NON LYAPUNOV ULTIMATE BOUND ESTIMATION

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***Abstract*— This paper presents a new method for estimating the ultimate bound in perturbed systems. The methodology, exploiting the perturbation structure and the system geometry, gives an implicit expression of the ultimate bound which can be solved by fixed point iterations. Thus, it is simpler than the classic Lyapunov analysis and, in many cases, less conservative.**

***Keywords*— Nonlinear Systems, Perturbed Systems, Ultimate Bound.**

I INTRODUCTION

The effects of perturbations are a common issue related to the study an analysis of dynamical systems. Perturbations could result from modeling errors, aging or uncertainties and disturbances which exist in any realistic problem (Khalil, 1996).

In a typical situation, the perturbation is unknown but it is supposed to be bounded in some way. In presence of *non-vanishing perturbations* –i.e., perturbations that do not disappear when the state go to the equilibrium point– asymptotic stability is not possible. However, under certain conditions, the *ultimately boundedness* of the trajectories can be guaranteed.

Non-vanishing perturbations can represent effects of quantization in A/D and D/A converters (Kofman, 2003), unknown disturbance signals (Koskouej and Zinober, 2000), unmodeled dynamics (Lee *et al.*, 1998), data rate limitations in networked control systems (Walsh *et al.*, 2002; Elia and Mitter, 2001), errors in numerical methods (Kofman, 2002*a*), etc. In all those problems, it is always important estimating the ultimate bound as a measure of the undesirable effect of the perturbations.

The usual tool for the estimation is based on the use of Lyapunov functions (Khalil, 1996), but it has the drawback that the resulting bound may be very conservative due to the loss of the system and perturbation structure during a generic analysis.

A different approach was introduced in (Kofman, 2002*b*) where the ultimate bound of a linear time invarying (LTI) system with a perturbation bounded by a constant was deduced based on geometrical principles. That study arrived to an explicit expression for the ultimate bound that, in the examples analyzed, was noticeably less conservative than what can be obtained with Lyapunov.

This work extends that approach to the LTI case where the perturbation is bounded by a function depending on the state and then to nonlinear systems. In both cases, the ultimate bound estimation is expressed as the solution of a fixed point problem. An estimation of the region of attraction is also provided by the methodology.

The examples analyzed illustrate the advantages of the methodology in terms of simplicity and the non conservative features of the estimation.

II PRELIMINARIES

A Notation

The symbol $|\cdot|$ will indicate the component-wise module of a matrix or vector. If T is a matrix with components $T_{1,1},\ldots,T_{n,m}$, then $|T|$ will be a new matrix of the same size than T with components $|T_{1,1}|,\ldots,|T_{n,m}|$.

For vectors having the same dimension, the vector inequality $x \leq y$ implies that $x_i \leq y_i$ for every component of x and y.

According to these definitions, it results that $|T \cdot x| \leq |T| \cdot |x|$.

B State Independent Perturbations

The analysis presented in (Kofman, 2002*b*) gives an estimation of the ultimate bound of a LTI system with state and input perturbations. Here, state perturbations will not be considered since they can be represented by equivalent input perturbations.

In order to deal with nonlinear systems, it will be also necessary to establish estimations of the region of attraction.

Thus, the results of (Kofman, 2002*b*) will be derived again taking into account the mentioned changes.

The following lemma and its corollary allow deducing Theorems 1 and 2, which establish the ultimate bound of a LTI system with input perturbations bounded component-wise by a constant vector.

Lemma 1. *Consider the following first order equation with complex coefficient*

$$\dot{z} = a \cdot z(t) + v(t) \tag{1}$$

where $a,\ z,\ v \in \mathbb{C}$. Assume that $\mathbb{R}e(a) < 0$, $|v(t)| \leq v_m \ \ \forall t \geq 0$ and define $z_m \triangleq |\mathbb{R}e(a)^{-1}| \cdot v_m$.

Then, the condition $|z(0)| \leq z_m$ implies that $|z(t)| \leq z_m$, $\forall t \geq 0$.

