The Ricci flow and its solitons for homogeneous manifolds

Jorge Lauret,
Univ. Nac. de Córdoba

Rosario, August 2nd, 2012

Contents

Contents

Contents

The space of homogeneous manifolds

The space of homogeneous manifolds

Fix $\mathfrak{g}$: real vector space.

The space of homogeneous manifolds

Fix $\mathfrak{g}$: real vector space.

$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$: direct sum decomposition,

The space of homogeneous manifolds

Fix $\mathfrak{g}$: real vector space.

$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$: direct sum decomposition, $\langle \cdot, \cdot \rangle$: inner product on $\mathfrak{p}$.

The space of homogeneous manifolds

Fix $\mathfrak{g}$: real vector space.

$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$: direct sum decomposition, $\langle \cdot, \cdot \rangle$: inner product on $\mathfrak{p}$.

Given $\mu \in \Lambda^2 \mathfrak{g}^* \otimes \mathfrak{g}$,

The space of homogeneous manifolds

Fix $\mathfrak{g}$: real vector space.

$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$: direct sum decomposition, $\langle \cdot, \cdot \rangle$: inner product on $\mathfrak{p}$.

Given $\mu \in \Lambda^2 \mathfrak{g}^* \otimes \mathfrak{g}$, i.e. $\mu : \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathfrak{g}$ bil. and skew-symm.,

The space of homogeneous manifolds

Fix $\mathfrak{g}$: real vector space.

$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$: direct sum decomposition, $\langle \cdot, \cdot \rangle$: inner product on $\mathfrak{p}$.

Given $\mu \in \Lambda^2 \mathfrak{g}^* \otimes \mathfrak{g}$, i.e. $\mu : \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathfrak{g}$ bil. and skew-symm.,

$\mu \rightsquigarrow (G_\mu / K_\mu, g_\mu)$ homogeneous space provided:

The space of homogeneous manifolds

Fix $\mathfrak{g}$: real vector space.

$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$: direct sum decomposition, $\langle \cdot, \cdot \rangle$: inner product on $\mathfrak{p}$.

Given $\mu \in \Lambda^2 \mathfrak{g}^* \otimes \mathfrak{g}$, i.e. $\mu : \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathfrak{g}$ bil. and skew-symm.,

$\mu \rightsquigarrow (G_\mu / K_\mu, g_\mu)$ homogeneous space provided:

(i) μ Jacobi ✓, $\mu(\mathfrak{k}, \mathfrak{k}) \subset \mathfrak{k}$, $\mu(\mathfrak{k}, \mathfrak{p}) \subset \mathfrak{p}$

The space of homogeneous manifolds

Fix $\mathfrak{g}$: real vector space.

$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$: direct sum decomposition, $\langle \cdot, \cdot \rangle$: inner product on $\mathfrak{p}$.

Given $\mu \in \Lambda^2 \mathfrak{g}^* \otimes \mathfrak{g}$, i.e. $\mu : \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathfrak{g}$ bil. and skew-symm.,

$\mu \rightsquigarrow (G_\mu / K_\mu, g_\mu)$ homogeneous space provided:

(i) μ Jacobi ✓, $\mu(\mathfrak{k}, \mathfrak{k}) \subset \mathfrak{k}$, $\mu(\mathfrak{k}, \mathfrak{p}) \subset \mathfrak{p}$ $(\rightsquigarrow G_\mu, K_\mu)$.

The space of homogeneous manifolds

Fix $\mathfrak{g}$: real vector space.

$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$: direct sum decomposition, $\langle \cdot, \cdot \rangle$: inner product on $\mathfrak{p}$.

Given $\mu \in \Lambda^2 \mathfrak{g}^* \otimes \mathfrak{g}$, i.e. $\mu : \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathfrak{g}$ bil. and skew-symm.,

$\mu \rightsquigarrow (G_\mu / K_\mu, g_\mu)$ homogeneous space provided:

(i) μ Jacobi ✓, $\mu(\mathfrak{k}, \mathfrak{k}) \subset \mathfrak{k}$, $\mu(\mathfrak{k}, \mathfrak{p}) \subset \mathfrak{p}$ $(\rightsquigarrow G_\mu, K_\mu)$.

(ii) K_μ closed in G_μ

The space of homogeneous manifolds

Fix $\mathfrak{g}$: real vector space.

$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$: direct sum decomposition, $\langle \cdot, \cdot \rangle$: inner product on $\mathfrak{p}$.

Given $\mu \in \Lambda^2 \mathfrak{g}^* \otimes \mathfrak{g}$, i.e. $\mu : \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathfrak{g}$ bil. and skew-symm.,

$\mu \rightsquigarrow (G_\mu / K_\mu, g_\mu)$ homogeneous space provided:

(i) μ Jacobi ✓, $\mu(\mathfrak{k}, \mathfrak{k}) \subset \mathfrak{k}$, $\mu(\mathfrak{k}, \mathfrak{p}) \subset \mathfrak{p}$ $(\rightsquigarrow G_\mu, K_\mu)$.

(ii) K_μ closed in G_μ $(\rightsquigarrow G_\mu / K_\mu$ manifold$)$.

The space of homogeneous manifolds

Fix $\mathfrak{g}$: real vector space.

$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$: direct sum decomposition, $\langle\cdot,\cdot\rangle$: inner product on $\mathfrak{p}$.

Given $\mu \in \Lambda^2\mathfrak{g}^* \otimes \mathfrak{g}$, i.e. $\mu : \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathfrak{g}$ bil. and skew-symm.,

$\mu \rightsquigarrow (G_\mu/K_\mu, g_\mu)$ homogeneous space provided:

(i) μ Jacobi ✓, $\mu(\mathfrak{k},\mathfrak{k}) \subset \mathfrak{k}$, $\mu(\mathfrak{k},\mathfrak{p}) \subset \mathfrak{p}$ ($\rightsquigarrow G_\mu$, K_μ).

(ii) K_μ closed in G_μ ($\rightsquigarrow G_\mu/K_\mu$ manifold).

(iii) $\{Z \in \mathfrak{k} : \mu(Z,\mathfrak{p}) = 0\} = 0$

The space of homogeneous manifolds

Fix $\mathfrak{g}$: real vector space.

$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$: direct sum decomposition, $\langle \cdot, \cdot \rangle$: inner product on $\mathfrak{p}$.

Given $\mu \in \Lambda^2\mathfrak{g}^* \otimes \mathfrak{g}$, i.e. $\mu : \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathfrak{g}$ bil. and skew-symm.,

$\mu \rightsquigarrow (G_\mu/K_\mu, g_\mu)$ homogeneous space provided:

(i) μ Jacobi ✓, $\mu(\mathfrak{k}, \mathfrak{k}) \subset \mathfrak{k}$, $\mu(\mathfrak{k}, \mathfrak{p}) \subset \mathfrak{p}$ $(\rightsquigarrow G_\mu, K_\mu)$.

(ii) K_μ closed in G_μ $(\rightsquigarrow G_\mu/K_\mu$ manifold).

(iii) $\{Z \in \mathfrak{k} : \mu(Z, \mathfrak{p}) = 0\} = 0$ $(\rightsquigarrow G_\mu/K_\mu$ almost-effective).

The space of homogeneous manifolds

Fix $\mathfrak{g}$: real vector space.

$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$: direct sum decomposition, $\langle\cdot,\cdot\rangle$: inner product on $\mathfrak{p}$.

Given $\mu \in \Lambda^2\mathfrak{g}^* \otimes \mathfrak{g}$, i.e. $\mu : \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathfrak{g}$ bil. and skew-symm.,

$\mu \rightsquigarrow (G_\mu/K_\mu, g_\mu)$ homogeneous space provided:

(i) μ Jacobi ✓, $\mu(\mathfrak{k},\mathfrak{k}) \subset \mathfrak{k}$, $\mu(\mathfrak{k},\mathfrak{p}) \subset \mathfrak{p}$ $(\rightsquigarrow G_\mu,\ K_\mu)$.

(ii) K_μ closed in G_μ $(\rightsquigarrow G_\mu/K_\mu$ manifold$)$.

(iii) $\{Z \in \mathfrak{k} : \mu(Z,\mathfrak{p}) = 0\} = 0$ $(\rightsquigarrow G_\mu/K_\mu$ almost-effective$)$.

(iv) $\langle\cdot,\cdot\rangle$ is $\mathrm{ad}_\mu\,\mathfrak{k}|_{\mathfrak{p}}$-invariant

The space of homogeneous manifolds

Fix $\mathfrak{g}$: real vector space.

$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$: direct sum decomposition, $\langle \cdot, \cdot \rangle$: inner product on $\mathfrak{p}$.

