On local normal forms of completely integrable systems

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Abstract

The purpose of this talk is to show that a differential system on $\mathbb{R}^n$ which admits a set of $n-1$ independent conservation laws defined on an open subset $\Omega \subseteq \mathbb{R}^n$, is essentially equivalent on an open and dense subset $U \subseteq \Omega$, with the linear differential system
$u_1' = u_1, \ u_2' = u_2, \ldots, \ u_n' = u_n$.
Consequently, the core of the (local) complete integrability is actually the integrability of the simplest linear differential equation, namely $u' = u$.
Another consequence of this result is that locally, a completely integrable differential system admits symmetries.

Completely integrable systems

Let us consider a differential system on $\mathbb{R}^n$:

$$\begin{cases} \dot{x}_1 = X_1(x_1, \ldots, x_n) \\ \dot{x}_2 = X_2(x_1, \ldots, x_n) \\ \ldots \\ \dot{x}_n = X_n(x_1, \ldots, x_n), \end{cases} \tag{1}$$

where $X_1, X_2, \ldots, X_n \in \mathscr{C}^\infty(\mathbb{R}^n, \mathbb{R})$ are arbitrary real functions.

Suppose that $C_1, \ldots, C_{n-2}, C_{n-1} : \Omega \subseteq \mathbb{R}^n \to \mathbb{R}$ are $n-1$ independent $\mathscr{C}^\infty$ integrals of motion of (1) defined on a nonempty open subset $\Omega \subseteq \mathbb{R}^n$.

Since $C_1, \ldots, C_{n-2}, C_{n-1} : \Omega \subseteq \mathbb{R}^n \to \mathbb{R}$ are integrals of motion of the vector field $X = X_1 \partial_{x_1} + \cdots + X_n \partial_{x_n} \in \mathfrak{X}(\mathbb{R}^n)$, we obtain that for each $i \in \{1, \ldots, n-1\}$

$$\mathscr{L}_X C_i = \langle \nabla C_i(x), X(x) \rangle = \sum_{j=1}^{n} \partial_{x_j} C_i \cdot \dot{x}_j = 0,$$

for every $x = (x_1, \ldots, x_n) \in \Omega$, where $\langle \cdot, \cdot \rangle$ stand for the canonical inner product on $\mathbb{R}^n$, and respectively ∇ stand for the gradient with respect to $\langle \cdot, \cdot \rangle$.

Hence, the vector field X can be written as the vector field $\star(\nabla C_1 \wedge \cdots \wedge \nabla C_{n-1})$ multiplied by a real function (rescaling function), where $\star$ stand for the Hodge star operator for multivector fields.

It may happen that the domain of definition for the rescaling function to be a proper subset of Ω. In the following we will consider the generic case when the rescaling function is defined on an open and dense subset of Ω. In order to simplify the notations, we will also denote this set by Ω.

Completely integrable systems

In coordinates, the completely integrable system $\dot{x} = X(x)$ can be written as:

$$\begin{cases} \dot{x}_1 = \nu \cdot \dfrac{\partial(C_1, \dots, C_{n-1}, x_1)}{\partial(x_1, \dots, x_n)} \\ \dot{x}_2 = \nu \cdot \dfrac{\partial(C_1, \dots, C_{n-1}, x_2)}{\partial(x_1, \dots, x_n)} \\ \dots \\ \dot{x}_n = \nu \cdot \dfrac{\partial(C_1, \dots, C_{n-1}, x_n)}{\partial(x_1, \dots, x_n)}. \end{cases} \tag{2}$$

where ν is the rescaling function.

A Poisson structure on a manifold P is a bilinear map (Poisson bracket)

$$\{\cdot,\cdot\}:\mathscr{C}^\infty(P,\mathbb{R})\times\mathscr{C}^\infty(P,\mathbb{R})\to\mathscr{C}^\infty(P,\mathbb{R})$$

that defines a Lie algebra structure on $\mathscr{C}^\infty(P,\mathbb{R})$ and that is a derivation in each entry.

The elements in the center of this algebra are called *Casimir* functions.

