The index of symmetry and naturally reductive spaces

Carlos Olmos

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## Carlos Olmos

FaMAF, Universidad Nacional de Córdoba

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In some sense, our philosophy is in the direction of the concept of co-polarity by Claudio Gorodski, that measures how a representation, orbit like, differ from a symmetric (isotropy) representation (and also we try to classify those spaces when the defect is small).

## The index of symmetry.

Let $M^{n}$ be a Riemannian manifold and denote by $\mathfrak{K}(M)$ the algebra of global Killing fields on $M$.
For $q \in M$, let us define the Cartan subspace $\mathfrak{p}^{q}$ at $q$, by

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$$
\mathfrak{p}^{q}:=\left\{X \in \mathfrak{K}(M):(\nabla X)_{q}=0\right\}
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The symmetric isotropy algebra at $q$ is defined by

$$
\mathfrak{k}^{q}:=\left\{[X, Y]: X, Y \in \mathfrak{p}^{q}\right\}
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Observe that $\mathfrak{k}^{q}$ is contained in the (full) isotropy subalgebra $\mathfrak{K}_{q}(M)$. In fact, if $X, Y \in \mathfrak{p}^{q}$,

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$[X, Y]_{q}=(\nabla X Y)_{q}-(\nabla Y X)_{q}=0$

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Observe that $\mathfrak{k}^{q}$ is contained in the (full) isotropy subalgebra $\mathfrak{K}_{q}(M)$. In fact, if $X, Y \in \mathfrak{p}^{q}$, $[X, Y]_{q}=\left(\nabla_{X} Y\right)_{q}-\left(\nabla_{Y} X\right)_{q}=0$. Moreover, since $\mathfrak{p}^{q}$ is left invariant by the isotropy at $q$,

$$
\mathfrak{g}^{q}:=\mathfrak{k}^{q} \oplus \mathfrak{p}^{q}
$$

is an involutive Lie algebra.

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The symmetric subspace at $q, \mathfrak{s}_{q} \subset T_{q} M$, is defined by

$$
\mathfrak{s}_{q}:=\left\{X . q: X \in \mathfrak{p}^{q}\right\}=\mathfrak{p}^{q} \cdot q
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The symmetric subspace at $q, \mathfrak{s}_{q} \subset T_{q} M$, is defined by symmetry and naturally reductive spaces

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For dealing with the distribution $q \mapsto \mathfrak{s}^{q}$ one needs to regard Killing fields

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For dealing with the distribution $q \mapsto \mathfrak{s}^{q}$ one needs to regard Killing fields as parallel sections of the so called canonical (vector) bundle over $M$, symmetry and naturally reductive spaces

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where $B$ acts on a tensor as a derivation.

## Lemma

Let $X, Y \in \mathfrak{p}^{q}$, regarded as Killing fields, and let $Z$ be an arbitrary tangent field of $M$. Then symmetry and naturally reductive spaces

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R_{X(q), Y(q)} Z(q)=-[[X, Y], Z](q)
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## Proposition <br> If $M$ is compact, then $G^{q}$ acts almost effectively on the orbit $G^{q} . q$.

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## Proposition <br> If $M$ is compact, then $G^{q}$ acts almost effectively on the orbit $G^{q} . q$.

Identify $T_{q}\left(G^{q} . q\right)=\mathfrak{s}_{q} \simeq \mathfrak{p}^{q}$ and decompose

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where $\mathfrak{p}_{0}$ corresponds to the Euclidean factor and $\mathfrak{p}_{i}$ corresponds to the irreducible factors, in the de Rham local decomposition of the orbit $G^{q} . q(i=1, \ldots, r)$.
Let, for $j=0, \ldots, r$,

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\mathfrak{k}_{j}:=\left[\mathfrak{p}_{j}, \mathfrak{p}_{j}\right] .
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But, if the action of $G^{q}$ is not almost effective on the orbit $G^{q} . q$, we cannot conclude neither that $\mathfrak{g}^{q}$ is spanned by

But, if the action of $G^{q}$ is not almost effective on the orbit $G^{q} . q$, we cannot conclude neither that $\mathfrak{g}^{q}$ is spanned by $\mathfrak{g}_{0}, \ldots, \mathfrak{g}_{r}$
nor that these subalgebras are in a direct sum

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But, if the action of $G^{q}$ is not almost effective on the orbit
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nor that these subalgebras are in a direct sum (and not even that $\mathfrak{k}_{0}$ is trivial or that $\mathfrak{g}_{i}$ are ideals).