Given $\mu \in \Lambda^2 \mathfrak{g}^* \otimes \mathfrak{g}$, i.e. $\mu : \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathfrak{g}$ bil. and skew-symm.,

$\mu \rightsquigarrow (G_\mu / K_\mu, g_\mu)$ homogeneous space provided:

(i) μ Jacobi ✓, $\mu(\mathfrak{k}, \mathfrak{k}) \subset \mathfrak{k}$, $\mu(\mathfrak{k}, \mathfrak{p}) \subset \mathfrak{p}$ $(\rightsquigarrow G_\mu, K_\mu)$.

(ii) K_μ closed in G_μ $(\rightsquigarrow G_\mu / K_\mu$ manifold).

(iii) $\{Z \in \mathfrak{k} : \mu(Z, \mathfrak{p}) = 0\} = 0$ $(\rightsquigarrow G_\mu / K_\mu$ almost-effective).

(iv) $\langle \cdot, \cdot \rangle$ is $\operatorname{ad}_\mu \mathfrak{k}|_{\mathfrak{p}}$-invariant ($\rightsquigarrow g_\mu$ G_μ-invariant), $g_\mu(o) = \langle \cdot, \cdot \rangle$).

The space of homogeneous manifolds

Fix $\mathfrak{g}$: real vector space.

$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$: direct sum decomposition, $\langle \cdot, \cdot \rangle$: inner product on $\mathfrak{p}$.

Given $\mu \in \Lambda^2 \mathfrak{g}^* \otimes \mathfrak{g}$, i.e. $\mu : \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathfrak{g}$ bil. and skew-symm.,

$\mu \rightsquigarrow (G_\mu / K_\mu, g_\mu)$ homogeneous space provided:

(i) μ Jacobi ✓, $\mu(\mathfrak{k}, \mathfrak{k}) \subset \mathfrak{k}$, $\mu(\mathfrak{k}, \mathfrak{p}) \subset \mathfrak{p}$ ($\rightsquigarrow G_\mu$, K_μ).
(ii) K_μ closed in G_μ ($\rightsquigarrow G_\mu / K_\mu$ manifold).
(iii) $\{Z \in \mathfrak{k} : \mu(Z, \mathfrak{p}) = 0\} = 0$ ($\rightsquigarrow G_\mu / K_\mu$ almost-effective).
(iv) $\langle \cdot, \cdot \rangle$ is $\operatorname{ad}_\mu \mathfrak{k}|_\mathfrak{p}$-invariant ($\rightsquigarrow g_\mu$ G_μ-invariant), $g_\mu(o) = \langle \cdot, \cdot \rangle$).

$q = \dim \mathfrak{k}$, $n = \dim \mathfrak{p}$,

The space of homogeneous manifolds

Fix $\mathfrak{g}$: real vector space.

$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$: direct sum decomposition, $\langle \cdot, \cdot \rangle$: inner product on $\mathfrak{p}$.

Given $\mu \in \Lambda^2 \mathfrak{g}^* \otimes \mathfrak{g}$, i.e. $\mu : \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathfrak{g}$ bil. and skew-symm.,

$\mu \rightsquigarrow (G_\mu / K_\mu, g_\mu)$ homogeneous space provided:

(i) μ Jacobi ✓, $\mu(\mathfrak{k}, \mathfrak{k}) \subset \mathfrak{k}$, $\mu(\mathfrak{k}, \mathfrak{p}) \subset \mathfrak{p}$ ($\rightsquigarrow G_\mu$, K_μ).

(ii) K_μ closed in G_μ ($\rightsquigarrow G_\mu / K_\mu$ manifold).

(iii) $\{Z \in \mathfrak{k} : \mu(Z, \mathfrak{p}) = 0\} = 0$ ($\rightsquigarrow G_\mu / K_\mu$ almost-effective).

(iv) $\langle \cdot, \cdot \rangle$ is $\operatorname{ad}_\mu \mathfrak{k}|_{\mathfrak{p}}$-invariant ($\rightsquigarrow g_\mu$ G_μ-invariant), $g_\mu(o) = \langle \cdot, \cdot \rangle$).

$q = \dim \mathfrak{k}$, $n = \dim \mathfrak{p}$,

$$\mathcal{H}_{q,n} := \left\{ \mu \in \Lambda^2 \mathfrak{g}^* \otimes \mathfrak{g} : \text{(i)-(iv)✓} \right\} \subset \mathcal{L}_{q+n}.$$

The space of homogeneous manifolds

Fix $\mathfrak{g}$: real vector space.

$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$: direct sum decomposition, $\langle \cdot, \cdot \rangle$: inner product on $\mathfrak{p}$.

Given $\mu \in \Lambda^2\mathfrak{g}^* \otimes \mathfrak{g}$, i.e. $\mu : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ bil. and skew-symm.,

$\mu \rightsquigarrow (G_\mu/K_\mu, g_\mu)$ homogeneous space provided:

(i) μ Jacobi ✓, $\mu(\mathfrak{k},\mathfrak{k}) \subset \mathfrak{k}$, $\mu(\mathfrak{k},\mathfrak{p}) \subset \mathfrak{p}$ ($\rightsquigarrow G_\mu$, K_μ).

(ii) K_μ closed in G_μ ($\rightsquigarrow G_\mu/K_\mu$ manifold).

(iii) $\{Z \in \mathfrak{k} : \mu(Z,\mathfrak{p}) = 0\} = 0$ ($\rightsquigarrow G_\mu/K_\mu$ almost-effective).

(iv) $\langle \cdot, \cdot \rangle$ is $\operatorname{ad}_\mu \mathfrak{k}|_{\mathfrak{p}}$-invariant ($\rightsquigarrow g_\mu$ G_μ-invariant), $g_\mu(o) = \langle \cdot, \cdot \rangle$).

$q = \dim \mathfrak{k}$, $n = \dim \mathfrak{p}$,

$$\mathcal{H}_{q,n} := \left\{\mu \in \Lambda^2\mathfrak{g}^* \otimes \mathfrak{g} : \text{(i)-(iv)}\checkmark\right\} \subset \mathcal{L}_{q+n}.$$

$\mathcal{H}_{q,n} \leftrightarrow$ all simply connected Riemannian homogeneous space of dimension n with a q-dimensional isotropy (up to isometry).

The space of homogeneous manifolds

Fix $\mathfrak{g}$: real vector space.

$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$: direct sum decomposition, $\langle \cdot, \cdot \rangle$: inner product on $\mathfrak{p}$.

Given $\mu \in \Lambda^2\mathfrak{g}^* \otimes \mathfrak{g}$, i.e. $\mu : \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathfrak{g}$ bil. and skew-symm.,

$\mu \rightsquigarrow (G_\mu/K_\mu, g_\mu)$ homogeneous space provided:

(i) μ Jacobi ✓, $\mu(\mathfrak{k},\mathfrak{k}) \subset \mathfrak{k}$, $\mu(\mathfrak{k},\mathfrak{p}) \subset \mathfrak{p}$ $(\rightsquigarrow G_\mu,\ K_\mu)$.

(ii) K_μ closed in G_μ $(\rightsquigarrow G_\mu/K_\mu$ manifold).

(iii) $\{Z \in \mathfrak{k} : \mu(Z,\mathfrak{p}) = 0\} = 0$ $(\rightsquigarrow G_\mu/K_\mu$ almost-effective).

(iv) $\langle \cdot, \cdot \rangle$ is $\operatorname{ad}_\mu \mathfrak{k}|_\mathfrak{p}$-invariant ($\rightsquigarrow g_\mu$ G_μ-invariant), $g_\mu(o) = \langle \cdot, \cdot \rangle$).

$q = \dim \mathfrak{k}$, $n = \dim \mathfrak{p}$,

$$\mathcal{H}_{q,n} := \left\{\mu \in \Lambda^2\mathfrak{g}^* \otimes \mathfrak{g} : \text{(i)-(iv)}\checkmark\right\} \subset \mathcal{L}_{q+n}.$$

$\mathcal{H}_{q,n} \leftrightarrow$ all simply connected Riemannian homogeneous space of dimension n with a q-dimensional isotropy (up to isometry).

$\mathcal{H}_{0,n} = \mathcal{L}_n$ variety of Lie algebras

The space of homogeneous manifolds

Fix $\mathfrak{g}$: real vector space.

$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$: direct sum decomposition, $\langle \cdot, \cdot \rangle$: inner product on $\mathfrak{p}$.