A pair consisting of a smooth manifold P together with a Poisson structure, is called a *Poisson manifold* and is usually denoted by $(P,\{\cdot,\cdot\})$.

The derivation property of a Poisson structure on P allows us to assign to each function $H \in \mathscr{C}^\infty(P,\mathbb{R})$ a vector field $X_H \in \mathscr{X}(P)$ via the equality

$$X_H(F) := \{F,H\} \quad \text{for every} \quad F \in \mathscr{C}^\infty(P,\mathbb{R}).$$

The vector field $X_H \in \mathscr{X}(P)$ is called the *Hamilton-Poisson vector field* or the *Hamiltonian vector field* associated to the *Hamiltonian function* H. The triple $(P,\{\cdot,\cdot\},H)$ is called *Hamilton-Poisson dynamical system*.

The derivation property of the Poisson bracket also implies that for any two functions $f,g \in \mathscr{C}^\infty(P,\mathbb{R})$, the value of the bracket $\{f,g\}(z)$ at an arbitrary point $z \in P$ depends on f only through $df(z)$ which allows us to define a contravariant antisymmetric two-tensor $\Pi \in \Lambda^2(P)$ by

$$\Pi(z)(\alpha_z,\beta_z) = \{f,g\}(z),$$

where $df(z) = \alpha_z \in T_z^*P$ and $dg(z) = \beta_z \in T_z^*P$. This tensor is called *Poisson tensor* or *Poisson structure* on P.

Hamilton-Poisson dynamical systems

The vector bundle map $\Pi^\sharp : T^*P \to TP$ naturally associated to Π is defined by

$$\Pi(z)(\alpha_z, \beta_z) = \left\langle \alpha_z, \Pi^\sharp(\beta_z) \right\rangle.$$

Note that for any $H \in \mathscr{C}^\infty(P, \mathbb{R})$ the Hamiltonian vector field X_H verifies that $X_H = \Pi^\sharp(dH)$.

Hamilton-Poisson dynamical systems

If one denote by n the dimension of the manifold P, the expression of the Poisson tensor relative to a local coordinates system $(x_1, ..., x_n)$ is given by the bivector field

$$\Pi = \sum_{1 \le k < \ell \le n} \Pi^{k\ell}(x_1, \ldots, x_n) \partial_{x_k} \wedge \partial_{x_\ell},$$

where $\Pi^{k\ell}(x_1, \ldots, x_n) := \{x_k, x_\ell\}$.

If there is no danger of confusion, the skew-symmetric matrix $\Pi := [\Pi^{k\ell}(x_1, \ldots, x_n)]_{1 \le k < \ell \le n}$ is also called the Poisson structure.

Hamilton-Poisson dynamical systems

Using the local expression of the map $\Pi^\sharp$ one get the local expression of the Hamiltonian vector field associated to a function $H \in \mathscr{C}^\infty(P,\mathbb{R})$ be given by

$$X_H = \sum_{1\leq i\leq n}\left(\sum_{1\leq j\leq n}\Pi^{ij}\partial_{x_j}H\right)\partial_{x_i}.$$

Hamilton-Poisson realization of completely integrable systems

Recall that the vector field X generating the completely integrable system (1) is proportional to the vector field

$$\star(\nabla C_1 \wedge \cdots \wedge \nabla C_{n-1})$$

where $\star$ stand for the Hodge star operator for multivector fields.

Consequently, the vector field X can be realized on the open set $\Omega \subseteq \mathbb{R}^n$ as the Hamilton-Poisson vector field $X_H \in \mathfrak{X}(\Omega)$ with respect to the Hamiltonian function $H := C_{n-1}$ and respectively the Poisson bracket defined by:

$$\{f, g\}_{\nu; C_1, \dots, C_{n-2}} dx_1 \wedge \cdots \wedge dx_n = \nu dC_1 \wedge \dots dC_{n-2} \wedge df \wedge dg,$$

where ν is a given real function (rescaling).

