# DYNAMICAL SYSTEMS: Basic Concepts and Mathematical Models

## SYSTEM

The concept of ***system*** can be defined in many different ways. At first it can be said that ***a system is an object or collection of objects whose properties we want to study***.

***Examples:*** A greenhouse, an electrical machine, a production process (e.g., paper production, petrochemical industry, champagne production, pharmaceutical industry, etc.), the solar system, a power plant, the european interconnected electrical power system, the european (macro)economic market, the (micro)economy of an enterprise, a computing center, a town's or country's public transportation system, internet, an enterprise intranet, etc.

### Some useful definitions

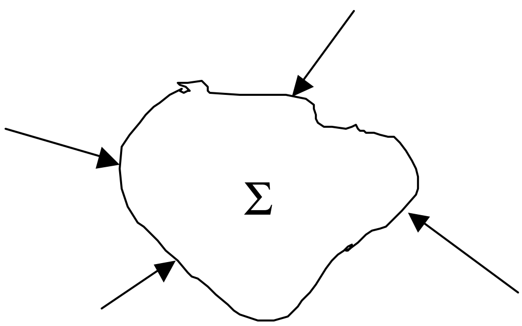
- ***SYSTEM*** **(**After **DIN 66201,** Deutsches Institut für Normung**):**

**A System (S) is a bounded array of interacting entities.**
***(1) bounded, (3) array, (2) interacting entities***

**Un Système (S) c'est une disposition delimitée d'entitées qui interagissent.**
***(3) disposition, (1) delimitée, (2) entitées qui interagissent***

We distinguish three terms in this definition:

***(1)*** ***Delimitation*** (spatial, conceptual) regarding the rest of the Universe. The elements of the rest of the Universe having relevant influence on the System are replaced by fictitious elements of equivalent action and incorporated to the System.



**(2)** ***Interacting Entities***, or System Components: ***Elements, processes, (sub)systems***.

- ***Process (DIN 66201:*** Transformation and/or transport of matter, energy, information, and/or any kind of (generalized) "goods".

***(3)*** ***Relative arrangement*** (of the components): It defines the ***System Structure***. Even having identical components two systems are different if their components are arranged according to different structures.

Taking for granted the delimitation from the rest of the Universe, an equivalent definition of system would be as follows:

***A System $S$ is an entity built-up by a set of components and a structure***:

$$S = \{Components, Structure\}$$

This definition can be specialized to different domains of Science, Technology, etc. , through the characterisation of the components involved in the system. For instance, a physical system could be defined as follows:

***PHYSICAL SYSTEM ($FS$)***: Is any system where the interaction involves exchange of ***matter and/or energy and/or information***.

***DYNAMICAL PHYSICAL SYSTEM ($DFS$)***: Is a $FS$ where the ***storage*** of matter and/or energy and/or information is taken into account (by the person defining the system).

This distinction is not always desirable or possible, then it is well known that very often systems are to be described using mixed domains of knowledge. For instance, describing physiological problems in living systems usually involves phenomenological laws of biology and some physical/chemical laws (heat/matter exchange, kinetics of reaction processes, mechanics, and so on).

***DYNAMICAL SYSTEM:*** Even if originated in Physics, the concept of ***Dynamical System*** has been generalized to any kind of ***systems*** where ***storage phenomena*** are taken into account.

# METHODOLOGY FOR SYSTEM ANALYSIS

## EXPERIMENTAL ANALYSIS

Science has developed methods useful to research open questions asked by humans about systems and their properties. Scientific methods rely on ***experimentation***, which consists in ***performing trials**** on the system and ***observing* its reactions** in order to ***obtain laws*** explaining the system behavior. As far as we are concerned here, these laws have mostly the form of mathematical expressions, in a wide sense.

The experimental method is not always feasible, for instance because of one or many of the following reasons:

- **Economical Cost:** For an industrial process some experiments could imply bringing the production to standstill during some time, which could be economically unacceptable in certain cases.
- **Risks:** The consequences of the experiment could be undesirable or intolerable. (*Examples:* Certain experiments in nuclear plants like trying to test "¿what would be the effect of a leak into the atmosfere of some amount of radioactive steam? "; or trying to answer "how would it impact on an ecosystem the extinction of a species?").
- **Impossibility of performing the experiment**
  - **System does not exist:** Typically, at the early stages of a new system design it is most important to know how some parameters could affect the overall system behavior, even before constructing some physical prototype. (*Ex.:* "¿what could be the aerodynamic performance of a new plane with a new type of wing profile?"; "¿what is the incidence of a new car suspension on the passanger's comfort?").
  - **Human inability to perform the experiment:** Even if technology allows the human breed to master many natural phenomena, (fortunately in some cases!) many of them remain still irreductible. (*Ex.*: "¿what would be the effect of a displacement of the magnetic axis of Earth?").

* Sometimes the trial limits itself to collecting some instrument data, without exerting any action on the system. This is the usual way in which Astronomy proceeds. The same procedure is followed for instance when monitoring technical systems through measurements during their normal operation.

When performing experiments is not possible or just inconvenient the solution is ***modeling***.

## *MODELS*

A ***model*** of a system is basically a tool which allows finding answers to question on the system without the need of conducting experiments on it. A ***model*** is always a simplified representation of reality (for an already existent $\boldsymbol{FS}$), or of a conceptual prototype (just a project of a new $\boldsymbol{FS}$).

## *CLASSIFICATIONS OF MODELS*

- **Physical Models**

  These are physical reproductions of the original systems at (usually) reduced scale. The results obtained through experimenting on the model are transferred to the original system thanks to the theory of Dimensional Analysis. (*Exs*.: wind-tunnels to study aerodynamic phenomena in airplanes; scale modeling of a river-bed in order to study hydrological phenomena, and so on).

- **Abstract Models**
  - **Mental Models:** (unconscious) image of a process behavior (*Ex.:* Everybody, even people without any knowledge of Physics, can manage its own body; particularlily athletes do it with high skills)
  - **Verbal/Textual Models:** describing constitution or behavior. (*Ex.:* operating instructions or functional description of a machine).
  - **Technical Models:** Most commonly given as plans, graphs, etc., they represent with specific symbology the constitution of man-made systems (Ex.: the scheme of the air conditioning system of a building; the diagram for constructing a printed-circuit board; etc)
  - **Mathematical Models:** hard models (commonly using number-valued or logical variables), soft/fuzzy (commonly using linguistic variables).