Given $\mu \in \Lambda^2 \mathfrak{g}^* \otimes \mathfrak{g}$, i.e. $\mu : \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathfrak{g}$ bil. and skew-symm.,

$\mu \rightsquigarrow (G_\mu / K_\mu, g_\mu)$ homogeneous space provided:

(i) μ Jacobi ✓, $\mu(\mathfrak{k}, \mathfrak{k}) \subset \mathfrak{k}$, $\mu(\mathfrak{k}, \mathfrak{p}) \subset \mathfrak{p}$ $(\rightsquigarrow G_\mu,\ K_\mu)$.
(ii) K_μ closed in G_μ $(\rightsquigarrow G_\mu / K_\mu$ manifold).
(iii) $\{Z \in \mathfrak{k} : \mu(Z, \mathfrak{p}) = 0\} = 0$ $(\rightsquigarrow G_\mu / K_\mu$ almost-effective).
(iv) $\langle \cdot, \cdot \rangle$ is $\operatorname{ad}_\mu \mathfrak{k}|_{\mathfrak{p}}$-invariant ($\rightsquigarrow g_\mu$ G_μ-invariant), $g_\mu(o) = \langle \cdot, \cdot \rangle$).

$q = \dim \mathfrak{k}$, $n = \dim \mathfrak{p}$,

$$\mathcal{H}_{q,n} := \left\{ \mu \in \Lambda^2 \mathfrak{g}^* \otimes \mathfrak{g} : \text{(i)-(iv)}\checkmark \right\} \subset \mathcal{L}_{q+n}.$$

$\mathcal{H}_{q,n} \leftrightarrow$ all simply connected Riemannian homogeneous space of dimension n with a q-dimensional isotropy (up to isometry).

$\mathcal{H}_{0,n} = \mathcal{L}_n$ variety of Lie algebras $\leftrightarrow$ left-invariant metrics on all n-dimensional s.c. Lie groups.

Convergence

Convergence

Lie injectivity radius of $(G_\mu/K_\mu, g_\mu)$, $\mu \in \mathcal{H}_{q,n}$,

Convergence

Lie injectivity radius of $(G_\mu/K_\mu, g_\mu)$, $\mu \in \mathcal{H}_{q,n}$,

$$r_\mu := \sup\left\{ r > 0 : \pi_\mu \circ \exp_\mu : B(0,r) \to G_\mu/K_\mu \quad \text{diffeomorphism} \right\}.$$

Convergence

Lie injectivity radius of $(G_\mu/K_\mu, g_\mu)$, $\mu \in \mathcal{H}_{q,n}$,

$$r_\mu := \sup\left\{r > 0 : \pi_\mu \circ \exp_\mu : B(0,r) \to G_\mu/K_\mu \quad \text{diffeomorphism}\right\}.$$

Theorem (JL 2010)

μ_k, $\lambda \in \mathcal{H}_{q,n}$.

Convergence

Lie injectivity radius of $(G_\mu/K_\mu, g_\mu)$, $\mu \in \mathcal{H}_{q,n}$,

$$r_\mu := \sup\left\{ r > 0 : \pi_\mu \circ \exp_\mu : B(0,r) \to G_\mu/K_\mu \quad \text{diffeomorphism} \right\}.$$

Theorem (JL 2010)

$\mu_k, \lambda \in \mathcal{H}_{q,n}$. *If* $\mu_k \to \lambda$ *and* $\inf_k r_{\mu_k} > 0$, *then,*

Convergence

Lie injectivity radius of $(G_\mu/K_\mu, g_\mu)$, $\mu \in \mathcal{H}_{q,n}$,

$$r_\mu := \sup\left\{ r > 0 : \pi_\mu \circ \exp_\mu : B(0,r) \to G_\mu/K_\mu \quad \text{diffeomorphism} \right\}.$$

Theorem (JL 2010)

$\mu_k, \lambda \in \mathcal{H}_{q,n}$. If $\mu_k \to \lambda$ and $\inf_k r_{\mu_k} > 0$, then, $(G_{\mu_k}/K_{\mu_k}, g_{\mu_k}) \to (G_\lambda/K_\lambda, g_\lambda)$ pointed (or Cheeger-Gromov), after passing to a subsequence.

Convergence

Lie injectivity radius of $(G_\mu/K_\mu, g_\mu)$, $\mu \in \mathcal{H}_{q,n}$,

$$r_\mu := \sup\left\{r > 0 : \pi_\mu \circ \exp_\mu : B(0,r) \to G_\mu/K_\mu \quad \text{diffeomorphism}\right\}.$$

Theorem (JL 2010)

$\mu_k, \lambda \in \mathcal{H}_{q,n}$. If $\mu_k \to \lambda$ and $\inf_k r_{\mu_k} > 0$, then, $(G_{\mu_k}/K_{\mu_k}, g_{\mu_k}) \to (G_\lambda/K_\lambda, g_\lambda)$ pointed (or Cheeger-Gromov), after passing to a subsequence.

In the case $\mathfrak{k} = 0$,

Convergence

Lie injectivity radius of $(G_\mu/K_\mu, g_\mu)$, $\mu \in \mathcal{H}_{q,n}$,

$$r_\mu := \sup\left\{ r > 0 : \pi_\mu \circ \exp_\mu : B(0,r) \to G_\mu/K_\mu \quad \text{diffeomorphism} \right\}.$$

Theorem (JL 2010)

$\mu_k, \lambda \in \mathcal{H}_{q,n}$. If $\mu_k \to \lambda$ and $\inf_k r_{\mu_k} > 0$, then, $(G_{\mu_k}/K_{\mu_k}, g_{\mu_k}) \to (G_\lambda/K_\lambda, g_\lambda)$ pointed (or Cheeger-Gromov), after passing to a subsequence.

In the case $\mathfrak{k} = 0$, condition $\inf_k r_{\mu_k} > 0$ automatically holds,

Convergence

Lie injectivity radius of $(G_\mu/K_\mu, g_\mu)$, $\mu \in \mathcal{H}_{q,n}$,

$$r_\mu := \sup\left\{ r > 0 : \pi_\mu \circ \exp_\mu : B(0,r) \to G_\mu/K_\mu \quad \text{diffeomorphism} \right\}.$$

Theorem (JL 2010)

$\mu_k, \lambda \in \mathcal{H}_{q,n}$. If $\mu_k \to \lambda$ and $\inf\limits_k r_{\mu_k} > 0$, then, $(G_{\mu_k}/K_{\mu_k}, g_{\mu_k}) \to (G_\lambda/K_\lambda, g_\lambda)$ pointed (or Cheeger-Gromov), after passing to a subsequence.

In the case $\mathfrak{k} = 0$, condition $\inf\limits_k r_{\mu_k} > 0$ automatically holds, and $g_{\mu_k} \to g_\lambda$ smoothly on $\mathbb{R}^n \equiv \mathfrak{g}$, provided all μ_k are completely solvable (e.g. nilpotent).

Examples of singular behavior

- A sequence $\mu_k \in \mathcal{H}_{1,7}$ of Aloff-Wallach spaces $(\mathrm{SU}(3)/S^1_{p,q})$

Examples of singular behavior

- A sequence $\mu_k \in \mathcal{H}_{1,7}$ of Aloff-Wallach spaces $(\mathrm{SU}(3)/S^1_{p,q})$ which converges to another Aloff-Wallach space λ, but such that it does not admit any pointed convergent subsequence.

Examples of singular behavior

- A sequence $\mu_k \in \mathcal{H}_{1,7}$ of Aloff-Wallach spaces $(\mathrm{SU}(3)/S^1_{p,q})$ which converges to another Aloff-Wallach space λ, but such that it does not admit any pointed convergent subsequence.
- A divergent sequence $\mu_k \in \mathcal{H}_{0,3}$ of left-invariant metrics on $\widetilde{\mathrm{SL}_2}(\mathbb{R})$

Examples of singular behavior

- A sequence $\mu_k \in \mathcal{H}_{1,7}$ of Aloff-Wallach spaces $(\mathrm{SU}(3)/S^1_{p,q})$ which converges to another Aloff-Wallach space λ, but such that it does not admit any pointed convergent subsequence.
- A divergent sequence $\mu_k \in \mathcal{H}_{0,3}$ of left-invariant metrics on $\widetilde{\mathrm{SL}_2}(\mathbb{R})$ which nevertheless pointed converges to $\mathbb{R} \times H^2$.

Examples of singular behavior

- A sequence $\mu_k \in \mathcal{H}_{1,7}$ of Aloff-Wallach spaces $(\mathrm{SU}(3)/S^1_{p,q})$ which converges to another Aloff-Wallach space λ, but such that it does not admit any pointed convergent subsequence.
- A divergent sequence $\mu_k \in \mathcal{H}_{0,3}$ of left-invariant metrics on $\widetilde{\mathrm{SL}_2}(\mathbb{R})$ which nevertheless pointed converges to $\mathbb{R} \times H^2$. μ_k is actually isometric to a convergent sequence in $\mathcal{H}_{1,3}$.

