The index of symmetry

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In this talk, based on joint work with Silvio Reggiani, we would like to draw the attention to some concept that we call *index of symmetry* $i_{\mathfrak{s}}(M)$ of a Riemannian manifold M^n

$$0 \leq i_{\mathfrak{s}}(M) \leq n$$

One has that M is symmetric if and only if $i_{\mathfrak{s}}(M) = n$

We are, of course, interested on non-symmetric spaces with positive index of symmetry. In this case one has that $i_{\mathfrak{s}}(M) \leq n - 2$, as we will see later (in other words the *co-index of symmetry* is at least 2).

These examples are known homogenous spaces but endowed with a very particular Riemannian metric.

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We are able to classify the spaces with small co-index of symmetry, after proving a bound on the dimension n of the space, for a fixed positive co-index of symmetry k (for irreducible spaces, since the product by a symmetric space does not change the co-index of symmetry, but increases the dimension).

The concept of index of symmetry came out from the study of compact naturally reductive spaces such that the isotropy has non-trivial fixed vectors (and so the full isometry group is bigger than the presentation group). For such spaces it is not hard to prove that the index of symmetry is al least the dimension of the fixed vectors of the isotropy representation.

Recently, with Reggiani and Tamaru, we could prove the equality, if the space is (irreducible, non-symmetric) presented with the transvections.

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The subjects of this talk may be regarded as an effort to explore Riemannian manifolds that are symmetric up to some defect (in the hope of finding distinguished non-symmetric homogeneous manifolds).

In some sense, our philosophy is in the direction of the concept of co-polarity by Claudio Gorodski, that measures how a representation, orbit like, differ from a symmetric (isotropy) representation (and also we try to classify those spaces when the defect is small).

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The index of symmetry.

Let M^n be a Riemannian manifold and denote by $\mathfrak{K}(M)$ the algebra of global Killing fields on M.
For $q \in M$, let us define the Cartan subspace $\mathfrak{p}^q$ at q, by

$$\mathfrak{p}^q := \{X \in \mathfrak{K}(M) : (\nabla X)_q = 0\}$$

The symmetric isotropy algebra at q is defined by

$$\mathfrak{k}^q := \{[X, Y] : X, Y \in \mathfrak{p}^q\}$$

Observe that $\mathfrak{k}^q$ is contained in the (full) isotropy subalgebra $\mathfrak{K}_q(M)$. In fact, if $X, Y \in \mathfrak{p}^q$, $[X, Y]_q = (\nabla_X Y)_q - (\nabla_Y X)_q = 0$. Moreover, since $\mathfrak{p}^q$ is left invariant by the isotropy at q,

$$\mathfrak{g}^q := \mathfrak{k}^q \oplus \mathfrak{p}^q$$

is an involutive Lie algebra.

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The symmetric subspace at q, $\mathfrak{s}_q \subset T_qM$, is defined by

$$\mathfrak{s}_q := \{X.q : X \in \mathfrak{p}^q\} = \mathfrak{p}^q.q$$

The local version, involving local Killing fields, can be equivalently defined as follows (from a joint work with Sergio Console, PAMS 09)

$$\mathfrak{s}_q^{loc} := \{v \in T_qM : \nabla^k_v R = 0, \ k = 0, \dots, n + \frac{1}{2}n(n-1)\},$$

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For dealing with the distribution $q \mapsto \mathfrak{s}^q$ one needs to regard Killing fields as parallel sections of the so called canonical (vector) bundle over M,

$$TM \oplus \Lambda^2(TM) \simeq TM \oplus \mathfrak{so}(TM)$$

where the connection $\bar{\nabla}$ in $TM \oplus \mathfrak{so}(TM)$ is given by

$$\bar{\nabla}_Y(Z, B) = (\nabla_Y Z - BY, \nabla_Y B - R_{Y,Z})$$

The bijection is given by

$$Z \leftrightarrow (Z, \nabla Z)$$

The curvature tensor $\bar{R}$ of $\bar{\nabla}$ is given by

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Lemma

Let $X, Y \in \mathfrak{p}^q$, regarded as Killing fields, and let Z be an arbitrary tangent field of M. Then