## *MATHEMATICAL MODELS: MM*

***MM are constituted by mathematical expressions describing the relationships among variables characterizing a system.***

Expressions of the following type can be found:

- **Systems of equations**
- **Inequalities**
- **Logic-mathematical expressions**
- **Linguistic expressions**

These expressions relate variables representing ***signals*** in the system.

- ***SIGNAL*** **(DIN 66201)**: Is the representation of an *information* through the values of a physical/system magnitude.

## *CLASSIFICATION OF MM*

Very often the talk is about *systems*, while in fact *models* are meant.

### *Continuous Time vs Discrete Time Systems*

In a *continuous time model* time as a variable is associated to (a subset of) the real numbers.

In a *discrete time model* time is associated to (a subset of) the natural numbers. (Ex.: sampled systems; numerical representation of a continuous time systems, etc.).

### *Continuous Systems*

*Continuous Systems* are *Continuous Time Systems* where the descriptive variables are represented by (piece-wise) *continuous* functions.

### *Discrete Event Systems*

*Discrete Event Systems* are *Continuous Time Systems* where the descriptive variables being themselves piece-wise *constant* functions have been represented by events.

### *Static vs Dynamical Systems*

If there are only ***instantaneous relationships*** among the descriptive variables of a system (*a model !*), then it is said to be a static model (*Ex*.: equations with only algebraic or trascendental expressions, or functions in general). In the paragraph on ***Causality*** this concept will be better explained.

If ***the relationships among the variables do require*** (not only their current value, i.e., the value at a given time instant but also) ***previous values***, then the model is said to be a ***dynamical model*** (*Ex.*: differential or difference equations having ***time*** as the fundamental ***variable*** –for details please see upcoming paragraph on ***Clasification of Variables***).

### *Deterministic vs Stochastic Models*

A model is said to be ***deterministic*** if it expresses without mathematical uncertainty the relationships among the variables. The information processed by the model is mapped to precisely determined values and/or functions.

A model is said to be ***stochastic*** if it expresses mathematical uncertainty in the relationships among the variables through probabilistic concepts using random variables.

## *Distributed vs Lumped Parameter Systems*

System magnitudes take values in time and space.

If a MM uses time-space functions to represent the system descriptive variables, then it is said to be a ***distributed parameter model***, because usually they use physical coefficients or parameters which are distributed on space (*Ex.:* the density of a fluid in gas pipeline; resistivity, inductivity and capacity per unit length in an electrical transmission line). The associated dynamic models typically are systems of *partial derivative equations*.

A ***lumped parameter model*** is obtained when replacing the space dependency of the variables using their average value in the space domain of their definition. Space disappears of the model, it is no longer present as one of the ***fundamental variables, only time is left*** as such. *System parameters are* now extensive model quantities, they are *lumped or concentrated on the space domain*. The associated dynamic models typically involve systems of ordinary differential equations.

## *Parametric and Nonparametric Systems*

***Parametric MM*** are fully specified or defined with a finite set of parameters (*Ex*.: a transfer function; an ordinary differential equation).

***Nonparametric MM*** cannot be fully specified with a finite set of parameters (*Ex*.: the step response of a linear time-invariant system; the frequency response of an amplifier or of a magnetic tape).

## *Linear vs Nonlinear Systems*

In ***Linear MM*** the superposition principle is valid, i.e., to a linear combination of causes (inputs and/or initial conditions for instance) the model reacts with the same linear combination of the original effects.

In ***Nonlinear MM*** the superposition principle is not valid.

## *Time Invariant vs Time Variant Systems*

A MM is time invariant if to time-shifted inputs time-shifted outputs do correspond.

## ***USING MM***

MM are built to study system properties and to predict its behavior under different operating conditions. Conceptually, two main groups of techniques devised to achieve these objectives can be distinguished:

- ***Theoretical Analysis of MM*** : Mathematical methods of qualitative (stability and other system properties, etc.) and quantitative (equation solving, etc.) analysis.

- ***Experimental Analysis of MM:*** Studying quantitative and qualitative properties of the system through experimenting with the model on computers. This is called ***Simulation or Experimental Mathematics.***

***SIMULATION: Digital / Analog / Hybrid***

***SIMULATION (general):*** Investigation of the system behavior conducting experiments on another system replacing and representing the first one.

***DIGITAL SIMULATION*** involves:

- Discrete Representation of Continuous Variables
- Function Approximation
- Numerical Methods
- Numerical (finite-length arithmetic calculation errors) errors.

## ***MODELING OR CONSTRUCTING MM***

There are two conceptually different techniques, which commonly are complementary used in practical applications:

- ***ANALYTICAL OR "PHYSICAL" MODELING***
- ***"EXPERIMENTAL" MODELING OR SYSTEM IDENTIFICATION***

***ANALYTICAL OR "PHYSICAL" MODELING*** is based on the knowledge of first principles (physical/systemic laws governing phenomena in the system) and of system constitution. ***Constitutive Relationships*** are associated to ***system components***; ***Structural Relationships*** are associated to ***system structure***. Building the system model consists in organizing all these separate mathematical expressions in a consistent mathematical system.

***"EXPERIMENTAL" MODELING OR SYSTEM IDENTIFICATION*** deals with the obtention of mathematical models and their parameterization using measurement data resulting from experiments conducted on the system *(you can find useful information on System Identification at http://www.eie.fceia.unr.edu.ar/~lsd/ )*

*For a concise and practical introduction to the methods of analytical modeling and identification see for instance the book by Lennart and Glad[*] .*

[*] Ljung, Lennart & Torkel Glad, *"Modeling of Dynamic Systems",* Prentice Hall, 1994, Englewood Cliffs, USA.

## CLASSIFICATION OF VARIABLES

***FUNDAMENTAL VARIABLES:*** **íSpace, Timeý**

Systems do exist in space-time, so space-time is a kind o scene where systems "live", processes happen, and variables evolve.

***DESCRIPTIVE VARIABLES*:** are all the variables representing the magnitudes associated to the system.

- ***PARAMETERS*:** *Constants* (could be also *Variables having predetermined trajectory* independent of the evolution of phenomena in the system) (*Ex.*: *Universal Constants, System Constants, Design Parameters)*.
- ***INPUTS / INDEPENDENT VARIABLES / CAUSES:*** Descriptive Variables having signals which completely unrelated to or independent of other signals in the system, and which are not predetermined. They represent external actions (generated in the environment) on the system.
    - ***MANIPULATED INPUTS***
    - ***DISTURBANCES***
- ***DEPENDENT VARIABLES / EFFECTS:*** Descriptive Variables having associated signals depending of other signals in the system.
    - ***OUTPUTS***: Any descriptive variable of interest can be considered as an output.

