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EFFECTS OF TIME QUANTIZATION AND NOISE IN LEVEL CROSSING SAMPLING STABILIZATION

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Abstract: A recent paper introduced a level crossing sampling (LCS) scheme that produces samples as triggered by the crossing of the system output through its quantization levels. That paper also proposed an LCS control strategy, by which an open loop unstable system can be stabilized using feedback over error and delay free binary communication channels. Assuming exact knowledge of the sampling times and output measurements, such LCS control strategy drives the system state to the origin in finite time. The present paper relaxes these assumptions and shows, for scalar systems, that although finite-time stabilization is no longer possible, practical stability can be achieved. Expressions for ultimate error bounds and conditions for closed-loop stability are given.

Keywords: Event-based sampling, Networked control systems.

1. INTRODUCTION

Motivated by recent developments in networked control theory (e.g. Antsaklis and Baillieul, 2004), Kofman and Braslavsky (2006) have introduced a novel level crossing sampling (LCS) scheme and a LCS control design strategy for feedback stabilization over communication channels with data-rate constraints. LCS direct design is an asynchronous sampled-data control technique that consists on the sequential application of two open-loop procedures: (1) finite-time estimation of the system state initial condition $x(0)$, (2) finite-time control to the origin by application of a bang-bang control procedure. Kofman and Braslavsky (2006) show that the system state can be driven to the origin in finite time from an arbitrary initial condition after the transmission of $2n + 2$ bits, where n is the order of the system. Such finite-time stabilization relies on the assumptions delay and error-free transmissions, the knowledge of the sampling times to infinite precision, and exact output measurements.

The present paper relaxes the assumptions in Kofman and Braslavsky (2006) in two ways: (1) we consider that sampling times are approximated to their nearest value in the set $\{0, T, 2T, 3T, \dots\}$, where $T > 0$ is the time quantization interval, and (2) we consider that the system output is corrupted by bounded measurement noise. The present preliminary results, developed for a first order plant, show that after the application of one cycle of the proposed estimation/control procedures, the state of the system is not driven to the origin but to some final state $x_f \neq 0$. We derive bounds on x_f that are useful in determining conditions for practical stability and estimating the average data-rates resulting by repeatedly applying LCS estimation/control cycles.

The general setup from Kofman and Braslavsky (2006) is given in Section 2, while Section 3 reviews the LCS direct control design strategy on a simple first order system. Section 4 studies effects of sampling time quantization, while Section 5 studies effects of measurement noise. Conclusions are given in Section 6.

2. LEVEL CROSSING SAMPLING FEEDBACK SCHEME

The general feedback scheme considered is shown in Figure 1. We model two communication links in the feedback loop: a sensor communication link, between the measured plant output and the controller input, and an actuator communication link, between controller output and plant input. These communication links are assumed to be error-free and have no noise disturbances nor transmission delays.

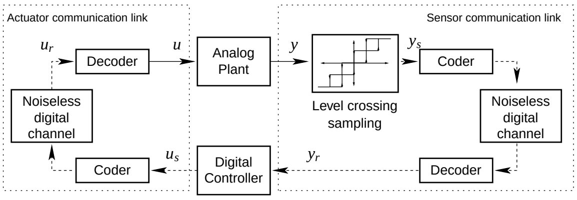

Fig. 1. Event-driven sampled-data scheme for feedback stabilization over digital channels

Plant: The plant is a continuous-time, finite dimensional LTI system given by the minimal realization

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t), \\ y(t) &= Cx(t). \end{aligned} \tag{1}$$

The matrices A, B, C are assumed known, but the system initial condition $x(0)$ is unknown.

Actuator: We consider an actuator that can only produce values in the set $\mathcal{U} \triangleq \{-U, 0, U\}$ where U is a positive real constant. We assume that the switching of the actuator is instantaneous and the values are held constant (in a zero-order-hold fashion) until a different value of $u(t)$ is generated.

Level crossing sampling device: The proposed LCS scheme generates asynchronous samples of the output $y(t)$ whenever it differs from the previous sample in a fixed quantity $h > 0$, which we call the output amplitude quantization interval. Let $y(t)$ be continuous-time and (scalar) real valued. Then the LCS device produces the quantized (but exact) samples

$$y_s(t_k) = y(t_k) \tag{2}$$

at the sampling instants t_k, $k = 0, 1, 2, \ldots$ defined by

$$t_k = \inf\{\tau \in (t_{k-1}, \tau] : |y(\tau) - y_s(t_{k-1})| > h\}. \tag{3}$$

Figure 2 illustrates an output signal produced by such LCS device, plotted together with the input signal that generates it, for a quantization interval $h = 1$.