Examples of singular behavior

- A sequence $\mu_k \in \mathcal{H}_{1,7}$ of Aloff-Wallach spaces $(\mathrm{SU}(3)/S^1_{p,q})$ which converges to another Aloff-Wallach space λ, but such that it does not admit any pointed convergent subsequence.
- A divergent sequence $\mu_k \in \mathcal{H}_{0,3}$ of left-invariant metrics on $\widetilde{\mathrm{SL}_2}(\mathbb{R})$ which nevertheless pointed converges to $\mathbb{R} \times H^2$. μ_k is actually isometric to a convergent sequence in $\mathcal{H}_{1,3}$.
- A sequence $\mu_k \in \mathcal{H}_{1,5}$ of homogeneous metrics on $S^3 \times S^2$ converging to $\lambda \notin \mathcal{H}_{1,5}$ (K_λ noncompact).

Examples of singular behavior

- A sequence $\mu_k \in \mathcal{H}_{1,7}$ of Aloff-Wallach spaces $(\mathrm{SU}(3)/S^1_{p,q})$ which converges to another Aloff-Wallach space λ, but such that it does not admit any pointed convergent subsequence.
- A divergent sequence $\mu_k \in \mathcal{H}_{0,3}$ of left-invariant metrics on $\widetilde{\mathrm{SL}_2}(\mathbb{R})$ which nevertheless pointed converges to $\mathbb{R} \times H^2$. μ_k is actually isometric to a convergent sequence in $\mathcal{H}_{1,3}$.
- A sequence $\mu_k \in \mathcal{H}_{1,5}$ of homogeneous metrics on $S^3 \times S^2$ converging to $\lambda \notin \mathcal{H}_{1,5}$ (K_λ noncompact). However, λ can be viewed as an element of $\mathcal{H}_{2,4}$, giving rise to a collapsing of the μ_k with bounded curvature to a metric on $S^2 \times S^2$.

Ricci flow

Ricci flow

$g(t)$ Ricci flow starting at the homogeneous manifold

$$(M, g_0) = (G_{\mu_0}/K_{\mu_0}, g_{\mu_0}), \qquad \mu_0 \in \mathcal{H}_{q,n},$$

Ricci flow

$g(t)$ Ricci flow starting at the homogeneous manifold

$$(M, g_0) = (G_{\mu_0}/K_{\mu_0}, g_{\mu_0}), \qquad \mu_0 \in \mathcal{H}_{q,n},$$

$$\boxed{\frac{\partial}{\partial t} g(t) = -2\,\mathrm{Rc}(g(t))}, \qquad g(0) = g_{\mu_0},$$

Ricci flow

$g(t)$ Ricci flow starting at the homogeneous manifold

$$(M, g_0) = (G_{\mu_0}/K_{\mu_0}, g_{\mu_0}), \qquad \mu_0 \in \mathcal{H}_{q,n},$$

$$\boxed{\frac{\partial}{\partial t} g(t) = -2\,\mathrm{Rc}(g(t))}, \qquad g(0) = g_{\mu_0},$$

$\Rightarrow G_{\mu_0} \subset \mathrm{Isom}(M, g(t))$ for all t

Ricci flow

$g(t)$ Ricci flow starting at the homogeneous manifold

$$(M, g_0) = (G_{\mu_0}/K_{\mu_0}, g_{\mu_0}), \qquad \mu_0 \in \mathcal{H}_{q,n},$$

$$\boxed{\frac{\partial}{\partial t} g(t) = -2\,\mathrm{Rc}(g(t))}, \qquad g(0) = g_{\mu_0},$$

$\Rightarrow G_{\mu_0} \subset \mathrm{Isom}(M, g(t))$ for all t

$\Rightarrow g(t) \leftrightarrow \langle \cdot, \cdot \rangle_t$: $\mathrm{Ad}(K_{\mu_0})$-invariant inner product on $\mathfrak{p}$ solving the ODE:

$$\frac{\mathrm{d}}{\mathrm{d}t} \langle \cdot, \cdot \rangle_t = -2\,\mathrm{Rc}(\langle \cdot, \cdot \rangle_t), \qquad \langle \cdot, \cdot \rangle_0 = \langle \cdot, \cdot \rangle,$$

Ricci flow

$g(t)$ Ricci flow starting at the homogeneous manifold

$$(M, g_0) = (G_{\mu_0}/K_{\mu_0}, g_{\mu_0}), \qquad \mu_0 \in \mathcal{H}_{q,n},$$

$$\boxed{\frac{\partial}{\partial t} g(t) = -2\,\mathrm{Rc}(g(t))}, \qquad g(0) = g_{\mu_0},$$

$\Rightarrow G_{\mu_0} \subset \mathrm{Isom}(M, g(t))$ for all t

$\Rightarrow g(t) \leftrightarrow \langle\cdot,\cdot\rangle_t$: $\mathrm{Ad}(K_{\mu_0})$-invariant inner product on $\mathfrak{p}$ solving the ODE:

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle\cdot,\cdot\rangle_t = -2\,\mathrm{Rc}(\langle\cdot,\cdot\rangle_t), \qquad \langle\cdot,\cdot\rangle_0 = \langle\cdot,\cdot\rangle,$$

$$t \in (T_-, T_+), \qquad -\infty \leq T_- < 0 < T_+ \leq T_+.$$

Ricci flow

$g(t)$ Ricci flow starting at the homogeneous manifold

$$(M, g_0) = (G_{\mu_0}/K_{\mu_0}, g_{\mu_0}), \qquad \mu_0 \in \mathcal{H}_{q,n},$$

$$\boxed{\frac{\partial}{\partial t} g(t) = -2\,\mathrm{Rc}(g(t))}, \qquad g(0) = g_{\mu_0},$$

$\Rightarrow G_{\mu_0} \subset \mathrm{Isom}(M, g(t))$ for all t

$\Rightarrow g(t) \leftrightarrow \langle\cdot,\cdot\rangle_t$: $\mathrm{Ad}(K_{\mu_0})$-invariant inner product on $\mathfrak{p}$ solving the ODE:

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle\cdot,\cdot\rangle_t = -2\,\mathrm{Rc}(\langle\cdot,\cdot\rangle_t), \qquad \langle\cdot,\cdot\rangle_0 = \langle\cdot,\cdot\rangle,$$

$t \in (T_-, T_+), \quad -\infty \leq T_- < 0 < T_+ \leq T_+.$

Ricci flow on $\mathcal{H}_{q,n}$???

Bracket flow

Bracket flow

Consider for a curve $\mu(t) \in \Lambda^2\mathfrak{g}^* \otimes \mathfrak{g}$ the bracket flow:

Bracket flow

Consider for a curve $\mu(t) \in \Lambda^2\mathfrak{g}^* \otimes \mathfrak{g}$ the bracket flow:

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}t}\mu = -\pi\left(\left[\begin{smallmatrix} 0 & 0 \\ 0 & \mathrm{Ric}_\mu \end{smallmatrix}\right]\right)\mu},$$

Bracket flow

Consider for a curve $\mu(t) \in \Lambda^2\mathfrak{g}^* \otimes \mathfrak{g}$ the bracket flow:

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}t}\mu = -\pi\left(\begin{bmatrix} 0 & 0 \\ 0 & \mathrm{Ric}_\mu \end{bmatrix}\right)\mu},$$

$$\pi : \mathfrak{gl}_{q+n} \longrightarrow \mathrm{End}(\Lambda^2\mathfrak{g}^* \otimes \mathfrak{g}),$$

Bracket flow

Consider for a curve $\mu(t) \in \Lambda^2\mathfrak{g}^* \otimes \mathfrak{g}$ the bracket flow:

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}t}\mu = -\pi\left(\left[\begin{smallmatrix} 0 & 0 \\ 0 & \mathrm{Ric}_\mu \end{smallmatrix}\right]\right)\mu},$$

$$\pi : \mathfrak{gl}_{q+n} \longrightarrow \mathrm{End}(\Lambda^2\mathfrak{g}^* \otimes \mathfrak{g}),$$

$$\pi(A)\mu := A\mu(\cdot,\cdot) - \mu(A\cdot,\cdot) - \mu(\cdot,A\cdot), \quad \forall A \in \mathfrak{gl}_{q+n},$$

Bracket flow

Consider for a curve $\mu(t) \in \Lambda^2\mathfrak{g}^* \otimes \mathfrak{g}$ the bracket flow:

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}t}\mu = -\pi\left(\left[\begin{smallmatrix} 0 & 0 \\ 0 & \mathrm{Ric}_\mu \end{smallmatrix}\right]\right)\mu},$$

$$\pi : \mathfrak{gl}_{q+n} \longrightarrow \mathrm{End}(\Lambda^2\mathfrak{g}^* \otimes \mathfrak{g}),$$

$$\pi(A)\mu := A\mu(\cdot,\cdot) - \mu(A\cdot,\cdot) - \mu(\cdot,A\cdot), \quad \forall A \in \mathfrak{gl}_{q+n},$$

$\mathrm{Ric}_\mu : \mathfrak{p} \to \mathfrak{p}$ Ricci operator,

Bracket flow

Consider for a curve $\mu(t) \in \Lambda^2\mathfrak{g}^* \otimes \mathfrak{g}$ the bracket flow:

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}t}\mu = -\pi\left(\left[\begin{smallmatrix} 0 & 0 \\ 0 & \mathrm{Ric}_\mu \end{smallmatrix}\right]\right)\mu},$$

$\pi : \mathfrak{gl}_{q+n} \longrightarrow \mathrm{End}(\Lambda^2\mathfrak{g}^* \otimes \mathfrak{g})$,

$\pi(A)\mu := A\mu(\cdot,\cdot) - \mu(A\cdot,\cdot) - \mu(\cdot, A\cdot), \quad \forall A \in \mathfrak{gl}_{q+n}$,

$\mathrm{Ric}_\mu : \mathfrak{p} \to \mathfrak{p}$ Ricci operator, $\mathrm{Ric}_\mu = M_\mu - \frac{1}{2}B_\mu - S(\mathrm{ad}_\mu H_\mu|_{\mathfrak{p}})$.