Proof. Let $z \triangleq \rho \cdot e^{j\theta}$ with $\rho,\ \theta \in \mathbb{R}$. Replacing and operating equation (1) becomes

$$\dot{\rho} + j\rho \cdot \dot{\theta} = a\rho + v \cdot e^{-j\theta}$$

Taking the real part it results that

$$\dot{\rho} = \mathbb{R}\mathrm{e}(a)\rho + \mathbb{R}\mathrm{e}(v \cdot e^{-j\theta}) \leq \mathbb{R}\mathrm{e}(a)\rho + v_m$$

Thus, when

$$\rho = |z(t)| = z_m$$

it results that $\dot{\rho} \leq 0$ and $|z(t)|$ cannot become greater than the given bound. □

Applying Lemma 1 to each component of a decoupled system, the following corollary is obtained

Corollary 1. *Consider the system*

$$\dot{z}(t) = \Lambda \cdot z(t) + v(t) \tag{2}$$

where $z,\ v \in \mathbb{C}^n$ and $\Lambda \in \mathbb{C}^{n\times n}$.

Assume that Λ is a diagonal matrix with $\mathbb{R}e(\Lambda_{i,i}) < 0$ and suppose that $|v(t)| \leq v_m$, $\forall t \geq 0$. Define $z_m \triangleq |\mathbb{R}e(\Lambda)^{-1}| \cdot v_m$.

Then, the condition $|z(0)| \leq z_m$ implies that $|z(t)| \leq z_m$, $\forall t \geq 0$.

The non–diagonal can be analyzed as follows.

Theorem 1. *Consider the system*

$$\dot{x}(t) = A \cdot x(t) + u(t) \tag{3}$$

where $x,\ u \in \mathbb{R}^n$, and $A \in R^{n\times n}$ is Hurwitz and diagonalizable.

Let Λ, V be a pair of eigenvalues and eigenvectors matrices, so that $\Lambda = V^{-1} A \cdot V$ is diagonal.

Suppose that $|u(t)| \leq u_m$. Then, the condition

$$|V^{-1} \cdot x(0)| \leq |\mathbb{R}e(\Lambda)^{-1}| \cdot |V^{-1}| \cdot u_m$$

implies that

$$|x(t)| \leq |V| \cdot |\mathbb{R}e(\Lambda)^{-1}| \cdot |V^{-1}| \cdot u_m \quad \forall t \geq 0$$

Proof. Let $x(t) = V \cdot z(t)$. Then, system (3) becomes

$$\dot{z}(t) = \Lambda \cdot z(t) + V^{-1}u(t)$$

which has the form of (2) with $v(t) \triangleq V^{-1}u(t)$. Notice also that

$$|v(t)| \leq |V^{-1}| \cdot u_m$$

According to Corollary 1, the condition

$$|z(0)| = |V^{-1}x(0)| \leq |\mathbb{R}\mathrm{e}(\Lambda)^{-1}| \cdot |V^{-1}| \cdot u_m$$

implies that

$$|z(t)| \leq |\mathbb{R}\mathrm{e}(\Lambda)^{-1}| \cdot |V^{-1}| \cdot u_m \ \ \forall t \tag{4}$$

and then

$$|x(t)| \leq |V| \cdot |z(t)| \leq |V| \cdot |\mathbb{R}\mathrm{e}(\Lambda)^{-1}| \cdot |V^{-1}| \cdot u_m$$

which completes the proof. □

This last result can be extended to arbitrary initial conditions as follows

Theorem 2. *Consider system (3) under the same hypothesis of Theorem 1.*

Then, for any arbitrary initial condition x_0 and a positive vector $\epsilon \in \mathbb{R}^n$, a finite time t_1 exists so that

$$|x(t)| \leq |V| \cdot |\mathbb{R}e(\Lambda)^{-1}| \cdot |V^{-1}| \cdot u_m + \epsilon \quad \forall t > t_1 \tag{5}$$

Proof. Consider the system

$$\dot{y}(t) = A \cdot y(t) \tag{6}$$

with $y(0) = x(0) = x_0$. Since A is Hurwitz, $lim_{t\to\infty} y(t) = 0$ and then a finite time t_1 exists so that

$$|y(t)| \leq \epsilon \quad \forall t > t_1$$

Define $\tilde{x}(t) \triangleq x(t) - y(t)$. Then, subtracting (6) from (3) we have

$$\dot{\tilde{x}}(t) = A \cdot \tilde{x}(t) + u(t)$$

with $\tilde{x}(0) = 0$. Thus, this last system verifies Theorem 1 and it results that

$$\tilde{x}(t) \leq |V| \cdot |\mathbb{R}\mathrm{e}(\Lambda)^{-1}| \cdot |V^{-1}| \cdot u_m \quad \forall t \geq 0$$

and

$$\begin{aligned} |x(t)| &\leq |\tilde{x}(t)| + |y(t)| \\ &\leq |V| \cdot |\mathbb{R}\mathrm{e}(\Lambda)^{-1}| \cdot |V^{-1}| \cdot u_m + \epsilon \end{aligned}$$