Notice that the proposed LCS scheme (2), (3) has *hysteresis*, which in general reduces the number of samples generated. Moreover, if the derivative of $y(t)$ is bounded, hysteresis guarantees that a nonzero time interval exists between successive samples. A similar sampling scheme (without hysteresis) is *Lebesgue* sampling, analyzed in Åström and Bernhardsson (2002) for the control of a double integrator.

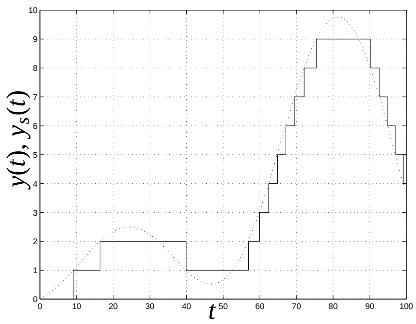

Fig. 2. Event-driven sampling/quantization

A fundamental property of the proposed LCS scheme is that successive samples always differ in $\pm h$. Consequently, each sample can be coded using only one bit, thus reducing the amount of information that needs to be transmitted.

Coder/digital channel. Sensor link: We assume that the digital channel is memory-less and error-free, and can only transmit one bit, 0 or 1, per asynchronous sample produced by the LCS device. The coding strategy is the following: When a sample is produced, at say $t = t_k$, the channel transmits

$$\begin{cases} 1, & \text{if } y_s(t_k) > y_s(t_{k-1}), \\ 0, & \text{if } y_s(t_k) < y_s(t_{k-1}). \end{cases} \tag{4}$$

Thus, a 1 is transmitted if the current sample of the output has increased with respect to its previous sample, or a 0 if it has decreased. If no samples are produced, no transmission takes place. However, notice that in the proposed scheme there is also information when no samples are produced; namely, the output $y(t)$ remains within its quantization band.

Decoder. Sensor link: The decoder receives the sequence of bits indicating the changes in y_s and calculates

$$y_r(t_k) = \begin{cases} h, & \text{when 1 is received,} \\ -h, & \text{when 0 is received.} \end{cases} \tag{5}$$

Notice that

$$y_r(t_k) = y_s(t_k) - y_s(t_{k-1}). \tag{6}$$

Digital controller: The controller receives an asynchronous sequence of values $y_r(t_k)$ and produces a sequence of control actions $u_s(\tau_j)$, where—in general—the time sequences t_k and τ_j are different.

We consider that $u_s(\tau_j) \in \mathcal{U}$, and $u_s(\tau_0) = 0$. We shall also restrict $|u_s(\tau_j) - u_s(\tau_{j-1})| \in \{U, 2pU\}$, that is, successive control actions differ in one quantization level or they can jump between the saturation limits. This restriction permits coding the successive control values using only one bit.

In Section 3 we describe the *LCS direct design* control strategy, from Kofman and Braslavsky (2006), which shows a way to calculate the sequence of control actions $u_s(\tau_j)$ based on a bang-bang control strategy.

Coder/digital channel. Actuator link: The coding strategy in the actuator link is similar to that of the sensor link, transmitting only when the controller produces a control signal that differs from its value at the previous sampling instant. Then, when a sample is produced at $t = \tau_j$, the channel transmits

$$\begin{cases} 1, & \text{if } u_s(\tau_j) - u_s(\tau_{j-1}) \in \{-2U, U\} \\ 0, & \text{if } u_s(\tau_j) - u_s(\tau_{j-1}) \in \{-U, 2U\} \end{cases} \tag{7}$$

Decoder. Actuator link: The actuator decoder receives the sequence of bits informing the changes in u_s, and it builds the signal $u_r(\tau_j)$ according to following logic

$$\begin{cases} u_r(\tau_{j-1}) + U, & \text{if } u_r(\tau_{j-1}) < U \text{ and 1 is received} \\ u_r(\tau_{j-1}) - U, & \text{if } u_r(\tau_{j-1}) > -U \text{ and 0 is received} \\ -U, & \text{if } u_r(\tau_{j-1}) = U \text{ and 1 is received} \\ U, & \text{if } u_r(\tau_{j-1}) = -U \text{ and 0 is received} \end{cases}$$

Notice that, provided that $u_r(\tau_0) = u_s(\tau_0) = 0$, it is always true that $u_r(\tau_j) = u_s(\tau_j)$.