Bracket flow

Consider for a curve $\mu(t) \in \Lambda^2\mathfrak{g}^* \otimes \mathfrak{g}$ the bracket flow:

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}t}\mu = -\pi\left(\left[\begin{smallmatrix} 0 & 0 \\ 0 & \mathrm{Ric}_\mu \end{smallmatrix}\right]\right)\mu},$$

$\pi : \mathfrak{gl}_{q+n} \longrightarrow \mathrm{End}(\Lambda^2\mathfrak{g}^* \otimes \mathfrak{g})$,

$\pi(A)\mu := A\mu(\cdot,\cdot) - \mu(A\cdot,\cdot) - \mu(\cdot,A\cdot), \quad \forall A \in \mathfrak{gl}_{q+n}$,

$\mathrm{Ric}_\mu : \mathfrak{p} \to \mathfrak{p}$ Ricci operator, $\mathrm{Ric}_\mu = M_\mu - \frac{1}{2}B_\mu - S(\mathrm{ad}_\mu H_\mu|_{\mathfrak{p}})$.

$\mu_0 \in \mathcal{H}_{q,n} \Rightarrow \mu(t) \in \mathcal{H}_{q,n}$ for all t,

Bracket flow

Consider for a curve $\mu(t) \in \Lambda^2\mathfrak{g}^* \otimes \mathfrak{g}$ the bracket flow:

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}t}\mu = -\pi\left(\left[\begin{smallmatrix} 0 & 0 \\ 0 & \mathrm{Ric}_\mu \end{smallmatrix}\right]\right)\mu},$$

$\pi : \mathfrak{gl}_{q+n} \longrightarrow \mathrm{End}(\Lambda^2\mathfrak{g}^* \otimes \mathfrak{g})$,

$\pi(A)\mu := A\mu(\cdot,\cdot) - \mu(A\cdot,\cdot) - \mu(\cdot,A\cdot), \quad \forall A \in \mathfrak{gl}_{q+n}$,

$\mathrm{Ric}_\mu : \mathfrak{p} \to \mathfrak{p}$ Ricci operator, $\mathrm{Ric}_\mu = M_\mu - \frac{1}{2}B_\mu - S(\mathrm{ad}_\mu H_\mu|_{\mathfrak{p}})$.

$\mu_0 \in \mathcal{H}_{q,n} \Rightarrow \mu(t) \in \mathcal{H}_{q,n}$ for all t,

$\mu(t) \rightsquigarrow (G_{\mu(t)}/K_{\mu(t)}, \langle\cdot,\cdot\rangle)$ curve of homogeneous spaces.

$$(M, g_0) = (G_{\mu_0}/K_{\mu_0}, g_{\mu_0}), \quad \mu_0 \in \mathcal{H}_{q,n}, \quad \text{(recall } \mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}\text{)},$$

$$(M,g_0) = (G_{\mu_0}/K_{\mu_0}, g_{\mu_0}), \quad \mu_0 \in \mathcal{H}_{q,n}, \quad (\text{recall } \mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}),$$

$$(M, g(t)), \quad \left(G_{\mu_0}/K_{\mu_0}, g_{\langle\cdot,\cdot\rangle_t}\right), \quad \left(G_{\mu(t)}/K_{\mu(t)}, g_{\mu(t)}\right),$$

$$(M, g_0) = (G_{\mu_0}/K_{\mu_0}, g_{\mu_0}), \quad \mu_0 \in \mathcal{H}_{q,n}, \quad (\text{recall } \mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}),$$

$$(M, g(t)), \quad \left(G_{\mu_0}/K_{\mu_0}, g_{\langle\cdot,\cdot\rangle_t}\right), \quad \left(G_{\mu(t)}/K_{\mu(t)}, g_{\mu(t)}\right),$$

Theorem (JL 2010)

$\exists\ \varphi(t) : M = G_{\mu_0}/K_{\mu_0} \longrightarrow G_{\mu(t)}/K_{\mu(t)}$ *such that*

$$(M, g_0) = (G_{\mu_0}/K_{\mu_0}, g_{\mu_0}), \quad \mu_0 \in \mathcal{H}_{q,n}, \quad (\text{recall } \mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}),$$

$$(M, g(t)), \quad \left(G_{\mu_0}/K_{\mu_0}, g_{\langle \cdot,\cdot \rangle_t}\right), \quad \left(G_{\mu(t)}/K_{\mu(t)}, g_{\mu(t)}\right),$$

Theorem (JL 2010)

$\exists\ \varphi(t) : M = G_{\mu_0}/K_{\mu_0} \longrightarrow G_{\mu(t)}/K_{\mu(t)}$ *such that*

$$\boxed{g(t) = \varphi(t)^* g_{\mu(t)}}, \qquad \forall t \in (T_-, T_+).$$

$$(M, g_0) = (G_{\mu_0}/K_{\mu_0}, g_{\mu_0}), \quad \mu_0 \in \mathcal{H}_{q,n}, \quad (\text{recall } \mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}),$$

$$(M, g(t)), \quad \left(G_{\mu_0}/K_{\mu_0}, g_{\langle\cdot,\cdot\rangle_t}\right), \quad \left(G_{\mu(t)}/K_{\mu(t)}, g_{\mu(t)}\right),$$

Theorem (JL 2010)

$\exists\ \varphi(t) : M = G_{\mu_0}/K_{\mu_0} \longrightarrow G_{\mu(t)}/K_{\mu(t)}$ *such that*

$$\boxed{g(t) = \varphi(t)^* g_{\mu(t)}}, \qquad \forall t \in (T_-, T_+).$$

Moreover, $\varphi(t) : G_{\mu_0}/K_{\mu_0} \longrightarrow G_{\mu(t)}/K_{\mu(t)}$ *equivariant diffeomorphism determined by the Lie group isomorphism between* G_{μ_0} *and* $G_{\mu(t)}$

$$(M, g_0) = (G_{\mu_0}/K_{\mu_0}, g_{\mu_0}), \quad \mu_0 \in \mathcal{H}_{q,n}, \quad (\text{recall } \mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}),$$

$$(M, g(t)), \quad \left(G_{\mu_0}/K_{\mu_0}, g_{\langle\cdot,\cdot\rangle_t}\right), \quad \left(G_{\mu(t)}/K_{\mu(t)}, g_{\mu(t)}\right),$$

Theorem (JL 2010)

$\exists\ \varphi(t) : M = G_{\mu_0}/K_{\mu_0} \longrightarrow G_{\mu(t)}/K_{\mu(t)}$ such that

$$\boxed{g(t) = \varphi(t)^* g_{\mu(t)}}, \qquad \forall t \in (T_-, T_+).$$

Moreover, $\varphi(t) : G_{\mu_0}/K_{\mu_0} \longrightarrow G_{\mu(t)}/K_{\mu(t)}$ equivariant diffeomorphism determined by the Lie group isomorphism between G_{μ_0} and $G_{\mu(t)}$ with derivative $\tilde{h} := \left[\begin{smallmatrix} I & 0 \\ 0 & h \end{smallmatrix}\right] : \mathfrak{g} \longrightarrow \mathfrak{g}$, where $h(t) := d\varphi(t)|_o : \mathfrak{p} \longrightarrow \mathfrak{p}$,

$$(M, g_0) = (G_{\mu_0}/K_{\mu_0}, g_{\mu_0}), \quad \mu_0 \in \mathcal{H}_{q,n}, \quad (\text{recall } \mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}),$$

$$(M, g(t)), \quad \left(G_{\mu_0}/K_{\mu_0}, g_{\langle\cdot,\cdot\rangle_t}\right), \quad \left(G_{\mu(t)}/K_{\mu(t)}, g_{\mu(t)}\right),$$