$\forall t > t_1$ completing the proof □

These theorems give explicit estimations of the ultimate bound of LTI systems where the perturbation bound does not depend on the state.

A less conservative estimation can be obtained from (4), which implies that

$$|V^{-1}x(t)| \leq |\mathbb{R}\mathrm{e}(\Lambda)^{-1}| \cdot |V^{-1}| \cdot u_m \ \ \forall t \tag{7}$$

Although it is less conservative, the explicit form is lost in this inequality.

III MAIN RESULTS

A State Dependent Perturbations (LTI)

Theorem 3. *Consider system* (3) *under the same hypothesis than before, except that now*

$$|u(t)| \leq \delta(x(t)) \quad \forall t \geq 0$$

where $\delta : \mathbb{R}^n \to \mathbb{R}^n$ *is a continuous map verifying*

$$|x_1| < |x_2| \Rightarrow \delta(x_1) < \delta(x_2) \tag{8}$$

Define

$$T(x) \triangleq |V| \cdot |\mathbb{R}e(\Lambda)^{-1}| \cdot |V^{-1}| \cdot \delta(x) \tag{9}$$

Let us suppose that $T(x)$ *is a contraction mapping in an open region* D *and let* b *be its unique fixed point in that region, i.e.,* $b = T(b)$*. Then,*

1. If $|V^{-1}x(0)| \leq |\mathbb{R}e(\Lambda)^{-1}| \cdot |V^{-1}| \cdot \delta(b)$ *it results that* $|x(t)| \leq b \ \forall t \geq 0$*.*

2. Let $x_m \in D$ *so that* $T(x_m) < |x_m|$*. If* $|V^{-1}x(0)| \leq |\mathbb{R}e(\Lambda)^{-1}| \cdot |V^{-1}| \cdot \delta(x_m)$ *then, given a positive vector* $\epsilon \in \mathbb{R}^n$*, a finite time* t_f *exists so that*

$$|x(t)| < b + \epsilon \quad \forall t > t_f$$

Proof. Let us define $x_0 \triangleq x(0)$.

1. First of all notice that

$$\begin{aligned} |x_0| &= |V \cdot V^{-1}x_0| \leq |V| \cdot |V^{-1}x_0| \\ &\leq |V| \cdot |\mathbb{R}\mathrm{e}(\Lambda)^{-1}| \cdot |V^{-1}| \cdot \delta(b) = T(b) = b \end{aligned}$$

Let $c > 0$ be a scalar sufficiently small so that $(1+c)b \in D$. Let $t_c > 0$ be the first instant of time in which the inequality $|x(t)| \leq (1+c)b$ becomes false. Then, in the interval $(0, t_c)$ we have

$$|x(t)| \leq (1+c)b \Rightarrow |u(t)| \leq \delta[(1+c)b]$$

Applying Theorem 1, it implies that

$$|x(t)| \leq |V| \cdot |\mathbb{R}\mathrm{e}(\Lambda)^{-1}| \cdot |V^{-1}| \cdot \delta[(1+c)b] = T[(1+c)b]$$

in $(0, t_c)$.

Since T is contractive, $T[(1+c)b] \in D$. Moreover, the property of $\delta(x)$ given by (8) implies that $b = T(b) < T[(1+c)b]$ and then

$$\begin{aligned} |V^{-1}x(0)| &\leq |\mathbb{R}\mathrm{e}(\Lambda)^{-1}| \cdot |V^{-1}| \cdot \delta(b) \\ &< |\mathbb{R}\mathrm{e}(\Lambda)^{-1}| \cdot |V^{-1}| \cdot \delta(T[(1+c)b]) \end{aligned}$$

Thus, we can repeat the analysis concluding that

$$|x(t)| < T(T[(1+c)b]) \quad \forall 0 \leq t < t_c$$

The recursive use of this procedure concludes that

$$|x(t)| < b \quad \forall t \in [0, t_c)$$

since b is the fixed point of T in D. Then, the continuity of $x(t)$ ensures that $|x(t_c)| \leq (1+c)b$ for any positive constant c contradicting the initial assumption that in t_c the inequality was false.