3. REVIEW OF THE LCS DIRECT DESIGN CONTROL STRATEGY

We consider the first order system

$$\dot{x}(t) = px(t) + u(t), \quad x(0) = x_0, \tag{8}$$

where $p > 0$ is known and x_0 is unknown. Without loss of generality, we assume that $x_0 > 0$. We next briefly illustrate the LCS direct design strategy on this system as presented in Kofman and Braslavsky (2006). As anticipated in Section 1, this strategy consists in two sequential open-loop cycles: a finite-time estimation of the initial condition $x(0)$ (while the control is set to zero), and a subsequent bang-bang control strategy to drive the state to the origin in finite-time (while measurements are ignored).

3.1 Finite-time estimation of x_0

As the system starts from $x(0) = x_0 > 0$, while the input is set at this stage to $u = 0$, the state will evolve following the increasing exponential

$$x(t) = e^{pt}x_0 > 0.$$

The first sample $x(t_1)$ is generated by the LCS device when $x(t)$ crosses its first quantization level,

$$e^{pt_1}x_0 = kh, \tag{9}$$

where $h > 0$ is the quantization interval for $x(t)$, and k is the positive integer

$$k \triangleq \lfloor x_0/h \rfloor + 1, \tag{10}$$

where $\lfloor x_0/h \rfloor$ denotes the integer part of x_0/h. Because $x(t_1) > 0$, a bit "1" is transmitted through the communication channel. The bit "1" is instantly received at the controller and the sampling time t_1 is registered *exactly* (to infinite precision).

The second sample $x(t_2)$ is generated when

$$x(t_2) - x(t_1) = (e^{pt_2} - e^{pt_1})x_0 = h. \tag{11}$$

A second bit "1" is sent through the channel and instantly received at the controller. The timer registers t_2 exactly.

The initial state is now estimated from the exact knowledge of t_1 and t_2 by

$$\hat{x}_0 = \frac{h}{(e^{pt_2} - e^{pt_1})} = x_0. \tag{12}$$

Because there are no transmission delays or errors in t_1 and t_2, the estimate is also exact. No more samples are taken from $x(t)$ at this point.

The *peak* data-rate in bits per second required for the estimation of x_0 is

$$R_e = \frac{2}{(t_2 - t_1)}.$$

Since t_1 and t_2 are the sampling times at which $x(t)$ crosses the first two quantization levels, we have from (9) and (11) that

$$x(t_2) = e^{p(t_2-t_1)}x(t_1) \;\Leftrightarrow\; (k+1)h = e^{p(t_2-t_1)}kh,$$

where k is given by (10), and thus follows that

$$t_2 - t_1 = \frac{1}{p}\log\left(\frac{k+1}{k}\right), \tag{13}$$

and hence,

$$R_e = \frac{2p}{\log\left(\frac{k+1}{k}\right)}. \tag{14}$$

From Equation (14) we see that the peak data-rate R_e will be the lowest when the initial condition is such that it falls within the first quantization level, that is, $k = 1$. Larger initial conditions will require higher data rates to obtain the estimate. Such lowest data-rate is

$$R_e^{\min} = \frac{2p}{\log 2} = (\log_2 e)\,2p,$$

which coincides with the lowest *average* data rate required for stability for this system (Baillieul, 2002).

3.2 Finite-time control to the origin

At time t_2 we apply the constant control signal $u(t) = -U$. From the exact knowledge of $x_0 > 0$, we compute the time t_3 until which this control signal should be applied,

$$u(t) = \begin{cases} -U & \forall t : t_2 \le t < t_3 \\ 0 & \forall t : t \ge t_3, \end{cases} \tag{15}$$

to drive $x(t)$ to the origin at $t = t_3$, that is,

$$0 = x(t_3) = e^{pt_3} x_0 - \int_{t_2}^{t_3} e^{p(t_3-\tau)}\, d\tau U$$
$$= e^{pt_3} x_0 - (e^{p(t_3-t_2)} - 1)\, \frac{U}{p}$$
$$\Leftrightarrow t_3 = t_2 + \frac{1}{p} \log\left(\frac{1}{1 - pe^{pt_2}\hat{x}_0/U}\right), \qquad (16)$$

where in this case, $\hat{x}_0 = x_0$ from (12). We can see from (16) that since we need $t_3 \geq t_2$, then U should satisfy

$$U > pe^{pt_2} x_0 = ph(k+1) \qquad (17)$$

to be able to drive the state to the origin. Thus, the design of a suitable value U requires the prior specification of a bound on the possible initial conditions for the system.