$$R_{X(q),Y(q)}Z(q) = -[[X, Y], Z](q)$$

Let $q \in M$ and assume that the index of symmetry at q is positive, i.e. $\dim \mathfrak{s}_q > 0$. Let us consider the Lie subalgebra $\mathfrak{g}^q$ of the full isometry algebra. One has that

$$\mathfrak{g}^q = \mathfrak{k}^q \oplus \mathfrak{p}^q$$

is an involutive Lie algebra. Let G^q be its associated Lie subgroup of $I(M)$. One has that the orbit $G^q.q$ is a global symmetric space, which is a totally geodesic immersed manifold of M.

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Let $X, Y \in \mathfrak{p}^q$, regarded as Killing fields, and let Z be an arbitrary tangent field of M. Then

$$R_{X(q),Y(q)}Z(q) = -[[X, Y], Z](q)$$

Let $q \in M$ and assume that the index of symmetry at q is positive, i.e. $\dim \mathfrak{s}_q > 0$. Let us consider the Lie subalgebra $\mathfrak{g}^q$ of the full isometry algebra. One has that

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Proposition

If M is compact, then G^q acts almost effectively on the orbit $G^q.q$.

Identify $T_q(G^q.q) = \mathfrak{s}_q \simeq \mathfrak{p}^q$ and decompose

$$\mathfrak{p}^q = \mathfrak{p}_0 \oplus \mathfrak{p}_1 \oplus \ldots \oplus \mathfrak{p}_r$$

where $\mathfrak{p}_0$ corresponds to the Euclidean factor and $\mathfrak{p}_i$ corresponds to the irreducible factors, in the de Rham local decomposition of the orbit $G^q.q$ ($i = 1, \ldots, r$).
Let, for $j = 0, \ldots, r$,

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Facts: *assume that M^n compact.*

(a) *G_i^q is a compact Lie subgroup of $I(M)$, if $i \geq 1$.*

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Theorem

Let M^n be a symply connected compact locally irreducible homogeneous Riemannian manifold, which is not locally symmetric, and let $k := n - i_{\mathfrak{s}}(M)$ be its co-index of symmetry. Then there is a subgroup of isometries $G \subset I(M)$, which acts transitively on M and such that $dim(G) \leq \frac{1}{2}k(k+1)$. Moreover, if the equality holds, then, up to a cover, $G = Spin(k+1)$ and G has non-trivial isotropy, if $k \geq 3$.

Corollary

Let M^n, $n \geq 3$, be a symply connected compact locally irreducible homogeneous Riemannian manifold with co-index of $k = 2$. Then $M = Spin(3) \simeq S^3$ with a left invariant metric that belongs to one of two families g_s^1 , g_t^2 described in the next.

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Examples.

- Left invariant metrics in Spin(3).

Since $Ad(\mathrm{Spin}(3)) = \mathrm{SO}(\mathfrak{so}(3)) \simeq \mathrm{SO}(3)$, with respect to the bi-invariant metric of curvature 1.

any left invariant metric, modulo isometries and rescaling, is determined by a triple of positive numbers

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The metrics with co-index of symmetry 2 are given exactly by the two families of metrics g_s^1 and g_t^2 associated to the triples

$$(1, s, 1-s), \qquad 0 < s < \frac{1}{2}$$

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$$(1, t, t), \qquad 0 < t \neq 1$$

The isometry group for the first family is Spin(3) and for the second family is Spin(3) $\times S^1$ (and the tranvections do not lie in Spin(3)), if $t \neq \frac{1}{2}$.

Observe that (Spin(3), g_t^2) is a Berger sphere. Or equivalently, up to a cover, it is the unit tangent bundle over the 2-sphere of constant curvature different from 1 (in which case the metric would be bi-invariant and the space symmetric).