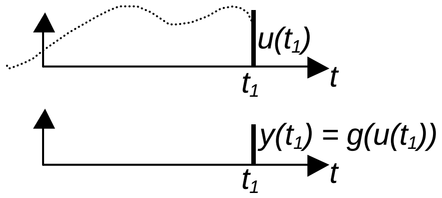
## CAUSALITY

***CAUSAL RELATIONSHIP***: A signal *y(·)* depends causally of other signal *u(·)* if:

i) *y(·)* depends of *u(·)*

ii) *y(·)* does not depend of future values of *u(·)*

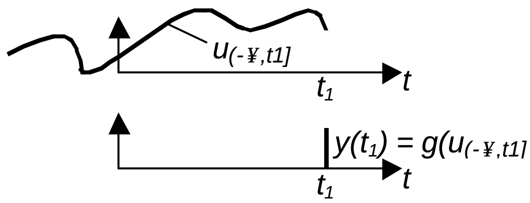
***STATIC CAUSAL RELATIONSHIP***: Causal relationship where for any generic time instant *t*, the value of the *effect y(t)* depends exclusively on the value of the *cause u(t)*, i.e., there is no dependency on previous values of *u(·)*.




$y(t) = g\,[\,u(t)\,]$ ,  $g\,[\bullet]$: is a function

***DYNAMICAL CAUSAL RELATIONSHIP***: Is a causal relationship where the value of the *effect y(t)* at any generic time instant ***t*** depends of at least one previous value of the *cause u(·)*.

*Memory* is associated to this phenomenon:




$y(t) = g\,[\,u_{(-\infty,\,t]}\,]$  ,  $g\,[\bullet]$: is a functional

***DYNAMICAL SYSTEM***: *A system* where there is at least one *dependent variable y(·)* being related through a *dynamical causal relationship* to at least one *input variable u(·)*.

## ***THE CONCEPT OF STATE***

*Regarding the following definitions it should be considered that the input signals are known to any time instant t.*

***VECTOR OF STATE VARIABLES***: Is any *set of dependent variables* such that its value at a generic time instant *t* statically *determines (i.e., through a function)* all other dependent variables (this definition is oriented to what happens in the system).

***VECTOR OF STATE VARIABLES***: Is any *set of dependent variables* such that its value at a generic time instant *t is sufficient to* statically *(i.e., through a function) compute* the value of any other dependent variable (this definition is oriented to the point of view of the modeler). Computation is done using constitutive and structural relationships.

***MINIMAL STATE VECTOR***: Is any State Vector whose value is *sufficient and necessary* to statically determine any other dependent variable of the system.

***MODEL ORDER***: Is equal to the minimal number of dependent variables whose values fully determine the values of all other dependent variables in the system.

***MODEL ORDER***: Equivalently, it is the maximal number of dependent variables whose value can be *arbitrarily* specified consistently with the restrictions in the model (static constitutive relationships and structural relationships).

***MODEL ORDER***: it is the *cardinal* number associated to any *Minimal State Vector*.

***¡¡THE MODEL ORDER IS UNIQUE!!***

***EXTERNAl VARIABLES*** := { Inputs, Outputs }

***INTERNAL VARIABLES*** º ***DEPENDENT VARIABLES***

*Note !! Þ with this definition, some internal variables could simultaneously be considered as external variables; this is the case of the output variables.*

***STATE VARIABLES*** – (restricted definition, as usually found in books mainly dealing with *differential equation models).*

Set of internal variables ***x(·)*** whose value ***x(t₀)*** (initial condition) at a generic time instant $t_0$ (initial time) suffices to calculate any other internal variable at any time $t \geq t_0$ (it assumes the knowledge of the input trajectory $u_{[t0, t]}$).

*This definition is inspired in the fact that under certain assumptions the solution of a set of differential equations is uniquely determined by the initial conditions and the input trajectory.*

***STATE EQUATIONS***: *CONCENTRATE THE DYNAMICS OF THE SYSTEM.*

***OUTPUT EQUATIONS*** : *STATIC EQUATIONS*.

## ***STATE SPACE MODELS*** (continuous systems)

### ***STATE EQUATION STANDARD FORM***

***(Vector) State Equation:*** $\dot{x}(t) = f(x(t), u(t), t)$

***(Vector) Output Equation:*** $y(t) = g(x(t), u(t), t)$

*x(t): State Vector, n-dimensional*

*u(t): Input Vector, m-dimensional*

*y(t):Output Vector, p-dimensional*

Componentwise written:

State Equations

$$
\begin{aligned}
\dot{x}_1(t) &= f_1(x_1(t),\dots,x_n(t),u_1(t),\dots,u_m(t),t) \\
\dot{x}_2(t) &= f_2(x_1(t),\dots,x_n(t),u_1(t),\dots,u_m(t),t) \\
&\vdots \\
\dot{x}_n(t) &= f_n(x_1(t),\dots,x_n(t),u_1(t),\dots,u_m(t),t)
\end{aligned}
$$

Output Equations

$$
\begin{aligned}
y_1(t) &= g_1(x_1(t),\dots,x_n(t),u_1(t),\dots,u_m(t),t) \\
y_2(t) &= g_2(x_1(t),\dots,x_n(t),u_1(t),\dots,u_m(t),t) \\
&\vdots \\
y_p(t) &= g_p(x_1(t),\dots,x_n(t),u_1(t),\dots,u_m(t),t)
\end{aligned}
$$

This is a general notation able to model *nonlinear systems* (in this case ***f*** y ***g*** will be nonlinear functions of ***x*** and/or ***u*** ) as well as *time-variant systems* (in this case the direct dependency of functions ***f*** y ***g*** upon time ***t*** allows for the presence of *variable parameters*).