Assuming that computation time is negligible and U satisfies (17) for all initial conditions in a predetermined set, at time t_2 we have the exact value of $x_0 > 0$, with which we compute t_3. We thus transmit a bit “0” through the actuator communication channel to indicate that the control action $u = -U$ should be applied from $t = t_2$ and then transmit a bit “1” at time t_3 to indicate that $u = 0$ should be applied from then on. The state has thus been driven to the origin with the transmission of two more bits (in addition to the two transmitted through the sensor channel for the estimation of x_0).

The resulting peak data-rate in bits per second for the control cycle is, from (16),

$$R_c = \frac{2}{t_3 - t_2} = \frac{2p}{\log\left(\frac{1}{1-ph(k+1)/U}\right)}. \qquad (18)$$

We see from (18) that, in contrast with the estimation case, the data-rate for control can be made *arbitrarily low* by designing U sufficiently close to $ph(k+1)$. For example, if we take $U = 2ph(k+1)$ we obtain

$$R_c = \frac{2p}{\log 2} = (\log_2 e)\, 2p = R_e^{\min}.$$

But if we take say

$$U = \left(\frac{2^m}{2^m - 1}\right) ph(k+1), \qquad (19)$$

with $m = 2, 3, 4, \ldots$, we obtain

$$R_c = \frac{2p}{m \log 2} = R_e^{\min}/m.$$

The larger the value of m, the “tighter” the resulting value of U and the lower the required data rate.

Of course, since x_0 is unknown, U cannot be predetermined accurately. We thus see that in achieving finite-time stability with low data-rates there is a trade-off between the minimum data-rate for control and the level of uncertainty in the bound that specifies the set of admissible initial conditions.

The combined estimation/control data-rate is then

$$R_{e/c} = \frac{4}{(t_3 - t_2) + (t_2 - t_1)} = \frac{4p}{\log\left(\frac{1}{1-ph(k+1)/U}\, \frac{k+1}{k}\right)}.$$

4. EFFECTS OF SAMPLING TIME QUANTIZATION

We will now consider that the times t_1, t_2, t_3 in the control strategy described in the previous section cannot be known exactly, but have to be *approximated* to a value in the set $\{0, T, 2T, 3T, \ldots\}$, for some suitable *time quantization interval* $T > 0$. The approximation of t_1, t_2, t_3 will affect the estimate $\hat{x}_0$, and thereby the computation of the control switch-off time t_3.

Denote by $\hat{t}_1$ and $\hat{t}_2$ the approximated values of t_1 and t_2. We then have that

$$\hat{t}_1 = \lceil t_1/T \rceil T, \qquad \hat{t}_2 = \lceil t_2/T \rceil T, \qquad (20)$$

where $\lceil t \rceil = \lfloor t \rfloor + 1$. Thus $\hat{t}_1, \hat{t}_2$ are multiples of T. Note that for the estimation procedure to work, we need that $\hat{t}_2 - \hat{t}_1 > T$, which is guaranteed by the following requirement, which we assume to hold throughout the present section.

Assumption 1. (Largest time quantisation interval T). For all initial conditions for the system (8) $x_0 : |x_0| < kh$, for some predetermined $k \geq 1, h > 0$, the time quantization interval $T > 0$ satisfies

$$t_2 - t_1 \geq 2T \Leftrightarrow T \leq \frac{1}{2p} \log\left(\frac{k+1}{k}\right). \qquad (21)$$

Assumption 1 means that the system evolution for all admissible initial conditions is such that the crossing of two levels of the system output will elapse for at least two time quantization intervals.