Hamilton-Poisson realization of completely integrable systems

Equivalently, the above Poisson bracket $\{\cdot,\cdot\}_{\nu;C_1,\dots,C_{n-2}}$ is given by:

$$\{f,g\}_{\nu;C_1,\dots,C_{n-2}} = \nu \cdot \frac{\partial(C_1,\dots,C_{n-2},f,g)}{\partial(x_1,\dots,x_n)}.$$

The functions $C_1,\dots,C_{n-2}$ form a complete set of Casimirs for the Poisson bracket $\{\cdot,\cdot\}_{\nu;C_1,\dots,C_{n-2}}$.

Hamilton-Poisson realization of completely integrable systems

The Hamiltonian vector field $X = X_H$ is acting on a smooth real function as:

$$X_H(f) = \{f, H\}_{\nu; C_1, \ldots, C_{n-2}}.$$

Hamilton-Poisson realization of completely integrable systems

Hence, the completely integrable system (1) can be locally written in Ω as a Hamilton-Poisson dynamical system of the type:

$$\begin{cases} \dot{x}_1 = \{x_1, H\}_{\nu; C_1, \dots, C_{n-2}} \\ \dot{x}_2 = \{x_2, H\}_{\nu; C_1, \dots, C_{n-2}} \\ \cdots \\ \dot{x}_n = \{x_n, H\}_{\nu; C_1, \dots, C_{n-2}}. \end{cases}$$

Hamilton-Poisson realization of completely integrable systems

Equivalently the above system can be written as:

$$\begin{cases} \dot{x}_1 = \nu \cdot \dfrac{\partial(C_1, \dots, C_{n-2}, x_1, H)}{\partial(x_1, \dots, x_n)} \\ \dot{x}_2 = \nu \cdot \dfrac{\partial(C_1, \dots, C_{n-2}, x_2, H)}{\partial(x_1, \dots, x_n)} \\ \dots \\ \dot{x}_n = \nu \cdot \dfrac{\partial(C_1, \dots, C_{n-2}, x_n, H)}{\partial(x_1, \dots, x_n)}. \end{cases} \tag{3}$$

Divergence free vector field associated with the vector filed X

Next result gives a method to construct a divergence free vector field out of the vector field X.
The divergence operator we will use in this approach is the divergence associated with the standard Lebesgue measure on $\mathbb{R}^n$, namely

$$\mathscr{L}_X(dx_1 \wedge \cdots \wedge dx_n) = (\operatorname{div} X) dx_1 \wedge \cdots \wedge dx_n.$$

Theorem

The vector field $\widetilde{X} := \frac{1}{\nu} \cdot X$ *is a divergence free vector field on* $\Omega \setminus \mathscr{Z}(\nu)$, *where* $\mathscr{Z}(\nu) = \{(x_1, \ldots, x_n) \in \Omega \mid \nu(x_1, \ldots, x_n) = 0\}$.

Generic Hypothesis

Assume that there exists a $\mathscr{C}^\infty$ rescaling function ν such that the Lebesgue measure of the set

$$\mathscr{O} := \left\{ x \in \Omega_0 \mid \operatorname{div}(X)(x) \cdot \frac{\partial(1/\nu, C_1, \ldots, C_{n-2}, H)}{\partial(x_1, \ldots, x_n)}(x) = 0 \right\}$$

in Ω_0 is zero.

Main Theorem

In the above hypothesis, the change of variables $((x_1,\ldots,x_n),t) \mapsto ((u_1,\ldots,u_n),s)$ given by

$$\begin{cases} u_1 = 1/v(x_1,\ldots,x_n) \\ u_2 = C_1(x_1,\ldots,x_n)/v(x_1,\ldots,x_n) \\ \ldots \\ u_{n-1} = C_{n-2}(x_1,\ldots,x_n)/v(x_1,\ldots,x_n) \\ u_n = H(x_1,\ldots,x_n)/v(x_1,\ldots,x_n) \\ ds = -\operatorname{div}(X)dt, \end{cases}$$

in the open and dense subset $\Omega_{00} := \Omega_0 \setminus \mathscr{O}$ of Ω_0, transforms the completely integrable system (3) restricted to Ω_{00} into the linear differential system

$$u_1' = u_1,\ u_2' = u_2, \ldots,\ u_n' = u_n,$$

where "prime" stand for the derivative with respect to the new time "s".