The particular case of *time-invariant nonlinear* systems would be specified by*:*

$$\dot{x}(t) = f(x(t),u(t))$$

$$y(t) = g(x(t),u(t))$$

If functions ***f*** y ***g*** are linear in ***x*** and ***u***, then the model is *linear* and is written as follows:

*Linear time-variant case* (*LTV-System*)

$$\dot{x}(t) = A(t)\cdot x(t) + B(t)\cdot u(t)$$

$$y(t) = C(t)\cdot x(t) + D(t)\cdot u(t)$$

where *A(t),B(t),C(t),D(t)* are real matrices whose entries are predefined time-functions:

$A(t): n\times n$ , *the evolution or system matrix*

$B(t): n\times m$ , *the input matrix*

$C(t): p\times n$ , *the output matrix*

$D(t): p\times m$ , *the static transmission matrix*

$t\in\Re$

*Linear time-invariant case* (*LTI-System*) (the entries of the matrices are constant)

$$\dot{x}(t) = A \cdot x(t) + B \cdot u(t)$$
$$y(t) = C \cdot x(t) + D \cdot u(t)$$

$A: n \times n \quad , \quad B: n \times m \quad , \quad C: p \times n \quad , \quad D: p \times m \quad , \quad t \in \Re$

## DIFFERENTIAL-ALGEBRAIC EQUATIONS OR DAE-SYSTEMS

Frequently, the system equations cannot be put into the neat **state equation standard form** just presented. Sometimes the interrelation among certain (group of) variables cannot be explicited, then equations of the following type are obtained:

**Differential Equation:** $\dot{x}(t) = f(x(t), z(t), u(t), t)$

**"Algebraic" Equation:** $\text{Ö}(x(t), z(t), u(t), t) = 0$

$z(t)$ is a set of internal variables depending on the state and input variables as specified by function $F$. $F$ is a static equation, but as most commonly it consists of algebraic (nonlinear) expressions, this *implicit equation* is usually called algebraic equation. If an explicit expression can be obtained solving $F$ for $z(t)$, then the original standard form can be recovered through substitution of the solution, say $z(t) = \varphi(x(t), u(t), t)$, into the differential equation and redefinition of function $f$.

## IMPLICIT STATE EQUATION FORM

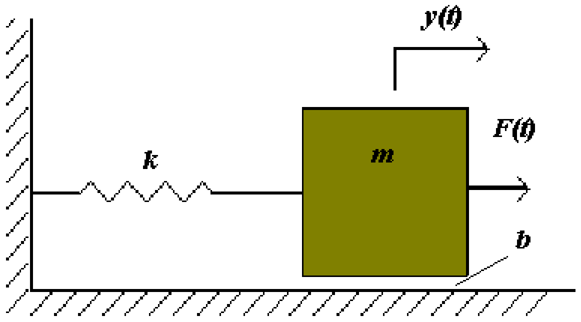
Sometimes not all state derivatives can be explicited, then the modeler is obliged to use a more general expression for the dynamics of the system, the so called

**Implicit State Equation Form**: $F(x(t), \dot{x}(t), u(t), t) = 0$

*Implicit and DAE forms are much harder to treat, they deserve special methods for analysis as well as for simulation !*

**DIFFERENT REPRESENTATION TYPES.**

**EXAMPLE. Model of a simple Physical System: Mass-Spring-Damper System.**




Under certain assumptions, the dynamics of the position *y*(*t*) of the mass point ***m*** can be described via the following equivalent models:

**Representation Type**: *Ordinary Differential Equation* (ODE):

$$m\,\ddot{y}(t) + b\,\dot{y}(t) + k\,y(t) = F(t)$$

**Representation Type:** *State Equations System* (SES)

Considering the ODE as the system, the variables $x_1$ and $x_2$ next defined satisfy the definition of *state variables*.

$$x_1(t) = y(t)$$
$$x_2(t) = \dot{y}(t)$$

$$\begin{aligned} \dot{x}_1(t) &= x_2(t) \\ \dot{x}_2(t) &= -\frac{k}{m}x_1(t) - \frac{b}{m}x_2(t) + \frac{1}{m}F(t) \\ y(t) &= x_1(t) \end{aligned}$$

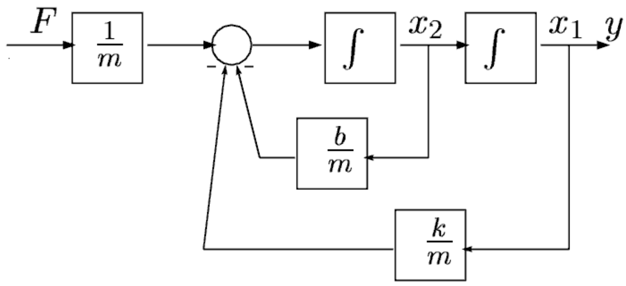
This is a linear time-invariant system, which can be represented by the matrix form:

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) + Du(t) \end{aligned}$$

with the following definition of the matrices A, B, C and D:

$$A = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{b}{m} \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix}, \quad C = [1 \;\; 0], \quad D = 0$$

**Representation Type:** *Block Diagram* (BD).




## NUMERICAL SOLUTION OF *ODEs* OR *STATE EQUATION SYSTEMS*

**BASICS of EULER'S METHOD.**

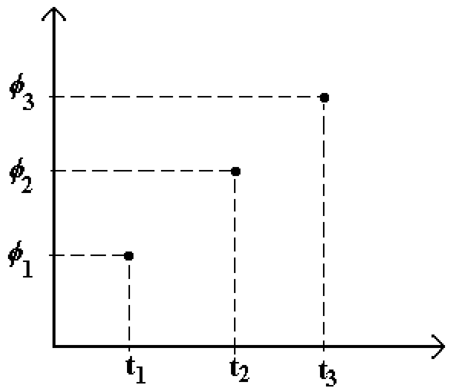
For simplicity, let's consider the scalar case (only one state variable) of the initial value problem:

$$\frac{dx(t)}{dt} = f[x(t), t] \qquad (1)$$
$$x(t_0) = x_0$$

A numerical approach to the problem of finding a solution to the previous problem demands the discrete representation of time through a sequence of points $t_k$, $k = 1,2,....,n$. Even if it not necessary and sometimes it could be inconvenient, let's assume to the effect of this introduction, that the points are equally spaced in time: $t_k = k \cdot h$

Let's now use the following notation for the exact solution of the problem: $f(t = k \cdot h) = f(t_k) = f_k$,

where $\phi_k$ stands for the value of the solution $f(t)$ at instant $t_k$. A set of pairs $(f_k, t_k)$ will constitute the ***exact*** discrete-time representation of the solution.




Numerical representation of the exact solution.

A numerical method will be next presented yielding a sequence of values $x_k$ constituting a good approximation of the exact values $f_k$. A set of pairs $(x_k, t_k)$ will constitute the ***approximate*** discrete-time representation of the solution.

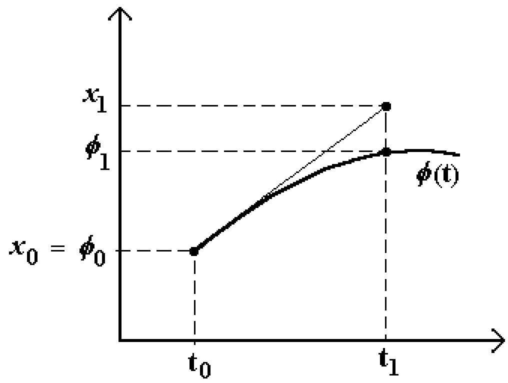
**Euler's Method**

There are a lot of methods allowing for the obtention of the numerical approximate solution of our problem. Numerical methods for solving differential equations are known as *integration methods*. The most simple and well known is Euler's Method.