Theorem (JL 2010)

$\exists\ \varphi(t) : M = G_{\mu_0}/K_{\mu_0} \longrightarrow G_{\mu(t)}/K_{\mu(t)}$ *such that*

$$\boxed{g(t) = \varphi(t)^* g_{\mu(t)}}, \qquad \forall t \in (T_-, T_+).$$

Moreover, $\varphi(t) : G_{\mu_0}/K_{\mu_0} \longrightarrow G_{\mu(t)}/K_{\mu(t)}$ *equivariant diffeomorphism determined by the Lie group isomorphism between* G_{μ_0} *and* $G_{\mu(t)}$ *with derivative* $\tilde{h} := \left[\begin{smallmatrix} I & 0 \\ 0 & h \end{smallmatrix}\right] : \mathfrak{g} \longrightarrow \mathfrak{g}$, *where* $h(t) := d\varphi(t)|_o : \mathfrak{p} \longrightarrow \mathfrak{p}$,

(i) $\frac{\mathrm{d}}{\mathrm{d}t} h = -h \operatorname{Ric}(\langle\cdot,\cdot\rangle_t), \quad h(0) = I.$

(ii) $\frac{\mathrm{d}}{\mathrm{d}t} h = -\operatorname{Ric}_{\mu(t)} h, \quad h(0) = I.$

$$(M, g_0) = (G_{\mu_0}/K_{\mu_0}, g_{\mu_0}), \quad \mu_0 \in \mathcal{H}_{q,n}, \quad (\text{recall } \mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}),$$

$$(M, g(t)), \quad \left(G_{\mu_0}/K_{\mu_0}, g_{\langle\cdot,\cdot\rangle_t}\right), \quad \left(G_{\mu(t)}/K_{\mu(t)}, g_{\mu(t)}\right),$$

Theorem (JL 2010)

$\exists\ \varphi(t) : M = G_{\mu_0}/K_{\mu_0} \longrightarrow G_{\mu(t)}/K_{\mu(t)}$ *such that*

$$\boxed{g(t) = \varphi(t)^* g_{\mu(t)}}, \qquad \forall t \in (T_-, T_+).$$

Moreover, $\varphi(t) : G_{\mu_0}/K_{\mu_0} \longrightarrow G_{\mu(t)}/K_{\mu(t)}$ *equivariant diffeomorphism determined by the Lie group isomorphism between* G_{μ_0} *and* $G_{\mu(t)}$ *with derivative* $\tilde{h} := \left[\begin{smallmatrix} I & 0 \\ 0 & h \end{smallmatrix}\right] : \mathfrak{g} \longrightarrow \mathfrak{g}$*, where* $h(t) := d\varphi(t)|_o : \mathfrak{p} \longrightarrow \mathfrak{p}$*,*

(i) $\frac{\mathrm{d}}{\mathrm{d}t} h = -h \operatorname{Ric}(\langle\cdot,\cdot\rangle_t), \quad h(0) = I.$

(ii) $\frac{\mathrm{d}}{\mathrm{d}t} h = -\operatorname{Ric}_{\mu(t)} h, \quad h(0) = I.$

(iii) $\langle\cdot,\cdot\rangle_t = \langle h\cdot, h\cdot\rangle.$

(iv) $\mu(t) = \tilde{h}\mu_0(\tilde{h}^{-1}\cdot, \tilde{h}^{-1}\cdot).$

$$(M, g(t)), \quad \left(G_{\mu_0}/K_{\mu_0}, g_{\langle\cdot,\cdot\rangle_t}\right), \quad \left(G_{\mu(t)}/K_{\mu(t)}, g_{\mu(t)}\right),$$

$$(M, g(t)), \quad \left(G_{\mu_0}/K_{\mu_0}, g_{\langle\cdot,\cdot\rangle_t}\right), \quad \left(G_{\mu(t)}/K_{\mu(t)}, g_{\mu(t)}\right),$$

$$g(t) = \varphi(t)^* g_{\mu(t)}$$

$$(M, g(t)), \quad \left(G_{\mu_0}/K_{\mu_0}, g_{\langle\cdot,\cdot\rangle_t}\right), \quad \left(G_{\mu(t)}/K_{\mu(t)}, g_{\mu(t)}\right),$$

$$g(t) = \varphi(t)^* g_{\mu(t)}$$

- Same behavior of the curvature and of any other Riemannian invariant.

$$(M, g(t)), \quad \left(G_{\mu_0}/K_{\mu_0}, g_{\langle\cdot,\cdot\rangle_t}\right), \quad \left(G_{\mu(t)}/K_{\mu(t)}, g_{\mu(t)}\right),$$

$$g(t) = \varphi(t)^* g_{\mu(t)}$$

- Same behavior of the curvature and of any other Riemannian invariant.
- Maximal interval of time where a solution exists is the same.

$$(M, g(t)), \quad \left(G_{\mu_0}/K_{\mu_0}, g_{\langle\cdot,\cdot\rangle_t}\right), \quad \left(G_{\mu(t)}/K_{\mu(t)}, g_{\mu(t)}\right),$$

$$g(t) = \varphi(t)^* g_{\mu(t)}$$

- Same behavior of the curvature and of any other Riemannian invariant.
- Maximal interval of time where a solution exists is the same.
- $\mu(t_k) \to \lambda \in \mathcal{H}_{q,n}$ (or a suitable normalization)

$$(M, g(t)), \quad \left(G_{\mu_0}/K_{\mu_0}, g_{\langle\cdot,\cdot\rangle_t}\right), \quad \left(G_{\mu(t)}/K_{\mu(t)}, g_{\mu(t)}\right),$$

$$g(t) = \varphi(t)^* g_{\mu(t)}$$

- Same behavior of the curvature and of any other Riemannian invariant.
- Maximal interval of time where a solution exists is the same.
- $\mu(t_k) \to \lambda \in \mathcal{H}_{q,n}$ (or a suitable normalization) $\Rightarrow$ convergence or subconvergence $g_{\mu_k} \to g_\lambda$ (infinitesimal, local or pointed).

$$(M, g(t)), \quad \left(G_{\mu_0}/K_{\mu_0}, g_{\langle\cdot,\cdot\rangle_t}\right), \quad \left(G_{\mu(t)}/K_{\mu(t)}, g_{\mu(t)}\right),$$

$$g(t) = \varphi(t)^* g_{\mu(t)}$$

- Same behavior of the curvature and of any other Riemannian invariant.
- Maximal interval of time where a solution exists is the same.
- $\mu(t_k) \to \lambda \in \mathcal{H}_{q,n}$ (or a suitable normalization) $\Rightarrow$ convergence or subconvergence $g_{\mu_k} \to g_\lambda$ (infinitesimal, local or pointed).
- $\mu(t)|_{\mathfrak{k}\times\mathfrak{g}} \equiv \mu_0|_{\mathfrak{k}\times\mathfrak{g}}$.

Application to nilmanifolds

Application to nilmanifolds

G nilpotent and s.c., $K = \{e\}$,

Application to nilmanifolds

G nilpotent and s.c., $K = \{e\}$, $\mathfrak{g} = \mathfrak{p} = \mathbb{R}^n = G$,

Application to nilmanifolds

G nilpotent and s.c., $K = \{e\}$, $\mathfrak{g} = \mathfrak{p} = \mathbb{R}^n = G$, μ nilpotent Lie bracket on $\mathfrak{g}$,

Application to nilmanifolds

G nilpotent and s.c., $K = \{e\}$, $\mathfrak{g} = \mathfrak{p} = \mathbb{R}^n = G$, μ nilpotent Lie bracket on $\mathfrak{g}$, $\boxed{\mathrm{Ric}_\mu = M_\mu}$,

Application to nilmanifolds

G nilpotent and s.c., $K = \{e\}$, $\mathfrak{g} = \mathfrak{p} = \mathbb{R}^n = G$, μ nilpotent Lie bracket on $\mathfrak{g}$, $\boxed{\operatorname{Ric}_\mu = M_\mu}$, $\operatorname{R}(g_\mu) = -\frac{1}{4}\|\mu\|^2$,

Application to nilmanifolds

G nilpotent and s.c., $K = \{e\}$, $\mathfrak{g} = \mathfrak{p} = \mathbb{R}^n = G$, μ nilpotent Lie bracket on $\mathfrak{g}$, $\boxed{\mathrm{Ric}_\mu = M_\mu}$, $\mathrm{R}(g_\mu) = -\frac{1}{4}\|\mu\|^2$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\mu = -\pi(\mathrm{Ric}_\mu)\mu$$

Application to nilmanifolds

G nilpotent and s.c., $K = \{e\}$, $\mathfrak{g} = \mathfrak{p} = \mathbb{R}^n = G$, μ nilpotent Lie bracket on $\mathfrak{g}$, $\boxed{\mathrm{Ric}_\mu = M_\mu}$, $\mathrm{R}(g_\mu) = -\frac{1}{4}\|\mu\|^2$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\mu = -\pi(\mathrm{Ric}_\mu)\mu = -\operatorname{grad}(\operatorname{tr}\mathrm{Ric}_\mu^2)_\mu$$

Application to nilmanifolds

G nilpotent and s.c., $K = \{e\}$, $\mathfrak{g} = \mathfrak{p} = \mathbb{R}^n = G$, μ nilpotent Lie bracket on $\mathfrak{g}$, $\boxed{\operatorname{Ric}_\mu = M_\mu}$, $\operatorname{R}(g_\mu) = -\frac{1}{4}\|\mu\|^2$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\mu = -\pi(\operatorname{Ric}_\mu)\mu = -\operatorname{grad}(\operatorname{tr}\operatorname{Ric}_\mu^2)_\mu$$

negative gradient flow of the square norm of the moment map for the action $\mathrm{GL}_n \circlearrowleft \Lambda^2\mathfrak{g}^* \otimes \mathfrak{g}$.