Local normal form of a completely integrable system

Consequently we obtained that the core of the complete integrability is the integrability of the simplest linear differential equation $\dot{u} = u$.

Another consequence is that the interesting dynamical behavior of a completely integrable system is generated generically by a set of zero Lebesgue measure.

If the rescaling function $\nu =: \nu_{cst.}$ is a constant function, then the Lebesgue measure of the set $\mathscr{O} = \Omega_0 = \Omega$ is nonzero in Ω_0, and hence the assumptions of the main theorem do not hold. In this case, we search for a new $\mathscr{C}^\infty$ rescaling function μ defined on an open and dense subset Ω_0 of Ω, such that the vector field $\mu \cdot X$ satisfies the assumptions of the main theorem.

The function μ satisfies:

$$\text{div}(\mu \cdot X) = \langle \nabla \mu, X \rangle + \mu \cdot \text{div}(X) = \langle \nabla \mu, X \rangle,$$

since in the case of a constant function $\nu = \nu_{cst.}$ we have that

$$0 = \text{div}(1/\nu_{cst.} \cdot X) = (1/\nu_{cst.}) \cdot \text{div}(X),$$

and hence $\text{div}(X) = 0$.
Consequently, we have to search for a rescaling function μ such that

$$\text{div}(\mu \cdot X) = \langle \nabla \mu, X \rangle$$

it is not identically zero in Ω_0.

The second condition that μ has to satisfy is that the function

$$\frac{\partial(1/\mu, C_1, \ldots, C_{n-2}, H)}{\partial(x_1, \ldots, x_n)}$$

it is not identically zero in Ω_0.

Local normal form of a completely integrable system

The transformation between the differential system (1) and the differential system generated by the vector field $\mu \cdot X$ is done by using the new time transformation $dt = \mu(x)dt'$. More exactly, the differential system (1) generated by X, namely:

$$\frac{dx}{dt} = X(x),$$

is transformed via the new time transformation $dt = \mu(x)dt'$ into the system:

$$\frac{dx}{dt'} = \mu(x) \cdot X(x). \tag{4}$$

The differential system (4) can also be realized as a Hamilton-Poisson dynamical system with respect to the Poisson bracket $\{\cdot,\cdot\}_{\mu\cdot\nu_{cst.};C_1,\ldots,C_{n-2}}$ and respectively the same Hamiltonian H as for the Hamilton-Poisson realization (3) of the vector field X.

One of the most important results in the literature that study the implication "symmetry $\Rightarrow$ integrability" in the context of a general dynamical system is a classical result due to Lie which says that if one have n linearly independent vector fields $X_1, \ldots, X_n$ on $\mathbb{R}^n$ that generates a solvable Lie algebra under commutation:
$[X_1, X_j] = c^1_{1,j} X_1$, $[X_2, X_j] = c^1_{1,j} X_1 + c^2_{2,j} X_2$, $\ldots$,
$[X_n, X_j] = c^1_{1,j} X_1 + c^2_{2,j} X_2 + \cdots + c^n_{n,j} X_n$, for $j \in \{1, \ldots, n\}$, where $c^k_{i,j}$ are the structural constants of the Lie algebra, then the differential equation $\dot{x} = X_1(x)$ is integrable by quadratures.

Moreover, by solving these quadratures one obtain $n-1$ functionally independent first integrals of the system $\dot{x} = X_1(x)$, and consequently one get his complete integrability.

This result was recently improved by Kozlov, by proving that in fact each of the differential equation $\dot{x} = X_j(x)$ is integrable by quadratures.

Recall also that the above mentioned solvable Lie algebra integrates to a solvable n-dimensional Lie group that acts freely on $\mathbb{R}^n$ and permutes the trajectories of the differential system, and hence is a symmetry group of the set of all trajectories of the system. Note that by "permutation" we mean that the group action maps trajectory to trajectory.