Euler's Method proceeds approximating the time derivative of $x(t)$ through the incremental quotient as follows:

$$X(t_k + \Delta t) = X(t_k) + f(X(t_k), t_k) \cdot \Delta t \quad \Rightarrow \quad X_{k+1} = X_k + f_k \cdot h \qquad (2)$$

where $\Delta t = h$ is called the *integration step*. See the following graphical interpretation of Euler's formula:




Graphical interpretation of Euler's formula.

Given the initial value problem (1) and Euler's formula (2) the approximate values $X_1, X_2, \ldots$ can be computed in a sequential way (first compute $X_1$ using the known value $X_0$, then use $X_1$ to calculate $X_2$, and so on).

**Errors:**

- The error due to the numerical approximation is called the ***truncation error***. Even if using an ideal computer this error will be present.
- The error due to the finite precision representation of numbers in real computers is called the ***round-off error***. It represents an additional source of errors.

**<u>Example</u>:**

Consider the following first order system:

$$\dot{x} = -3x^3 + t$$
$$x(0) = 1$$

Euler's method yields:

$$x_{k+1} = x_k + (-3x_k^3 + t_k) \cdot h,$$

whose succesive application produces

$x_1 = 1 - 3h \quad x_2 = x_1 + (-3x_1^3 + h) \cdot h \quad x_3 = x_2 + (-3x_2^3 + 2h) \cdot h$, and so on.

**EXERCISES**

1 – Find the explicit expression for $x_2$ above and use it to calculate the explicit expression for the next value $x_3$.

2 – Determine the numerical approximation (i.e., the particular expression for formula (2) above) of the second order State Equation System of the *mass-spring-damper* example via Euler's method. Apply the rule (2) to each of both state equations.

3 – The nonlinear second order SES constitutes a posible version of the famous Lotka-Volterra's model. The variables $x_1$ and $x_2$ represent respectively the population of Preys and Predators in a common habitat. This is a simple ***nonlinear*** case where the numerical approach is a must !

$$\begin{aligned} \dot{x}_1 &= \epsilon x_1 - \alpha x_1 x_2 - \sigma x_1^2 \\ \dot{x}_2 &= -m x_2 + \beta x_1 x_2 \end{aligned}$$

Apply the rule (2) of Euler's method to each of both state equations. Then particularize the result for the following set of parameter:

$$\epsilon = 0.1,\ \alpha = 0.01,\ \sigma = 0.01,\ m = 0.4,\ \beta = 0.5$$

Find numerically the three equilibrium points. If you like, write a program to implement the recursive algorithm given by Euler's method and draw the solutions in the $x_1$-$x_2$ plane for some set of initial value pairs $(x_{10}\ ,\ x_{20})$.

Recall the restriction of the solutions to the first quadrant.

## SOME ADVANCED ISSUES ON EULER'S METHOD.

### Forward or Explicit Euler

The technique previously described is known as forward or explicit Euler. As already seen, it is based on the following approximation of the time derivative of *x(t)*, which corresponds to the so-called ***forward** incremental quotient* (see Fig. "Graphical interpretation of Euler's Formula"):

$$\left.\frac{dx(t)}{dt}\right|_{t=t_k} \approx \left.\frac{\Delta x(t)}{\Delta t}\right|_{t=t_k} = \frac{x(t_k + h_k) - x(t_k)}{h_k} = \frac{x(t_{k+1}) - x(t_k)}{h_k} = \frac{x_{k+1} - x_k}{h_k}$$

As a result, when applied to the dynamical model

$$\boxed{\frac{d\,x(t)}{d\,t} = f\left(x(t), u(t), t\right)}$$

it yields the approximation

$$x_{k+1} \approx x_k + h_k \,.\, f\left(x_k, u_k, t_k\right)$$

which is handled as the identity

$$\boxed{x_{k+1} = x_k + h_k \,.\, f\left(x_k, u_k, t_k\right)}$$

in order to compute the numerical approximation to the solution of the differential equation.

The latter formula ***explicitely*** calculates the actualization of the state vector as a function of known values.

## Backward or Implicit Euler

In this case, the so-called ***backward*** *incremental quotient* is used in order to approximate the time derivative of *x(t)*:

$$\left.\frac{dx(t)}{dt}\right|_{t=t_k} \approx \left.\frac{\Delta x(t)}{\Delta t}\right|_{t=t_k} = \frac{x(t_k)-x(t_k-h_{k-1})}{h_{k-1}} = \frac{x(t_k)-x(t_{k-1})}{h_{k-1}} = \frac{x_k-x_{k-1}}{h_{k-1}}$$

As a result, when applied to the dynamical model

$$\boxed{\frac{d\,x(t)}{d\,t} = f\left(x(t), u(t), t\right)}$$

it yields the approximation

$$x_k \approx x_{k-1} + h_{k-1}\,.\,f\left(x_k, u_k, t_k\right)$$

which, incrementing the time index in one unit is shown to be equivalent to

$$x_{k+1} \approx x_k + h_k\,.\,f\left(x_{k+1}, u_{k+1}, t_{k+1}\right)$$

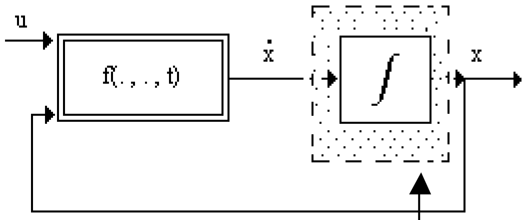
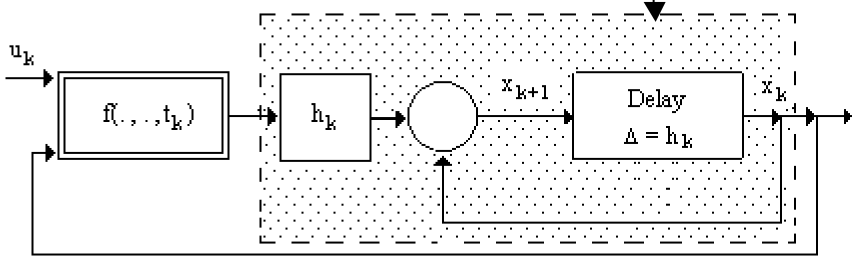
which is handled as the identity

$$\boxed{x_{k+1} = x_k + h_k\,.\,f\left(x_{k+1}, u_{k+1}, t_{k+1}\right)}$$

The latter formula ***implicitely*** defines the actualization of the state vector, because the right-hand side contains the unknown value $x_{k+1}$ of the state vector and not –as in the previous case– only known variable values ($u_{k+1}$ , $t_{k+1}$). Thus, the unknown $x_{k+1}$ cannot in general be calculated through a direct evaluation of the right-hand side, but it should be determined with the help of some implicit method.