4.1 *Effects on the estimation of x_0*

Assuming that two samples of $x(t)$ have already been registered during the estimation cycle, now at the approximated sample times $\hat{t}_1, \hat{t}_2$, the formula (12) then yields the initial condition estimate

$$\hat{x}_0 = \frac{h}{e^{p\hat{t}_2} - e^{p\hat{t}_1}}. \qquad (22)$$

Since now the sample times $\hat{t}_1, \hat{t}_2$ only approximate the true sample values t_1, t_2 according to (20), there will be an error in the estimate $\hat{x}_0$. The following bounds are easily obtained from (12) and (22).

Lemma 2. (Error in $\hat{x}_0$). The error in the estimation $\hat{x}_0$ using (22) satisfies

$$-\frac{e^{pT} - 1}{e^{p(t_2-t_1)} - e^{pT}} \leq \frac{x_0 - \hat{x}_0}{x_0} \leq \frac{e^{pT} - 1}{e^{pT} - e^{-p(t_2-t_1)}}, \qquad (23)$$

where x_0, t_1, t_2 are the system initial condition and exact sample times.

4.2 *Effects on the final state*

The open-loop control procedure described in Section 3.2 will generally yield a nonzero final state $x_f = x(\hat{t}_3)$ after the application of the control signal

$$u(t) = \begin{cases} -U & \forall t : \hat{t}_2 \le t < \hat{t}_3 \\ 0 & \forall t : t \ge \hat{t}_3, \end{cases} \tag{24}$$

where $\hat{t}_3$ is computed from (16) using the estimate $\hat{x}_0$ from (22) and the registered sample time approximation $\hat{t}_2$,

$$\hat{t}_3 = \hat{t}_2 + \frac{1}{p} \log\left(\frac{1}{1 - pe^{p\hat{t}_2}\hat{x}_0/U}\right). \tag{25}$$

Depending on whether the computed estimate $\hat{x}_0$ is a lower or upper bound of x_0, the control action (24) will be insufficient or excessive in bringing the state to the origin. The following proposition gives bounds on the resulting final state $x(\hat{t}_3)$.

Proposition 3. (Bounds on final state). Consider the system (8) with initial state $x(0) = x_0$ such that $|x_0| < kh$ for some positive integer k, and let $T > 0$ satisfy (21). Then, if U in (24) is such that $U > ph(k+1)(e^{pT} + 1)$, the final state $x(\hat{t}_3)$ obtained after the application of one cycle of the LCS estimation/control procedure satisfies

$$\frac{-\beta}{(k+1)/k - e^{pT}} \le x(\hat{t}_3) \le \frac{\beta}{e^{pT} - k/(k+1)}, \tag{26}$$

where

$$\beta = \frac{h(k+1)\, e^{pT}(e^{pT} - 1)}{1 - ph(k+1)(e^{pT} + 1)/U}. \tag{27}$$

PROOF. First note that we can show from (25) that

$$\left(e^{p(\hat{t}_3 - \hat{t}_2)} - 1\right) \frac{U}{p} = e^{p\hat{t}_3}\hat{x}_0, \tag{28}$$

which then implies that

$$\begin{aligned} x(\hat{t}_3) &= e^{p\hat{t}_3} x_0 - \int_{\hat{t}_2}^{\hat{t}_3} e^{p(\hat{t}_3 - \tau)}\, U\, d\tau, \\ &= e^{p\hat{t}_3} x_0 - \left(e^{p(\hat{t}_3 - \hat{t}_2)} - 1\right) \frac{U}{p}, \\ &= e^{p\hat{t}_3}(x_0 - \hat{x}_0) = \frac{e^{p\hat{t}_2}(x_0 - \hat{x}_0)}{1 - pe^{p\hat{t}_2}\hat{x}_0/U}, \end{aligned} \tag{29}$$

where in the last step we have used the identity

$$e^{p\hat{t}_3} e^{-p\hat{t}_2} = \frac{1}{1 - pe^{p\hat{t}_2}\hat{x}_0/U},$$

which also follows from (25).