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The distribution of symmetry $\mathfrak{s}$, of the unit tangent bundle M^{2n-1} of the sphere S_2^n of curvature 2, coincides with the vertical distribution ν. In particular, $i_{\mathfrak{s}} = n - 1$, where $i_{\mathfrak{s}} = \dim(\mathfrak{s})$ is the index of symmetry (or equivalently, the co-index of symmetry is equals to n).

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- Naturally reductive spaces whose isotropy has fixed vectors

Let $M = G/H$ be a homogeneous compact Riemannian manifold with a G-invariant metric $\langle\ ,\ \rangle$.

The space M is said to be *naturally reductive* if there exists a reductive decomposition

$$\mathcal{G} = \mathfrak{h} \oplus \mathfrak{m},$$

where $\mathcal{G} = \mathrm{Lie}(G)$, $\mathfrak{h} = \mathrm{Lie}(H)$, $\mathrm{Ad}(H)\mathfrak{m} \subset \mathfrak{m}$, such that the geodesics by $p = [e]$ are given by

$$\gamma_{X.p} = \mathrm{Exp}(tX).p$$

for al $X \in \mathfrak{m}$. In other words, the Riemannian geodesics coincide with the ∇^c-geodesics, where ∇^c is the canonical connection, which is a metric connection, of M associated to the reductive decomposition. This is in fact equivalent to the property that $[X, \cdot\,]_{\mathfrak{m}} : \mathfrak{m} \to \mathfrak{m}$ is skew-symmetric, for all $X \in \mathfrak{m}$ ($\mathfrak{m} \simeq T_pM$).

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$$\nabla_v \tilde{w} = \tfrac{1}{2}[\tilde{v}, \tilde{w}]_p,$$

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The tensor D is totally skew, i.e. $\langle D_v w, z \rangle$ is a 3-form.

Let M be a compact locally irreducible (non-symmetric) naturally reductive space. Let now, keeping the previous notation,

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be the set of fixed vectors of the isotropy at q.

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Let $\hat{w}$ denote the G-invariant vector with $\hat{w}(q) = w \in \mathfrak{m}_0$.
Such a field is parallel with respect to the canonical connection. In fact, any G-invariant tensor is ∇^c-parallel. Then, for any $v \in \mathfrak{m} \simeq T_pM$, $w \in \mathfrak{m}_0$,

$$(\nabla_v \hat{w})_q = D_v w$$

Observe, since D is totally skew, that $\hat{w}$ satisfies the Killing equation and hence it is a Killing field.

Remark. There are no more new Killing fields in M, since the canonical connection is unique (unless M is round sphere, or a Lie group, with a bi-invariant metric). This is by making use of the so-called *skew-torsion holonomy theorem* (O.- Reggiani, Crelle's 2011)

$$\mathrm{Lie}(I(M)) = \mathfrak{g} \oplus \hat{\mathfrak{m}}_0$$

(direct sum of ideals).

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Observe, since D is totally skew, that $\hat{w}$ satisfies the Killing equation and hence it is a Killing field.

Remark. There are no more new Killing fields in M, since the canonical connection is unique (unless M is round sphere, or a Lie group, with a bi-invariant metric). This is by making use of the so-called *skew-torsion holonomy theorem* (O.- Reggiani, Crelle's 2011)

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Therefore, $\mathfrak{m}_0 \subset \mathfrak{s}_q$, and thus the distribution of symmetry is non-trivial.

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Corollary (O.-Reggiani-Tamaru) Let $M = G/H$ be a simply connected compact normal homogeneous space. Then the index of symmetry of M coincides with the dimension of the fixed vectors of the isotropy H.

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Open questions.

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– Find new examples.

– Classify the case of co-index of symmetry equals to 3 and 4 (in which case de dimension is at most 6 or 10).

Or, more generally, classify the compact simply connected, irreducible, Riemannian homogeneous manifolds with a positive index of symmetry.

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