Application to nilmanifolds

G nilpotent and s.c., $K = \{e\}$, $\mathfrak{g} = \mathfrak{p} = \mathbb{R}^n = G$, μ nilpotent Lie bracket on $\mathfrak{g}$, $\boxed{\mathrm{Ric}_\mu = M_\mu}$, $\mathrm{R}(g_\mu) = -\frac{1}{4}\|\mu\|^2$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\mu = -\pi(\mathrm{Ric}_\mu)\mu = -\operatorname{grad}(\operatorname{tr}\mathrm{Ric}_\mu^2)_\mu$$

negative gradient flow of the square norm of the moment map for the action $\mathrm{GL}_n \circlearrowleft \Lambda^2\mathfrak{g}^* \otimes \mathfrak{g}$.

[JL 2001] g_μ Ricci soliton $\Leftrightarrow$ μ critical point of $\mu \mapsto \operatorname{tr}\mathrm{Ric}_\mu^2$ on the sphere.

Application to nilmanifolds

G nilpotent and s.c., $K = \{e\}$, $\mathfrak{g} = \mathfrak{p} = \mathbb{R}^n = G$, μ nilpotent Lie bracket on $\mathfrak{g}$, $\boxed{\mathrm{Ric}_\mu = M_\mu}$, $\mathrm{R}(g_\mu) = -\frac{1}{4}\|\mu\|^2$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\mu = -\pi(\mathrm{Ric}_\mu)\mu = -\operatorname{grad}(\operatorname{tr}\mathrm{Ric}_\mu^2)_\mu$$

negative gradient flow of the square norm of the moment map for the action $\mathrm{GL}_n \circlearrowleft \Lambda^2\mathfrak{g}^* \otimes \mathfrak{g}$.

[JL 2001] g_μ Ricci soliton $\Leftrightarrow$ μ critical point of $\mu \mapsto \operatorname{tr}\mathrm{Ric}_\mu^2$ on the sphere.

Theorem (JL 2009)

- *The Ricci flow $g(t)$ is a type-III solution*

Application to nilmanifolds

G nilpotent and s.c., $K = \{e\}$, $\mathfrak{g} = \mathfrak{p} = \mathbb{R}^n = G$, μ nilpotent Lie bracket on $\mathfrak{g}$, $\boxed{\operatorname{Ric}_\mu = M_\mu}$, $\operatorname{R}(g_\mu) = -\frac{1}{4}\|\mu\|^2$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\mu = -\pi(\operatorname{Ric}_\mu)\mu = -\operatorname{grad}(\operatorname{tr}\operatorname{Ric}_\mu^2)_\mu$$

negative gradient flow of the square norm of the moment map for the action $\mathrm{GL}_n \circlearrowleft \Lambda^2\mathfrak{g}^* \otimes \mathfrak{g}$.

[JL 2001] g_μ Ricci soliton $\Leftrightarrow$ μ critical point of $\mu \mapsto \operatorname{tr}\operatorname{Ric}_\mu^2$ on the sphere.

Theorem (JL 2009)

- *The Ricci flow $g(t)$ is a type-III solution (i.e. $t \in [0, \infty)$ and $\|\operatorname{Rm}(g(t))\| \leq \frac{C}{t}$).*

Application to nilmanifolds

G nilpotent and s.c., $K = \{e\}$, $\mathfrak{g} = \mathfrak{p} = \mathbb{R}^n = G$, μ nilpotent Lie bracket on $\mathfrak{g}$, $\boxed{\mathrm{Ric}_\mu = M_\mu}$, $\mathrm{R}(g_\mu) = -\frac{1}{4}\|\mu\|^2$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\mu = -\pi(\mathrm{Ric}_\mu)\mu = -\operatorname{grad}(\operatorname{tr}\mathrm{Ric}_\mu^2)_\mu$$

negative gradient flow of the square norm of the moment map for the action $\mathrm{GL}_n \circlearrowleft \Lambda^2\mathfrak{g}^* \otimes \mathfrak{g}$.

[JL 2001] g_μ Ricci soliton $\Leftrightarrow$ μ critical point of $\mu \mapsto \operatorname{tr}\mathrm{Ric}_\mu^2$ on the sphere.

Theorem (JL 2009)

- *The Ricci flow $g(t)$ is a type-III solution (i.e. $t \in [0, \infty)$ and $\|\mathrm{Rm}(g(t))\| \leq \frac{C}{t}$).*
- *$g(t)$ converges in C^∞ to a flat metric uniformly on compact sets in $\mathbb{R}^n$.*

Application to nilmanifolds

G nilpotent and s.c., $K = \{e\}$, $\mathfrak{g} = \mathfrak{p} = \mathbb{R}^n = G$, μ nilpotent Lie bracket on $\mathfrak{g}$, $\boxed{\mathrm{Ric}_\mu = M_\mu}$, $\mathrm{R}(g_\mu) = -\frac{1}{4}\|\mu\|^2$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\mu = -\pi(\mathrm{Ric}_\mu)\mu = -\operatorname{grad}(\operatorname{tr}\mathrm{Ric}_\mu^2)_\mu$$

negative gradient flow of the square norm of the moment map for the action $\mathrm{GL}_n \circlearrowleft \Lambda^2\mathfrak{g}^* \otimes \mathfrak{g}$.

[JL 2001] g_μ Ricci soliton $\Leftrightarrow$ μ critical point of $\mu \mapsto \operatorname{tr}\mathrm{Ric}_\mu^2$ on the sphere.

Theorem (JL 2009)

- *The Ricci flow $g(t)$ is a type-III solution (i.e. $t \in [0,\infty)$ and $\|\mathrm{Rm}(g(t))\| \leq \frac{C}{t}$).*
- *$g(t)$ converges in C^∞ to a flat metric uniformly on compact sets in $\mathbb{R}^n$.*
- *After rescaling ($\mathrm{R} \equiv -1$),*

Application to nilmanifolds

G nilpotent and s.c., $K = \{e\}$, $\mathfrak{g} = \mathfrak{p} = \mathbb{R}^n = G$, μ nilpotent Lie bracket on $\mathfrak{g}$, $\boxed{\operatorname{Ric}_\mu = M_\mu}$, $\mathrm{R}(g_\mu) = -\frac{1}{4}\|\mu\|^2$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\mu = -\pi(\operatorname{Ric}_\mu)\mu = -\operatorname{grad}(\operatorname{tr}\operatorname{Ric}_\mu^2)_\mu$$

negative gradient flow of the square norm of the moment map for the action $\mathrm{GL}_n \circlearrowleft \Lambda^2\mathfrak{g}^* \otimes \mathfrak{g}$.

[JL 2001] g_μ Ricci soliton $\Leftrightarrow$ μ critical point of $\mu \mapsto \operatorname{tr}\operatorname{Ric}_\mu^2$ on the sphere.

Theorem (JL 2009)

- *The Ricci flow $g(t)$ is a type-III solution (i.e. $t \in [0,\infty)$ and $\|\operatorname{Rm}(g(t))\| \leq \frac{C}{t}$).*
- *$g(t)$ converges in C^∞ to a flat metric uniformly on compact sets in $\mathbb{R}^n$.*
- *After rescaling ($\mathrm{R} \equiv -1$), $g(t)$ converges to a Ricci soliton metric g_∞,*

Application to nilmanifolds

G nilpotent and s.c., $K = \{e\}$, $\mathfrak{g} = \mathfrak{p} = \mathbb{R}^n = G$, μ nilpotent Lie bracket on $\mathfrak{g}$, $\boxed{\operatorname{Ric}_\mu = M_\mu}$, $\operatorname{R}(g_\mu) = -\frac{1}{4}\|\mu\|^2$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\mu = -\pi(\operatorname{Ric}_\mu)\mu = -\operatorname{grad}(\operatorname{tr}\operatorname{Ric}_\mu^2)_\mu$$

negative gradient flow of the square norm of the moment map for the action $\mathrm{GL}_n \circlearrowleft \Lambda^2\mathfrak{g}^* \otimes \mathfrak{g}$.

