = 1, 2, 3, ...\}$ as $k \rightarrow$ ¥).

7.1.2 ***i)*** Obtain the one-step approximation applying backward-Euler. ***ii)*** Then, obtain the expression for y_k as function (not of the previous value but) of y_0. ***iii)*** Using this latter expression, find the set of values of the step $h > 0$ which assure the stability of the approximation.

7.1.3 Compare both results.

7.2 For the following system:

$$\dot{y}(t) = a.y(t) - b.y^2(t) - c.y(t).u(t)$$ where a, b, c are positive constants.

7.2.1 Obtain the one-step approximation resulting from backward-Euler method.

7.2.2 Put the previous result in the standard form for a second-degree algebraic equation: $\alpha \cdot y_{k+1}^2 + \beta \cdot y_{k+1} + g = 0$. This is the implicit algebraic equation to be solved at each interation step of the numerical integration algorithm. Clearly **indicate** the dependence of the parameters α, β, g on the values of the differential equation coefficients, on the integration step h, and on the corresponding values of y and of u !