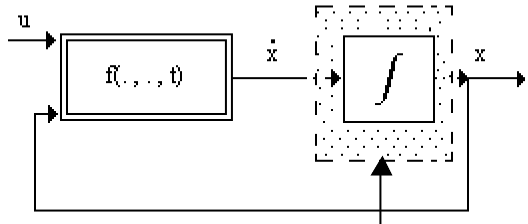
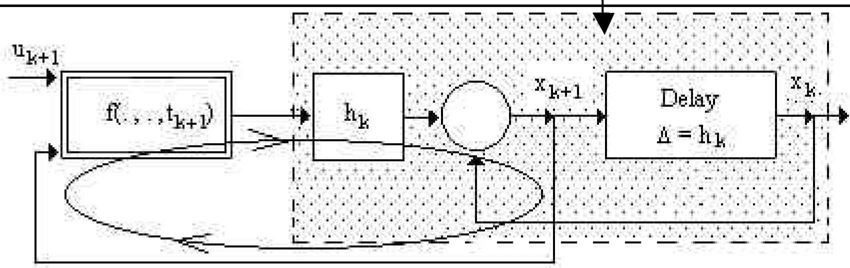
**BLOCK DIAGRAM (BD) REPRESENTATION OF BOTH FORMULAE, FORWARD AND BACKWARD EULER**.

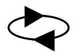
## Forward Euler

| | |
|---|---|
| $\boxed{\frac{d\,x(t)}{d\,t} = f\left(x(t), u(t), t\right)}$ | <br>continuous-time BD |
| $\boxed{x_{k+1} = x_k + h_k\,.\,f\left(x_k, u_k, t_k\right)}$<br>Forward Euler | <br>***Fig. FE-BD*****:** Forward-Euler discrete-time BD |

As shown in the figures above, the method Forward Euler assigns the *subsystem containing the discrete-time delay* as the numerical approximation to the *continuous-time integrator*.

## Backward Euler

| | |
|---|---|
| $\frac{d\,x(t)}{d\,t} = f\left(x(t), u(t), t\right)$ |  continuous-time BD |
| $x_{k+1} = x_k + h_k \,.\, f\left(x_{k+1}, u_{k+1}, t_{k+1}\right)$ Backward Euler |  ***Fig. BE-BD*:** Backward-Euler discrete-time BD |

As shown in the new set of figures, when using the method Backward Euler, a different discrete-time BD corresponds to the *continuous-time integrator*. Observe the *algebraic loop*  in the BD, which is the *graphical expression of an implicit equation*. An algebraic loop is a signal path without dynamical components building a closed loop in a BD, i.e., it has no integrators (or continuous-time delays) in the case of a continuous-time BD, and no discrete-time delays in a discrete-time BD.

**N.B. 1**: In this case, the algebraic loop (implicit equation) is a consequence of the numerical approximation method used (implicit Euler). It does not exist in the original state-equation system.

**N.B. 2**: Recall that there exist *continuous* state-equation systems with implicit equations. This is the case for instance of Differential-Algebraic Systems, which when put into the *continuous* BD form will contain algebraic loops due to the algebraic equations.

## *EXERCISES*

***First Exercise.*** Given the continuous-time model

$$\dot{x}(t) = a\,x(t) + bu(t)$$ (scalar variables and coefficients !)

a. Obtain both explicit and implicit discrete-time approximations after forward- <u>and</u> backward-Euler formulae, respectively.
b. The original continuous-problem being linear, it is possible to solve the implicit equation for $x_{k+1}$ , and in this way, to convert the implicit problem into an explicit one. Obtain the explicit

solution for $x_{k+1}$ , and analize on it the stability inherent to the backward-Euler method for the free system, i.e., for $u(t) = 0$. (Stability means that if the true solution converges for $t \rightarrow \infty$ – as it is the case for $a < 0$ – , then, the approximate solution converges for $k \rightarrow \infty$. In general, the stability of a numerical method will depend on the choice of $h$).

c. Draw the block diagram version of the three previous results.

***Second Exercise.*** Given the continuous-time (CT) model

$$\ddot{y}(t) + a_1 \dot{y}(t) + a_2\, y(t) = b\, u(t) \qquad \text{(scalar variables and coefficients !)}$$

a. Obtain both explicit and implicit discrete-time (DT) approximation after forward- and backward-Euler, respectively.

*Help*: a possible technique to solve this problem consists in converting the second order differential equation into a system of two state equations, which are to be discretized later (for instance, with the definitions $x_1 = y$ , $x_2 = y\text{-}dot$).

b. Draw the block diagram version of the two previous results.

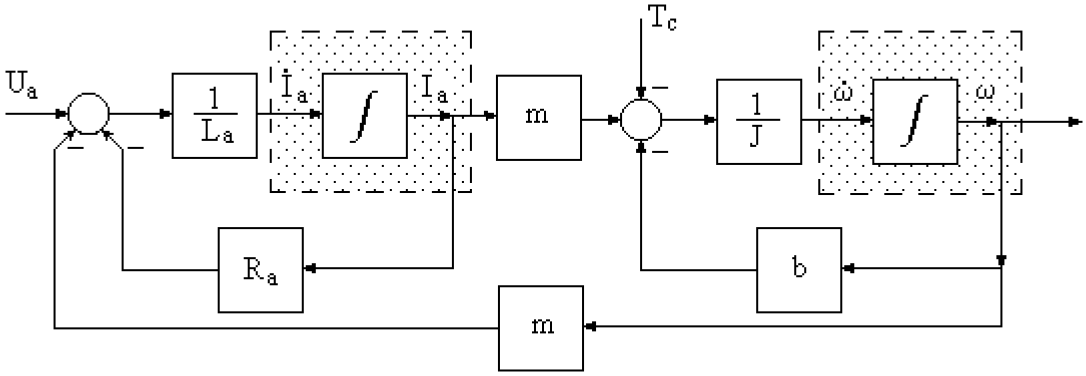
## CT-BD → DT-BD → DT-EQUATIONS

**Remark**: *This section is just for you to gain more confidence on manipulation of block-diagrams and other representation formalisms, and on the problems associated to discretization of continuous systems. In the praxis you wouldn't proceed in this way, then you would simply specify your CT-problem under the form of an ODE, or a set of CT–State Equations, or as a CT–BD. Programs embedded in the software would choose and implement a numerical approximation to solve the simulation problem.*