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If $M$ is compact then $\mathfrak{k}_{0}=0,\left[\mathfrak{g}_{i}, \mathfrak{g}_{j}\right]=0$, if $i \neq j$ and so $\mathfrak{g}^{q}$ is the direct sum of the ideals $\mathfrak{g}_{1}, \ldots, \mathfrak{g}_{s}$.

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where $\operatorname{Lie}\left(G_{i}^{q}\right)=\mathfrak{g}_{i}$.

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If $M^{n}$ is compact then, if $i \geq 1, G_{i}^{q}$ is a compact Lie subgroup of $I(M)$.

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Facts: assume that $M^{n}$ compact.

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Facts: assume that $M^{n}$ compact.
(a) $G_{i}^{q}$ is a compact Lie subgroup of $I(M)$, if $i \geq 1$.
(b) If $R_{u, v} \mid \mathfrak{s}_{q}=0$, then $R_{u, v}=0$, for any $u, v \in \mathfrak{s}_{q}$.

## Theorem

Let $M^{n}$ be a symply connected compact locally irreducible homogeneous Riemannian manifold, which is not locally

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- Left invariant metrics in Spin(3).

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The metrics with co-index of symmetry 2 are given exactly by the two families of metrics

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The metrics with co-index of symmetry 2 are given exactly by the two families of metrics $g_{s}^{1}$ and $g_{t}^{2}$

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The metrics with co-index of symmetry 2 are given exactly by the two families of metrics $g_{s}^{1}$ and $g_{t}^{2}$ associated to the triples

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Observe that $\left(\operatorname{Spin}(3), g_{t}^{2}\right)$ is a Berger sphere. Or equivalently, up to a cover, it is the unit tangent bundle over the 2 -sphere of constant curvature different from 1 (in which case the metric would be bi-invariant and the space symmetric).

- The unit tangent bundle over the sphere of curvature 2 .

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- The unit tangent bundle over the sphere of curvature 2 .

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- Naturally reductive spaces whose isotropy has fixed vectors

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for al $X \in \mathfrak{m}$. In other words, the Riemannian geodesics coincide with the $\nabla^{c}$-geodesics, where $\nabla^{c}$ is the canonical connection, which is a metric connection, of $M$ associated to the reductive decomposition. This is in fact equivalent to the property that $[X, \cdot]_{\mathfrak{m}}: \mathfrak{m} \rightarrow \mathfrak{m}$ is skew-symmetric, for all $X \in \mathfrak{m}\left(\mathfrak{m} \simeq T_{p} M\right)$.

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## The Levi-Civita connection is given by

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## The Levi-Civita connection is given by

$$
\nabla_{\nu} \tilde{w}=\frac{1}{2}[\tilde{v}, \tilde{w}]_{\rho},
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The tensor $D$ is totally skew, i.e. $\left\langle D_{v} w, z\right\rangle$ is a 3-form.

Let $M$ be a compact locally irreducible (non-symmetric) naturally reductive space. Let now, keeping the previous notation,

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\mathfrak{m}_{0} \subset \mathfrak{m} \simeq T_{p} M
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The difference tensor between both connections is given by

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D_{v} w=\nabla_{v} \tilde{w}-\nabla_{v}^{c} \tilde{w}=-\frac{1}{2}[\tilde{v}, \tilde{w}]_{p}=-\nabla_{v} \tilde{w} .
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Let $\hat{w}$ denote the $G$-invariant vector with $\hat{w}(q)=w \in \mathfrak{m}_{0}$.

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Therefore, $\mathfrak{m}_{0} \subset \mathfrak{s}_{q}$, and thus the distribution of symmetry is non-trivial.

Theorem (O.-Reggiani-Tamaru). Let $M$ be a simply connected compact homogeneous naturally reductive space. Then the index of symmetry of $M$ coincides with the dimension of the fixed vectors of the isotropy of the group of transvections.

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Theorem (O.-Reggiani-Tamaru). Let $M$ be a simply connected compact homogeneous naturally reductive space. Then the index of symmetry of $M$ coincides with the dimension of the fixed vectors of the isotropy of the group of transvections.

Corollary (O.-Reggiani-Tamaru) Let $M=G / H$ be a simply connected compact normal homogeneous space. Then the index of symmetry of $M$ coincides with the dimension of the fixed vectors of the isotropy $H$.

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Assume that $M^{n}$ is a compact simply connected irreducible The dimension bound Riemannian manifold with a positive index of symmetry.

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- Are the leaves of the distribution of symmetry compact (or equivalently, is the flat factor compact?).
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- If $n>3$, does the metric on $M$ projects down to the quotient by the symmetric foliation? (if the space is locally irreducible) The situation, for $n>3$, seems to be very rigid.

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Or, more generally, classify the compact simply connected, irreducible, Riemannian homogeneous manifolds with a positive index of symmetry.