2. Let $c > 0$ be a scalar sufficiently small so that $(1+c)T(x_m) \leq |x_m|$. We have,

$$\begin{aligned} |x_0| &= |V \cdot V^{-1}x_0| \leq |V| \cdot |V^{-1}x_0| \\ &\leq |V| \cdot |\mathbb{R}\mathrm{e}(\Lambda)^{-1}| \cdot |V^{-1}| \cdot \delta(x_m) = T(x_m) \\ &\leq \frac{|x_m|}{1+c} \end{aligned}$$

Let $t_c > 0$ be the first instant of time in which the inequality $|x(t)| \leq |x_m|$ becomes false. Thus, in $[0, t_c)$ we have

$$|x(t)| \leq |x_m| \Rightarrow |u(t)| \leq \delta(x_m)$$

Applying Theorem 1, it implies that

$$|x(t)| \leq T(x_m) \leq \frac{|x_m|}{1+c}$$

in $[0, t_c)$. Then, we can ensure that $|x(t_c)| \leq |x_m|$ contradicting the initial assumption. Thus,

$$|x(t)| \leq |x_m| \quad \forall t \geq 0$$

and $|u(t)| \leq \delta(x_m) \forall t \geq 0$

Then, Theorem 2 concludes that, given a positive vector $\gamma \in \mathbb{R}^n$, a finite time t_1 exist so that

$$\begin{aligned} |x(t)| &\leq |V| \cdot |\mathbb{R}\mathrm{e}(\Lambda)^{-1}| \cdot |V^{-1}| \cdot \delta(x_m) + \gamma \\ &= T(x_m) + \gamma \quad \forall t > t_1 \end{aligned}$$

Then, when $t > t_1$ it is also true that $|u(t)| \leq \delta(T(x_m) + \gamma)$. Applying again Theorem 2 we conclude that a positive time t_2 exists so that

$$|x(t)| \leq T(T(x_m) + \gamma) + \gamma \quad \forall t > t_1 + t_2$$

Defining $T_\gamma(x) \triangleq T(x) + \gamma$ and $T_\gamma^{(k)} = T_\gamma(T_\gamma^{(k-1)})$ the recursive use of this procedure arrives at

$$|x(t)| \leq T_\gamma^{(k)}(x_m) \ \ \forall t > \sum_{i=1}^{k} t_i$$

Since $T_\gamma(x_m) = T(x_m) + \gamma$ and $T(x_m) < x_m$, taking γ enough small it results that

$$T(x_m) < T_\gamma(x_m) < x_m$$

The property of T ensures also that

$$\begin{aligned} T(T(x_m)) &< T(T_\gamma(x_m)) < T(x_m) \Rightarrow \\ T(T(x_m)) &< T(T_\gamma(x_m)) + \gamma < T(x_m) + \gamma \end{aligned}$$

and

$$T^{(2)}(x_m) < T_\gamma^{(2)}(x_m) < T_\gamma(x_m)$$

Thus, applying this recursively we have that

$$b < T^{(k)}(x_m) < T_\gamma^{(k)}(x_m) < T_\gamma^{(k-1)}(x_m)$$

Clearly, the succession $T_\gamma^{(k)}(x_m)$ is monotonic decreasing and has a lower bound b given by the fixed point of T. Thus, it converges to certain point b_γ.

The continuity of T also ensures that

$$\lim_{\gamma \to 0} b_\gamma = b$$

Then, given $\epsilon > 0$, a value of $\gamma > 0$ can be found so that $b_\gamma < b + \epsilon/2$.