The purpose of this section is to study the converse implication: "integrability $\Rightarrow$ symmetry". The main result states that for a completely integrable differential system $\dot{x} = X(x)$, one can associate a vector field $\widetilde{X}$, defined on an open and dense set, such that to each $\overline{Y} \in \ker\left(\mathrm{ad}_{\widetilde{X}}\right)$ it corresponds a Lie symmetry of the vector field X; more exactly, there exists a real scalar function $\mu = \mu(\overline{Y})$ and a vector field $Y = Y(\overline{Y})$ such that $[X, Y] = \mu \cdot X$.

Since the existence of a pair consisting of a vector field Y and a scalar real function μ such that $[X, Y] = \mu \cdot X$ imply the existence of a one-parameter Lie group which permutes the trajectories of the differential system $\dot{x} = X(x)$, one obtains that on an open and dense set, "integrability $\Rightarrow$ symmetry".

Let us start this section by recalling a classical result concerning Lie symmetries of dynamical systems. In order to do that, consider a n-dimensional dynamical system, $\dot{x} = X(x)$, generated by a vector field $X = X_1(x_1,\ldots,x_n)\partial_{x_1} + \cdots + X_n(x_1,\ldots,x_n)\partial_{x_n}$, and a one-parameter Lie group of transformations, G, given by

$$\begin{cases} x_1^\star(x_1,\ldots,x_n;\varepsilon) = x_1 + \varepsilon\xi_1(x_1,\ldots,x_n) + O(\varepsilon^2) \\ \cdots \\ x_n^\star(x_1,\ldots,x_n;\varepsilon) = x_n + \varepsilon\xi_n(x_1,\ldots,x_n) + O(\varepsilon^2), \end{cases}$$

acting on an open subset $U \subseteq \mathbb{R}^n$ with associated infinitesimal generator, $Y = \xi_1(x_1,\ldots,x_n)\partial_{x_1} + \cdots + \xi_n(x_1,\ldots,x_n)\partial_{x_n} \in \mathfrak{X}(U)$.

In the above hypothesis, a classical result states that G is a symmetry group for the dynamical system $\dot{x} = X(x)$ (i.e., his action maps trajectory to trajectory), if Y is a Lie symmetry of X, in the sense that $[X, Y] = \mu \cdot X$, for some scalar function $\mu = \mu(x_1, \ldots, x_n)$.

Complete integrability versus symmetry

Theorem

In the hypothesis of main theorem, to each vector field $\overline{Y} \in \ker\left(\mathrm{ad}_{\widetilde{X}}\right)$, *where* $\widetilde{X} = u_1\partial_{u_1} + \cdots + u_n\partial_{u_n}$, *the vector field* $Y = \Phi^{\star}\overline{Y}$ *is a Lie symmetry of the vector field* X *that generates the dynamical system* (1), *where* $\Phi^{\star}\overline{Y}$ *is the pull-back of the vector field* Y *through the local diffeomorphism* Φ *defined be the change of variables*

$$(x_1, \ldots, x_n) \mapsto (u_1, \ldots, u_n) = \Phi(x_1, \ldots, x_n)$$

which brings the completely integrable system generated by the vector filed X *to the linear differential system generated by the vector field* $\widetilde{X}$.

Three dimensional Lotka-Volterra system

The 3D Lotka-Volterra system we is given by the following differential system:

$$\begin{cases} \dfrac{dx_1}{dt} = x_1(x_2 + x_3) \\ \dfrac{dx_2}{dt} = x_2(-x_1 + x_3) \\ \dfrac{dx_3}{dt} = x_3(-x_1 - x_2). \end{cases} \tag{5}$$

If one denote:

$$X = [x_1(x_2 + x_3)]\partial_{x_1} + [x_2(-x_1 + x_3)]\partial_{x_2} + [x_3(-x_1 - x_2)]\partial_{x_3},$$

then $\mathrm{div}(X) = 2(x_3 - x_1)$.