## DISCRETIZING DIRECTLY ON THE BLOCK DIAGRAMS, THEN OBTAINING THE DT-EQUATIONS:

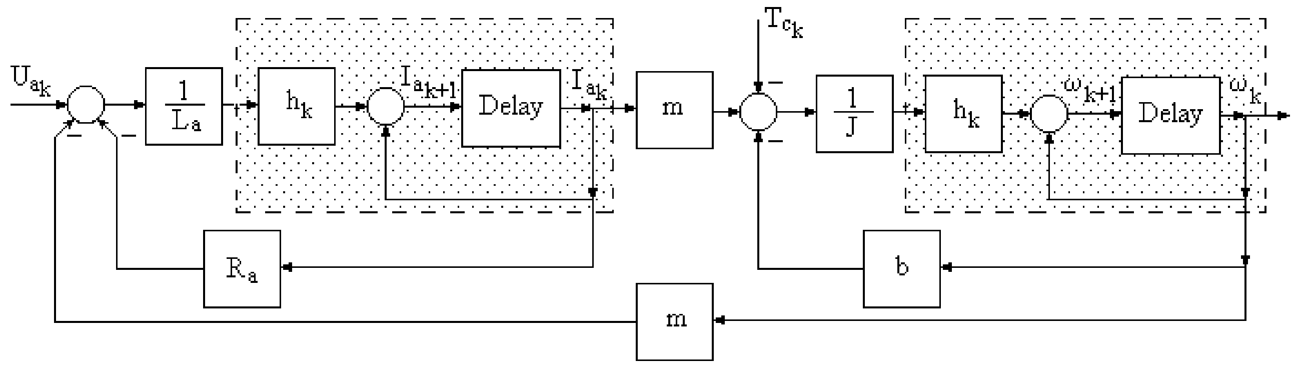
**Example**: ***PMDCM*** (Permanent Magnet DC Motor)

***i) CT Block diagram***



***ii)*** ***Forward Euler DT-BD*** *is obtained after the* ***BD*** *in* ***Fig. FE-BD above***:

***Forward Euler DT-BD***




The DT ***Explicit*** State Equations can be directly read from the previous BD as follows:

$$\begin{cases} I_{ak+1} = I_{ak} - h_k \dfrac{R_a}{L_a} I_{ak} - h_k \dfrac{m}{L_a} w_k + h_k \dfrac{1}{L_a} U_{ak} \\ w_{k+1} = w_k - h_k \dfrac{b}{J} w_k + h_k \dfrac{m}{J} I_{ak} - h_k \dfrac{1}{J} T_{ck} \end{cases}$$

or, in matrix form:

$$\begin{bmatrix} I_{ak+1} \\ w_{k+1} \end{bmatrix} = \begin{bmatrix} 1 - h_k \dfrac{R_a}{L_a} & -h_k \dfrac{m}{L_a} \\ h_k \dfrac{m}{J} & 1 - h_k \dfrac{b}{J} \end{bmatrix} \begin{bmatrix} I_{ak} \\ w_k \end{bmatrix} + \begin{bmatrix} h_k \dfrac{1}{L_a} & 0 \\ 0 & -h_k \dfrac{1}{J} \end{bmatrix} \begin{bmatrix} U_{ak} \\ T_{ck} \end{bmatrix}$$

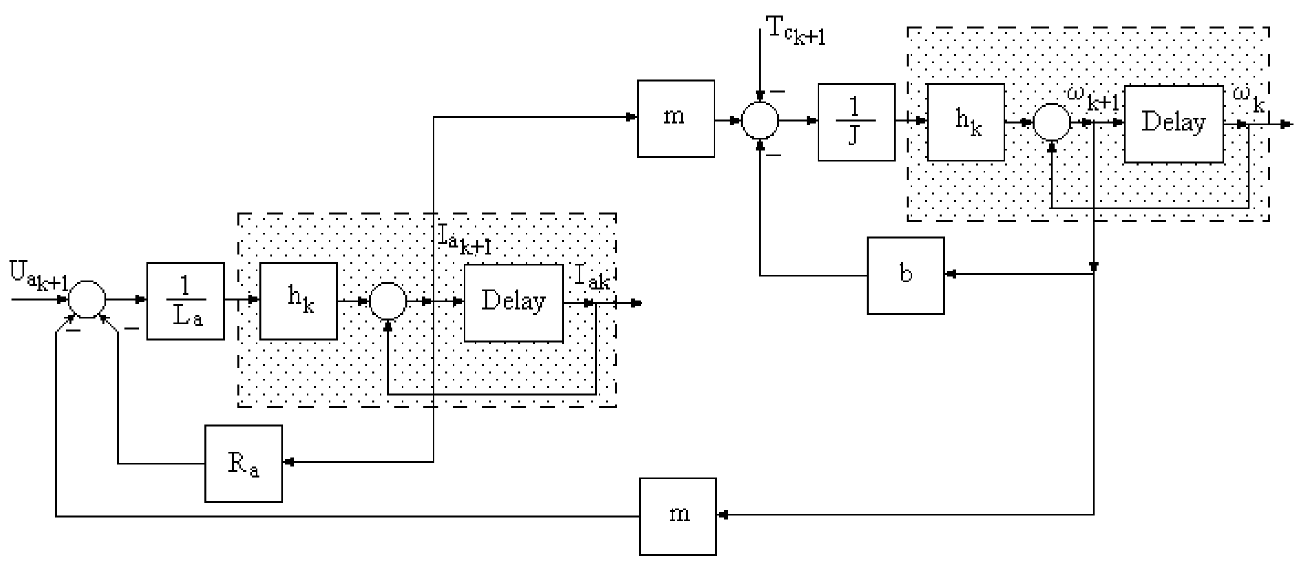
$$\underbrace{\begin{bmatrix} I_{ak+1} \\ \omega_{k+1} \end{bmatrix}}_{X_{k+1}} = \underbrace{\begin{bmatrix} 1 - h_k \dfrac{R_a}{L_a} & -h_k \dfrac{m}{L_a} \\ h_k \dfrac{m}{J} & 1 - h_k \dfrac{b}{J} \end{bmatrix}}_{A_k = A(h_k)} \underbrace{\begin{bmatrix} I_{ak} \\ \omega_k \end{bmatrix}}_{X_k} + \underbrace{\begin{bmatrix} h_k \dfrac{1}{L_a} & 0 \\ 0 & -h_k \dfrac{1}{J} \end{bmatrix}}_{B_k = B(h_k)} \underbrace{\begin{bmatrix} U_{ak} \\ T_{ck} \end{bmatrix}}_{U_k}$$