On the other hand, the lower bound of $(x_0 - \hat{x}_0)/x_0$ in (23), together with the fact that $e^{p(t_2 - t_1)} > e^{2pT}$ (from Assumption 1) implies that

$$\hat{x}_0 \le (1 + e^{-pT}) x_0,$$

which, in turn, yields (using $\hat{t}_2 \le t_2 + T$)

$$e^{p\hat{t}_2}\hat{x}_0 \le e^{pt_2} e^{pT}\hat{x}_0 \le e^{pt_2}(e^{pT} + 1) x_0. \tag{30}$$

Finally, the bounds (26) follow from using the bounds in (23) on the term $x_0 - \hat{x}_0$ in the numerator of (29), the bound (30) for $e^{p\hat{t}_2}\hat{x}_0$ in the denominator of (29), and the equalities $e^{p(t_2 - t_1)} = (k+1)/k$ and $e^{pt_2} x_0 = h(k+1)$ (which follow from (13) and (9)). □

5. EFFECTS OF BOUNDED MEASUREMENT NOISE

5.1 Effect on the initial state estimation

We consider again the system of Equation (8), but we now assume that the state measurement is affected by an unknown but bounded additive noise $\eta(t)$, $|\eta(t)| \le \overline{\eta}$. For simplicity, in contrast with the previous section, we will assume throughout this section that sample times are known exactly.

The measured output is

$$y(t) = x(t) + \eta(t) = e^{pt} x_0 + \eta(t). \tag{31}$$

The samples are now generated whenever $y(t)$—rather than $x(t)$—crosses the amplitude quantization levels. Although successive samples still satisfy

$$h = y(t_2) - y(t_1) = e^{p\, t_2} x_0 + \eta(t_2) - e^{p\, t_1} x_0 - \eta(t_1),$$

the computed estimation of the initial condition incurs an estimation error Δx_0,

$$\hat{x}_0 = \frac{h}{e^{p\, t_2} - e^{p\, t_1}} = x_0 - \Delta x_0, \tag{32}$$

which can be written as

$$\Delta x_0 = \frac{\eta(t_1) - \eta(t_2)}{e^{p\, t_2} - e^{p\, t_1}} = \frac{\eta(t_1) - \eta(t_2)}{h + \eta(t_1) - \eta(t_2)}\, x_0. \tag{33}$$

Notice from (32) and (33) that the amplitude quantization interval h must be at least twice the maximum noise amplitude $\overline{\eta}$. Otherwise, the estimation error could be infinite.

5.2 Error after one estimation/control cycle

At time t_2 the controller has the estimate $\hat{x}_0$ computed from (32), and thus the estimated state at time t_2 is

$$\hat{x}(t_2) = e^{p\, t_2}\hat{x}(0). \tag{34}$$

The applied control is aimed at driving the state to the origin in time t_3. However, due to the estimation error induced by the measurement noise, the true state will follow the solution

$$x(t_3) = e^{p(t_3 - t_2)} x(t_2) + \int_{t_2}^{t_3} e^{p(t-\tau)} u(\tau) d\tau,$$

where t_3 was calculated from (16) so that

$$e^{p(t_3 - t_2)}\hat{x}(t_2) - \int_{t_2}^{t_3} e^{p(t-\tau)} U d\tau = 0, \tag{35}$$

and thus the final state will deviate from the origin as

$$x(t_3) = e^{p(t_3 - t_2)}(x(t_2) - \hat{x}(t_2)) = e^{p(t_3 - t_2)}\Delta x_2. \tag{36}$$

Proposition 4. (Bound on final state). Consider the system (8) with measured output (31), with $|\eta(t)| < \overline{\eta}$, and initial state $x_0, |x_0| < kh$. Then, assuming that $U > ph^2(k+1)/(h - 2\overline{\eta})$, the final state error after one cycle of LCS estimation/control satisfies

$$|x(t_3)| \le \frac{1}{1 - \dfrac{ph^2(k+1)}{U(h - 2\overline{\eta})}} \frac{2\overline{\eta} h(k+1)}{h - 2\overline{\eta}}, \tag{37}$$

where t_3 is computed with (16) with $\hat{x}_0$ from (32).