Three dimensional Lotka-Volterra system

The system (5) admits a Hamilton-Poisson realization of the type (3), where:

$$v(x_1, x_2, x_3) = -\frac{x_1^2 x_3^2}{x_1 + x_2 + x_3},$$

$$C(x_1, x_2, x_3) = \frac{x_2(x_1 + x_2 + x_3)}{x_1 x_3},$$

$$H(x_1, x_2, x_3) = x_1 + x_2 + x_3.$$

Three dimensional Lotka-Volterra system

The sets introduced in the main theorem in the case of the Lotka-Volterra system are given by:

$$\Omega = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1 x_3 \neq 0\},$$
$$\Omega_0 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1 x_3 \neq 0;\ x_1 + x_2 + x_3 \neq 0\},$$
$$\mathscr{O} = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1 = x_3;\ x_1 x_3 \neq 0;\ x_1 + x_2 + x_3 \neq 0\},$$
$$\Omega_{00} = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1 \neq x_3;\ x_1 x_3 \neq 0; x_1 + x_2 + x_3 \neq 0\}.$$

Three dimensional Lotka-Volterra system

Then by the main theorem, the change of variables $((x_1, x_2, x_3), t) \mapsto ((u_1, u_2, u_3), s)$ defined by:

$$\begin{cases} u_1 = -\dfrac{x_1 + x_2 + x_3}{x_1^2 x_3^2} \\ u_2 = -\dfrac{x_2(x_1 + x_2 + x_3)^2}{x_1^3 x_3^3} \\ u_3 = -\dfrac{(x_1 + x_2 + x_3)^2}{x_1^2 x_3^2} \\ ds = 2(x_1 - x_3)dt, \end{cases}$$

for $(x_1, x_2, x_3) \in \Omega_{00}$ transforms the Lotka-Volterra system (5) into the linear differential system: $\dfrac{du_1}{ds} = u_1$, $\dfrac{du_2}{ds} = u_2$, $\dfrac{du_3}{ds} = u_3$.

3D Euler's equations of rigid body dynamics

Recall that Euler's equations from the free rigid body dynamics are given by the differential system:

$$\begin{cases} \dfrac{dx_1}{dt} = \dfrac{I_2 - I_3}{I_2 I_3} x_2 x_3 \\ \dfrac{dx_2}{dt} = \dfrac{I_3 - I_1}{I_1 I_3} x_1 x_3 \\ \dfrac{dx_3}{dt} = \dfrac{I_1 - I_2}{I_1 I_2} x_1 x_2, \end{cases} \tag{6}$$

where the nonzero real numbers I_1, I_2, I_3 are the components of the inertia tensor. In the following we consider the case when $I_2 \neq I_3$.

3D Euler's equations of rigid body dynamics

If one denote:

$$X = (\frac{I_2 - I_3}{I_2 I_3} x_2 x_3)\partial_{x_1} + (\frac{I_3 - I_1}{I_1 I_3} x_1 x_3)\partial_{x_2} + (\frac{I_1 - I_2}{I_1 I_2} x_1 x_2)\partial_{x_3},$$

then $\mathrm{div}(X) = 0$, and hence the assumptions of the main theorem do not hold.

The system (6) admits a Hamilton-Poisson realization of type (3), where:

$$v(x_1, x_2, x_3) =: v_{cst.}(x_1, x_2, x_3) = -1,$$
$$C(x_1, x_2, x_3) = \frac{1}{2}(x_1^2 + x_2^2 + x_3^2),$$
$$H(x_1, x_2, x_3) = \frac{1}{2}(\frac{x_1^2}{I_1} + \frac{x_2^2}{I_2} + \frac{x_3^2}{I_3}).$$

In order to correct the vector field X such that one can apply the main theorem, we choose a rescaling function $\mu(x_1, x_2, x_3) = x_1$, and the associated new time transformation $dt = \mu(x_1, x_2, x_3)dt'$ defined on the open and dense subset of $\mathbb{R}^3$ given by $\{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1 \neq 0\}$.