On the other hand, as

$$\lim_{k \to \infty} T_\gamma^{(k)}(x_m) = b_\gamma$$

given $\epsilon > 0$, a natural number N can be found so that

$$T_\gamma^{(k)}(x_m) < b_\gamma + \frac{\epsilon}{2} \quad \forall k \geq N$$

Then,

$$|x(t)| \leq T_\gamma^{(N)}(x_m) < b_\gamma + \frac{\epsilon}{2} < b + \epsilon \quad \forall t > \sum_{i=1}^{N} t_i$$

which completes the proof. □

A less conservative estimation of the ultimate bound can be obtained using (7), which yields,

$$|V^{-1}x(t)| \leq |\mathbb{R}\mathrm{e}(\Lambda)^{-1}| \cdot |V^{-1}| \cdot \delta(b) \ \ \forall t$$

Also, the estimation of the region of attraction might result conservative. Notice that when $\delta(x)$ is small, the estimation is also small. However a function like $\delta(x) + \delta_1$ with $\delta_1 > 0$ is also a bound for $|u(t)|$ and it can result in a bigger estimation of the region.

In fact, if a positive vector δ_1 satisfies

$$|V| \cdot |\mathbb{R}\mathrm{e}(\Lambda)^{-1}| \cdot |V^{-1}| \cdot \delta_1 < x_m - T(x_m)$$

then we can use $\delta(x)+\delta_1$ instead of $\delta(x)$ to prove that $|x(t)| \leq x_m \ \ \forall t \geq 0$ and then the region of attraction is defined by

$$|V^{-1}x(0)| \leq |\mathbb{R}\mathrm{e}(\Lambda)^{-1}| \cdot |V^{-1}| \cdot (\delta(x_m) + \delta_1)$$

B Application to Nonlinear Systems

Consider a nonlinear system

$$\dot{x}(t) = f(x(t), u(t))$$

where the origin with $u(t) = 0$ is an exponentially stable equilibrium point.

Let us define

$$A \triangleq \left. \frac{\partial f}{\partial x} \right|_{(0,0)}$$

which is Hurwitz and we will suppose that it is also diagonalizable.

We can rewrite,

$$\dot{x}(t) = Ax(t) + (f(x(t), u(t)) - Ax(t))$$

If we can find a function $\delta(x)$ so that

$$|f(x(t), u(t)) - Ax(t)| \leq \delta(x)$$

then we can use Theorem 3 to estimate the ultimate bound and the region of attraction.

IV EXAMPLES

A Pendulum with perturbations

The system

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -sin(x_1) - 10 \cdot x_2 + \tau(t) \end{aligned}$$

represents the dynamics of a pendulum with a big friction coefficient. We shall assume that $\tau(t)$ is a perturbation bounded by $|\tau(t)| \leq \tau_m$.

The system can be rewritten as

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -1 & -10 \end{bmatrix} \cdot x + \begin{bmatrix} 0 \\ x_1 - sin(x_1) + \tau(t) \end{bmatrix}$$

which has the form $A \cdot x + u(t)$. The perturbation term can be bounded by

$$|u(t)| \leq \delta(x) \triangleq \begin{bmatrix} 0 \\ \tau_m + \frac{|x_1|^3}{6} \end{bmatrix}$$

Defining $R \triangleq |V| \cdot |\mathbb{R}\mathrm{e}\Lambda^{-1}| \cdot |V^{-1}|$, the map T, according to (9), is $T(x) = R \cdot \delta(x)$. Its fixed point can be calculated as

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} R_{12}(\tau_m + \frac{|x_1|^3}{6}) \\ R_{22}(\tau_m + \frac{|x_1|^3}{6}) \end{bmatrix}$$

Let us suppose that $\tau_m = 0.1$. Then the least positive solution of the equation is given by $x_1 = 0.102243$ and $x_2 = 0.020448$. If that point is inside a region in which $T(x)$ is a contraction then

$$|x(t)| \leq \begin{bmatrix} 0.102243 \\ 0.020448 \end{bmatrix}$$

The contraction property of map T can be easily verified for $0 \leq x_1 < 1.95$ and for all x_2. Thus, the ultimate bound calculated holds.

An even less conservative estimation is given by

$$|V^{-1}x(t)| \leq |\mathbb{R}\mathrm{e}(\Lambda)^{-1}| \cdot |V^{-1}| \cdot \delta(b) = \begin{bmatrix} 0.1017 \\ 0.0103 \end{bmatrix}$$

Let us try to obtain a similar result with Lyapunov.