It can be seen that the DT ***Explicit*** State Equations are of the general form:

$$X_{k+1} = A_k X_k + B_k U_k$$

***iii)*** ***Backward Euler DT-BD*** *is obtained after the* ***BD*** *in* ***Fig. BE-BD*** *above*:

***Backward Euler DT-BD***




The DT ***Implicit*** State Equations can be directly read from the previous BD as follows:

$$\begin{cases} I_{ak+1} = I_{ak} - h_k \dfrac{R_a}{L_a} I_{ak+1} - h_k \dfrac{m}{L_a} w_{k+1} + h_k \dfrac{1}{L_a} U_{ak+1} \\ w_{k+1} = w_k - h_k \dfrac{b}{J} w_{k+1} + h_k \dfrac{m}{J} I_{ak+1} - h_k \dfrac{1}{J} T_{ck+1} \end{cases}$$

or, in matrix ***Implicit*** form:

$$\begin{bmatrix} I_{ak+1} \\ w_{k+1} \end{bmatrix} = \begin{bmatrix} I_{ak} \\ w_k \end{bmatrix} + \begin{bmatrix} -h_k \dfrac{R_a}{L_a} & -h_k \dfrac{m}{L_a} \\ h_k \dfrac{m}{J} & -h_k \dfrac{b}{J} \end{bmatrix} \begin{bmatrix} I_{ak+1} \\ w_{k+1} \end{bmatrix} + \begin{bmatrix} h_k \dfrac{1}{L_a} & 0 \\ 0 & -h_k \dfrac{1}{J} \end{bmatrix} \begin{bmatrix} U_{ak+1} \\ T_{ck+1} \end{bmatrix}$$

$$\underbrace{\begin{bmatrix} I_{ak+1} \\ \omega_{k+1} \end{bmatrix}}_{X_{k+1}} = \underbrace{\begin{bmatrix} I_{ak} \\ \omega_k \end{bmatrix}}_{X_k} + \underbrace{\begin{bmatrix} -h_k \dfrac{R_a}{L_a} & -h_k \dfrac{m}{L_a} \\ h_k \dfrac{m}{J} & -h_k \dfrac{b}{J} \end{bmatrix}}_{A_k = A(h_k)} \underbrace{\begin{bmatrix} I_{ak+1} \\ \omega_{k+1} \end{bmatrix}}_{X_{k+1}} + \underbrace{\begin{bmatrix} h_k \dfrac{1}{L_a} & 0 \\ 0 & -h_k \dfrac{1}{J} \end{bmatrix}}_{B_k = B(h_k)} \underbrace{\begin{bmatrix} U_{ak+1} \\ T_{ck+1} \end{bmatrix}}_{U_{k+1}}$$

$$X_{k+1} = X_k + A_k \, X_{k+1} + B_k U_{k+1}$$

<u>As the model is linear, an explicit expression can be recovered</u>, as follows:

$$\begin{bmatrix} I_{ak+1} \\ w_{k+1} \end{bmatrix} = \begin{bmatrix} 1-h_k \frac{R_a}{L_a} & -h_k \frac{m}{L_a} \\ h_k \frac{m}{J} & 1-h_k \frac{b}{J} \end{bmatrix}^{-1} \left( \begin{bmatrix} I_{ak} \\ w_k \end{bmatrix} + \begin{bmatrix} h_k \frac{1}{L_a} & 0 \\ 0 & -h_k \frac{1}{J} \end{bmatrix} \begin{bmatrix} U_{ak+1} \\ T_{ck+1} \end{bmatrix} \right)$$

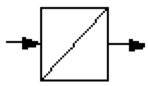
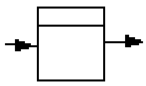
$$X_{k+1} = (I - A_k)^{-1} (X_k + B_k U_{k+1})$$

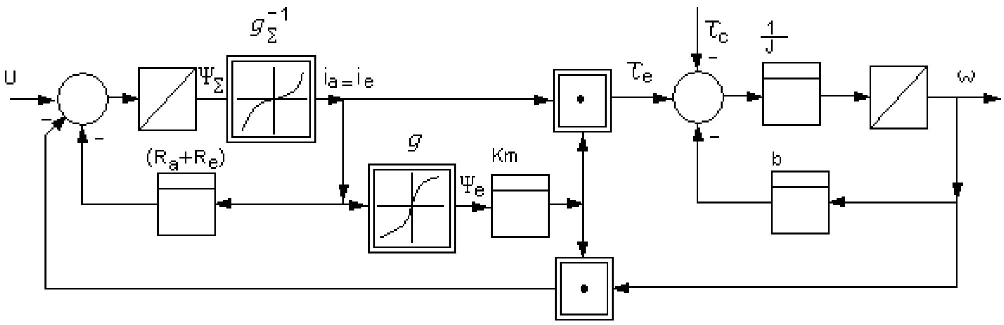
***Third Exercise.*** Consider the previously handled Lotka-Volterra (non-linear) model. Obtain the DT State Equations following the method in the preceding example, id est:

a. Construct the corresponding CT-BD.
b. Construct both, the explicit and the implicit DT-BD´s.
c. Write-down the DT State Equations through reading of the BD´s.

***Fourth Exercise.*** The following is the (non-linear) CT-BD of a Series Connected DC-Motor with full excitation(*). Do the same exercise as in both previous cases. Consider $g$ and $g_S$ as known non-linear functions, and $g_S^{-1}$ as the inverse of the latter.

*Meaning of the symbols in the BD :*

| This block is an *integrator*: | and this one is a *gain*, the value of its gain being Km: |
|---|---|
|  |   |




(*) Just for information, find below the equivalent circuit of the DC-Motor (if you are not interested, ignore it). $g$ is a non-linear function representing the dependence of the magnetic excitation flux $y_e$ on the excitation current $Ie$ : $Ø_e = g(I_e)$. In a full series connection of both the armature and the field coils, the armature and the excitation currents are the same: $Ia = Ie$. This situation is modeled as having a unique coil with $g_\Sigma(I_a) = Ø_e + Ø_a = g(I_e = I_a) + L_a I_a$ as its magnetic characteristic.