This new time transformation, transforms the system (6) into the differential system:

$$\begin{cases} \dfrac{dx_1}{dt'} = \dfrac{I_2 - I_3}{I_2 I_3} x_1 x_2 x_3 \\ \dfrac{dx_2}{dt'} = \dfrac{I_3 - I_1}{I_1 I_3} x_1^2 x_3 \\ \dfrac{dx_3}{dt'} = \dfrac{I_1 - I_2}{I_1 I_2} x_1^2 x_2, \end{cases} \tag{7}$$

3D Euler's equations of rigid body dynamics

Recall that the vector field which generates the differential system (7) is given by:

$$\mu \cdot X = (\frac{I_2 - I_3}{I_2 I_3} x_1 x_2 x_3)\partial_{x_1} + (\frac{I_3 - I_1}{I_1 I_3} x_1^2 x_3)\partial_{x_2} + (\frac{I_1 - I_2}{I_1 I_2} x_1^2 x_2)\partial_{x_3}.$$

One note that the divergence of the vector field $\mu \cdot X$ is given by:

$$\mathrm{div}(\mu \cdot X) = \frac{I_2 - I_3}{I_2 I_3} x_2 x_3.$$

For $I_2 \neq I_3$ we have that $\mathrm{div}(\mu \cdot X)$ it is not identically zero.

The system (7) admits a Hamilton-Poisson realization of the type (3), where:

$$\nu(x_1,x_2,x_3) = \nu_{cst.}(x_1,x_2,x_3)\cdot\mu(x_1,x_2,x_3) = -x_1,$$
$$C(x_1,x_2,x_3) = \frac{1}{2}(x_1^2+x_2^2+x_3^2),$$
$$H(x_1,x_2,x_3) = \frac{1}{2}(\frac{x_1^2}{I_1}+\frac{x_2^2}{I_2}+\frac{x_3^2}{I_3}).$$

3D Euler's equations of rigid body dynamics

The sets introduced in the main theorem in the case of the system (7) are given by:

$$
\begin{aligned}
\Omega &= \mathbb{R}^3, \\
\Omega_0 &= \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1 \neq 0\}, \\
\mathscr{O} &= \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1 \neq 0;\; x_2 x_3 = 0\}, \\
\Omega_{00} &= \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1 x_2 x_3 \neq 0\}.
\end{aligned}
$$

Then by the main theorem, the change of variables $((x_1, x_2, x_3), t') \mapsto ((u_1, u_2, u_3), s)$ defined by:

$$\begin{cases} u_1 = -\dfrac{1}{x_1} \\ u_2 = -\dfrac{x_1^2 + x_2^2 + x_3^2}{2x_1} \\ u_3 = -\dfrac{x_1}{2I_1} - \dfrac{x_2^2}{2x_1 I_2} - \dfrac{x_3^2}{2x_1 I_3} \\ ds = \dfrac{I_3 - I_2}{I_2 I_3} x_2 x_3 dt', \end{cases}$$

for $(x_1, x_2, x_3) \in \Omega_{00}$ transforms the system (7) into the linear differential system: $\dfrac{du_1}{ds} = u_1$, $\dfrac{du_2}{ds} = u_2$, $\dfrac{du_3}{ds} = u_3$.

3d quadratic integrable systems

In this section we show that any first-order autonomous three-dimensional differential equation possessing two independent quadratic constants of motion which admits a positive/negative definite linear combination, is affinely equivalent to the classical Euler's equations of the free rigid body dynamics with linear controls.

3d quadratic integrable systems as Hamilton-Poisson systems

Let K be a 3×3 skew-symmetric matrix and $\mathbf{k} \in \mathbb{R}^3$. Let us introduce the Poisson manifold $(\mathbb{R}^3, \{\cdot,\cdot\}_{(K,\mathbf{k})})$, where the Poisson bracket $\{\cdot,\cdot\}_{(K,\mathbf{k})}$ is defined by:

$$\{f,g\}_{(K,\mathbf{k})} := -\nabla C_{(K,\mathbf{k})} \cdot (\nabla f \times \nabla g),$$

for any $f, g \in C^\infty(\mathbb{R}^3, \mathbb{R})$, and the smooth function $C_{(K,\mathbf{k})} \in C^\infty(\mathbb{R}^3, \mathbb{R})$ is given by

$$C_{(K,\mathbf{k})}(u) := \frac{1}{2} u^T K u + u^T \mathbf{k}.$$

3d quadratic integrable systems as Hamilton-Poisson systems

The center of the Poisson algebra $C^\infty(\mathbb{R}^3, \mathbb{R})$ is generated by the Casimir invariant function $C_K \in C^\infty(\mathbb{R}^3, \mathbb{R})$,