A quadratic Lyapunov function with level surfaces that fit the form of the ultimate bound previously obtained is

$$U(x) = x_1^2 + 5x_2^2 + x_1x_2$$

with derivative

$$\begin{aligned} \dot{U}(x) &= -x_1 sin(x_1) - 99x_2^2 - 8x_1x_2 - \\ &\quad -10x_2 sin(x_1) + (x_1 + 10x_2)\tau(t) \end{aligned}$$

In this case, if we put $x_1 = 0.1462$ and $x_2 = -0.083$ we get that $U(x) = 0.0205$ and $\dot{U}(x) = 2.02 \times 10^{-6} >$

0 and it is impossible to ensure that the solutions finish inside the region $\{x|U(x) \le 0.0205\}$.

The maximum value of x_1 on $U(x) = 0.0205$ is 0.1469 while de maximum value of x_2 is 0.0657 which are both considerable bigger than what was obtained with the new method. Figure 1 compares the bounds.

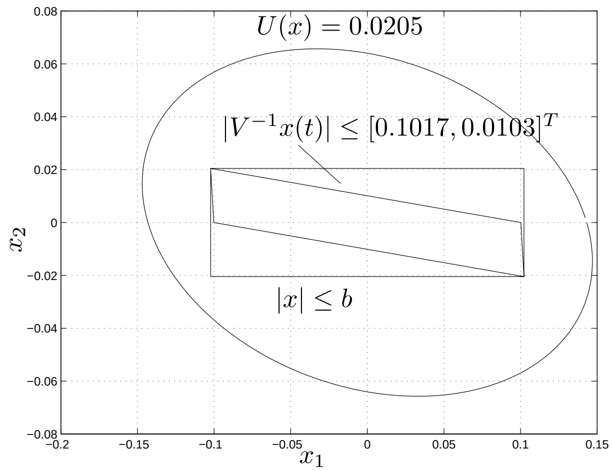

Figure 1: Lyapunov and contraction bounds in the pendulum system

B Khalil's example

System

$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= -4 \cdot x_1 - 2 \cdot x_2 + \beta x_2^3 + d(t)\end{aligned}$$

where $0 \le \beta \le 0.4(1-\xi)/c$ and $|d(t)| \le d_m$ was introduced in (Khalil, 1996) to establish its ultimate bound. The result was

$$\|x\|_2 \le \frac{\sqrt{29\lambda_{\max}(P)}d_m}{8\psi\theta\sqrt{\lambda_{\min}(P)}} \qquad (10)$$

where $x^T P x$ is the Lyapunov function of the nominal system and $0 < \theta < 1$.

For parameters $\xi = 0.5$, $d_m = 1$, and $c = 2.75$ the minimum bound (making $\theta = 1$) using the Lyapunov function provided there is 3.026 (in norm 2).

The new methodology, in this case, gives $b = [0.5777 \quad 1.1554]^T$ (using Matlab's eigenvalues and eigenvectors). The bound is again sensibly better.

Besides being more conservative, finding the appropriate Lyapunov function and evaluating its derivative along several level surfaces until being sure that it is negative is a difficult task, even for a simple second order system like the one studied. Any attempt to generalize this Lyapunov analysis obtaining a kind of formula would imply the lost of the problem structure yielding more conservative results. Eq.(10) is a typical expression of this sort (with the ratio of the eigenvalues of P) and, as it was shown in (Kofman, 2002*b*) and verified in the last example, it is conservative.

On the contrary, the new methodology can be easily programmed (it takes a few lines of Matlab code). Besides some matrix operations, the calculation of the bound only involves the solution of an equation of the form $b = T(b)$ which can be solved by fixed point iterations. Moreover, if the iterations start from a point in which $T(b) < b$, the value obtained after any number of iterations gives an estimation of the ultimate bound.

V CONCLUSIONS

A new method to estimate the ultimate bound of linear and nonlinear systems was presented. The method provides an alternative to the classic Lyapunov based analysis, resulting simpler and –sometimes– less conservative.

A disadvantage of the methodology, which should be consider for future research, is that it is unable to exploit the presence of stabilizing nonlinear terms (it treats any nonlinearity as perturbation).

Another case that was not yet considered is the non–diagonalizable one.

Finally, applications of the method to different practical problems should be studied. Particularly, we believe that its use to study the effects of quantization in sampled data control systems can improve the existing results, which are mainly based on Lyapunov theory (Ishii and Francis, 2003; Bullo and Liberzon, 2003).

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