PROOF. Since $x(t_2) = e^{pt_2}x_0$, operating on Equations (33) and (34), the estimation error at t_2 can be written as

$$\Delta x_2 = \frac{\eta(t_1) - \eta(t_2)}{h + \eta(t_1) - \eta(t_2)}\, x(t_2)$$
$$\Rightarrow |\Delta x_2| \leq \frac{2\overline{\eta}}{h - 2\overline{\eta}}\, |x(t_2)|. \quad (38)$$

Replacing (38) in (36), and using $e^{p(t_3-t_2)} = 1/(1 - px(t_2)/U)$ and $x(t_2) = e^{pt_2}x_0 = (k+1)h$ yields (37). □

5.3 Stability and ultimate bounds after successive LCS estimation/control cycles

When U is much larger than $ph^2(k + 1)/(h - 2\overline{\eta})$ in (37), after one cycle of estimation/control the state $|x_f| = |x(t_3)|$ is approximately bounded by $|\Delta x_2|$. If we take x_f as initial state for a new estimation/control cycle, and continue successively to drive the state near the origin, we should ask that $|x(t_3)| < |x(t_2)| - 2h$. Using the bound (37), we obtain the condition

$$|x(t_3)| \leq \frac{2\overline{\eta}}{h - 2\overline{\eta}}\, |x(t_2)| < |x(t_2)| - 2h,$$

and then it results that

$$|x(t_2)| > 2h\frac{h - 2\overline{\eta}}{h - 4\overline{\eta}}. \quad (39)$$

Thus, whenever the relationship (39) between $x(t_2)$, h and $\overline{\eta}$ is achieved, the control cycle will drive the state below $|x(t_2)| - 2h$ (i.e., at least one level below x_0). Of course, it is impossible to achieve this for $|x(t_2)| \leq 2h$, since we need at least two samples to estimate the state.

Notice also that to satisfy the stability condition (39) we need a more stringent condition $h > 4\overline{\eta}$ than that required for estimation $h > 2\overline{\eta}$ in Section 5.1.

The strongest stability condition that we can ask is that the state reaches the *first* quantization level after a number estimation/control cycles. To do that, we need the condition $|x(t_3)| < |x(t_2)| - 2h$ to be accomplished for $x(t_2) = 3h$ (when the second sample corresponds to the level $3h$, we must ensure that the control brings the state to the first quantization interval). This leads to the condition

$$|\Delta x_2| \leq \frac{2\overline{\eta}}{h - 2\overline{\eta}}\, |x(t_2)| = \frac{2\overline{\eta}}{h - 2\overline{\eta}}\, 3h < h$$

which yields the requirement $h > 8\overline{\eta}$.

Proposition 5. (Ultimate state bound and data rate). Under the conditions of Proposition 4, suppose further that $h/\overline{\eta} > 8$. Then, if n denotes the number of LCS estimation/control successive cycles,

$$\lim_{n\to\infty} |x(t)| < 2h + \overline{\eta}, \quad (40)$$

which is achieved with an *average* combined estimation/control data rate $R_{e/c}$ in bits per second that satisfies

$$\lim_{n\to\infty} R_{e/c} \leq \frac{4p}{\log\left(\frac{2h-\overline{\eta}}{2h+\overline{\eta}}\;\frac{h-2\overline{\eta}}{2\overline{\eta}}\right)}. \quad (41)$$

PROOF. Once the state reaches the first quantization interval, in successive cycles $|\hat{x}(t_2)| = 2h$ and thus $|x(t_2)| \leq 2h + \overline{\eta}$, which yields the ultimate bound (40). This implies that

$$|x(t_3)| \leq \frac{2\overline{\eta}}{h - 2\overline{\eta}}(2h + \overline{\eta})$$

From this worst case final state value, the minimum time for the output to reach the value $2h$ (i.e., to complete a new estimation cycle) is

$$\Delta t = \frac{1}{p}\log\left(\frac{2h - \overline{\eta}}{2h + \overline{\eta}}\;\frac{h - 2\overline{\eta}}{2\overline{\eta}}\right). \quad (42)$$

Since for U arbitrarily large the control action reduces to an pulse with very short duration, the lowest data rate will tend to be bounded by $4/\Delta t$ (2 bits for estimation + 2 bits for control, over a time lapse of at least Δt), which yields (41). □

Proposition 5 shows that the successive application of LCS estimation/control cycles can drive the state to a neighborhood of the origin, whose size is given by the level of noise and the quantization interval.

6. CONCLUSIONS

This paper presented preliminary results that relax assumptions in a level-crossing sampling control strategy introduced in Kofman and Braslavsky (2006). For simplicity, these results were presented for a first order system, We have shown that practical stability can be achieved with the proposed control strategy under quantized sampling time information and bounded measurement noise. Ultimate bounds and conditions for closed-loop stability have been derived. Extension of these results to higher order systems are currently under development.

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