$C_K(u) = \frac{1}{2}u^T K u + u^T \mathbf{k}.$

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Consequently, a quadratic Hamilton-Poisson system on $(\mathbb{R}^3, \{\cdot,\cdot\}_{(K,\mathbf{k})})$, is generated by a smooth function $H_{(A,\mathbf{a})} \in C^\infty(\mathbb{R}^3, \mathbb{R})$, given by

$$H_{(A,\mathbf{a})}(u) := \frac{1}{2} u^T A u + u^T \mathbf{a},$$

where $A \in Sym(3)$ is an arbitrary real symmetric matrix, and $\mathbf{a} \in \mathbb{R}^3$.

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Hence, the associated Hamiltonian system is governed by the following differential equation:

$$\dot{u} = (Ku + \mathbf{k}) \times (Au + \mathbf{a}), \ \ u \in \mathbb{R}^3. \tag{8}$$

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In the above settings, if there exists $\alpha, \beta \in \mathbb{R}$ such that $\alpha A + \beta K$ is positive (or negative) definite, then the system (8) is affinely equivalent to Euler's equations of the free rigid body dynamics with linear controls, namely:

$$\dot{u} = u \times (Du + \mathbf{d}), \;\; u \in \mathbb{R}^3, \tag{9}$$

where $D = \text{diag}(\lambda_1, \lambda_2, \lambda_3)$ is a real diagonal 3×3 matrix, and $\mathbf{d} \in \mathbb{R}^3$. In the case when $\mathbf{d} = 0$, the equations (9) are exactly the Euler's equations of the free rigid body dynamics.

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Note that in coordinates, the system (9) become:

$$\begin{cases} \dot{x}_1 = (\lambda_3 - \lambda_2)x_2x_3 + d_3x_2 - d_2x_3, \\ \dot{x}_2 = (\lambda_1 - \lambda_3)x_1x_3 - d_3x_1 + d_1x_3, \\ \dot{x}_3 = (\lambda_2 - \lambda_1)x_1x_2 + d_2x_1 - d_1x_2, \end{cases}$$

where (d_1, d_2, d_3) are the coordinates of $\mathbf{d}$.

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If the matrices A and K commutes, then the system (8) is orthogonally equivalent to the dynamical system:

$$\dot{v} = (D_K v + \hat{\mathbf{k}}) \times (D_A v + \hat{\mathbf{a}}), \;\; v \in \mathbb{R}^3, \tag{10}$$

where the real diagonal 3×3 matrices D_A, D_K are given by $D_A = R^T A R$, $D_K = R^T K R$, where $R \in O(Id) = O(3, \mathbb{R})$ is a 3×3 orthogonal matrix, and $\hat{\mathbf{k}} := \det(R) R^T \mathbf{k}$, $\hat{\mathbf{a}} := \det(R) R^T \mathbf{a}$.

Using coordinates, the system (10) becomes:

$$\begin{cases} \dot{x}_1 = (K_2A_3 - K_3A_2)x_2x_3 + (K_2a_3 - k_3A_2)x_2 + (k_2A_3 - K_3a_2)x_3 \\ +k_2a_3 - k_3a_2 \\ \dot{x}_2 = (K_3A_1 - K_1A_3)x_1x_3 + (k_3A_1 - K_1a_3)x_1 + (K_3a_1 - k_1A_3)x_3 \\ +k_3a_1 - k_1a_3 \\ \dot{x}_3 = (K_1A_2 - K_2A_1)x_1x_2 + (K_1a_2 - k_2A_1)x_1 + (k_1A_2 - K_2a_1)x_2 \\ +k_1a_2 - k_2a_1, \end{cases} \tag{11}$$

where $D_A = \mathrm{diag}(A_1, A_2, A_3)$, $D_K = \mathrm{diag}(K_1, K_2, K_3)$, $\hat{\mathbf{a}} = (a_1, a_2, a_3)$, and $\hat{\mathbf{k}} = (k_1, k_2, k_3)$.

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