[JL 2001] g_μ Ricci soliton $\Leftrightarrow$ μ critical point of $\mu \mapsto \operatorname{tr}\operatorname{Ric}_\mu^2$ on the sphere.

Theorem (JL 2009)

- *The Ricci flow $g(t)$ is a type-III solution (i.e. $t \in [0, \infty)$ and $\|\operatorname{Rm}(g(t))\| \leq \frac{C}{t}$).*
- *$g(t)$ converges in C^∞ to a flat metric uniformly on compact sets in $\mathbb{R}^n$.*
- *After rescaling ($\operatorname{R} \equiv -1$), $g(t)$ converges to a Ricci soliton metric g_∞, which is also invariant under a transitive nilpotent Lie group, though possibly non-isomorphic to G.*

Other applications

Other applications

[Guzhvina 2008] Bracket flow for nilmanifolds with applications to almost-flat manifolds.

Other applications

[Guzhvina 2008] Bracket flow for nilmanifolds with applications to almost-flat manifolds.

[Payne 2010] Qualitative behavior of bracket flow solutions for nilmanifolds.

Other applications

[Guzhvina 2008] Bracket flow for nilmanifolds with applications to almost-flat manifolds.

[Payne 2010] Qualitative behavior of bracket flow solutions for nilmanifolds.

[Glickenstein-Payne 2010] Ricci flow of 3-dim unimodular Lie groups.

Other applications

[Guzhvina 2008] Bracket flow for nilmanifolds with applications to almost-flat manifolds.

[Payne 2010] Qualitative behavior of bracket flow solutions for nilmanifolds.

[Glickenstein-Payne 2010] Ricci flow of 3-dim unimodular Lie groups.

[Arroyo 2012] Application to Ricci flow of 4-dim homogeneous manifolds and to Ricci flow of solvmanifolds.

Example in dim $= 3$

Example in dim = 3

$\mu = \mu_{a,b} \in \mathcal{H}_{1,3}$ defined by

Example in dim $= 3$

$\mu = \mu_{a,b} \in \mathcal{H}_{1,3}$ defined by

$$\begin{cases} \mu(X_3, Z_1) = X_2, \\ \mu(Z_1, X_2) = X_3, \\ \mu(X_2, X_3) = aX_1 + bZ_1. \end{cases}$$

Example in $\dim = 3$

$\mu = \mu_{a,b} \in \mathcal{H}_{1,3}$ defined by

$$\begin{cases} \mu(X_3, Z_1) = X_2, \\ \mu(Z_1, X_2) = X_3, \\ \mu(X_2, X_3) = aX_1 + bZ_1. \end{cases}$$

Bracket flow:

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t} a = (-\frac{3}{2}a^2 + 2b)a, \\ \\ \frac{\mathrm{d}}{\mathrm{d}t} b = (-a^2 + 2b)b. \end{cases}$$

Example in dim = 3

$\mu = \mu_{a,b} \in \mathcal{H}_{1,3}$ defined by

$$\begin{cases} \mu(X_3, Z_1) = X_2, \\ \mu(Z_1, X_2) = X_3, \\ \mu(X_2, X_3) = aX_1 + bZ_1. \end{cases}$$

Bracket flow:

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t} a = (-\frac{3}{2}a^2 + 2b)a, \\ \frac{\mathrm{d}}{\mathrm{d}t} b = (-a^2 + 2b)b. \end{cases}$$

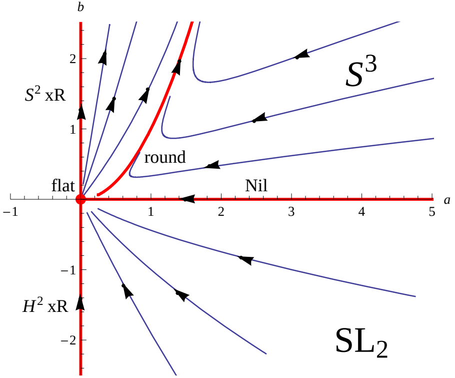

Figure: Phase plane for the ODE system

Example in dim = 3

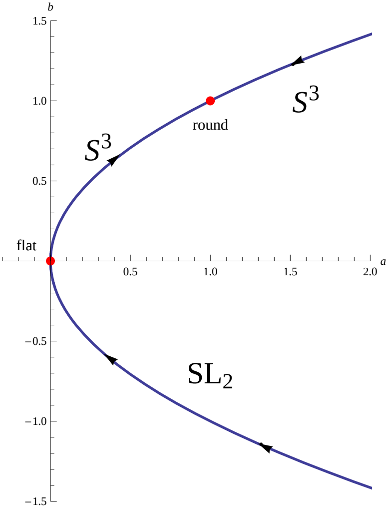

Figure: Volume-normalized bracket flow

Example in dim = 3

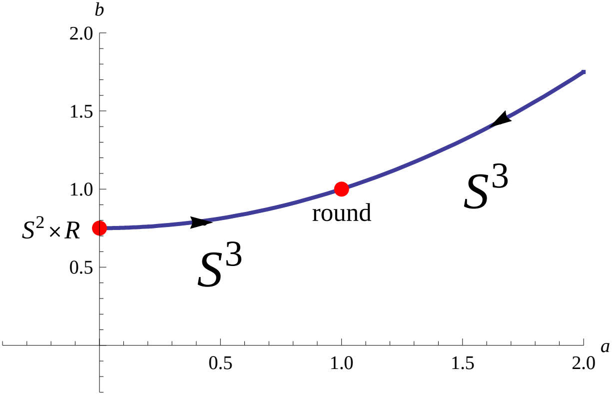

Figure: R-normalized bracket flows: $R \equiv \frac{3}{2}$.

Example in dim = 3

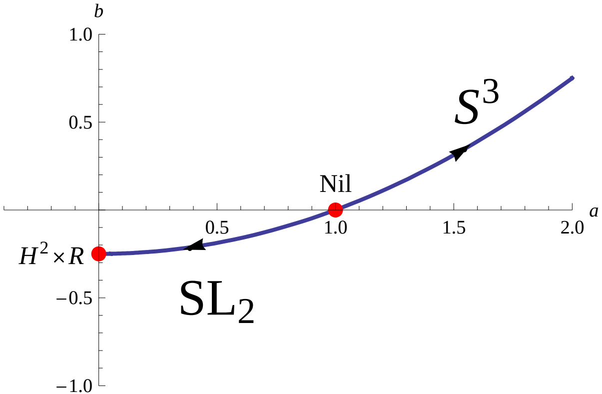

Figure: R-normalized bracket flows: $R \equiv -\frac{1}{2}$.

Example in simple Lie groups.

Example in simple Lie groups.

G/H irreducible compact symmetric space,

Example in simple Lie groups.

G/H irreducible compact symmetric space, $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$.

Example in simple Lie groups.

G/H irreducible compact symmetric space, $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$.

$$\text{Bracket flow: } \begin{cases} \frac{\mathrm{d}}{\mathrm{d}t} a = \frac{1}{4}(\alpha a^2 + (1-\alpha) b^2) a, \\ \\ \frac{\mathrm{d}}{\mathrm{d}t} b = -\frac{1}{4}(\alpha a^2 + (3-\alpha) b^2 - 4ab) b. \end{cases}$$

Example in simple Lie groups.

G/H irreducible compact symmetric space, $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$.

$$\text{Bracket flow: } \begin{cases} \frac{\mathrm{d}}{\mathrm{d}t} a = \frac{1}{4}(\alpha a^2 + (1-\alpha) b^2) a, \\ \\ \frac{\mathrm{d}}{\mathrm{d}t} b = -\frac{1}{4}(\alpha a^2 + (3-\alpha) b^2 - 4ab) b. \end{cases}$$

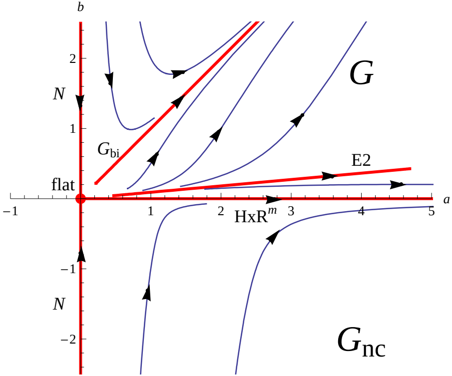

Figure: Phase plane for the ODE system

Example in simple Lie groups

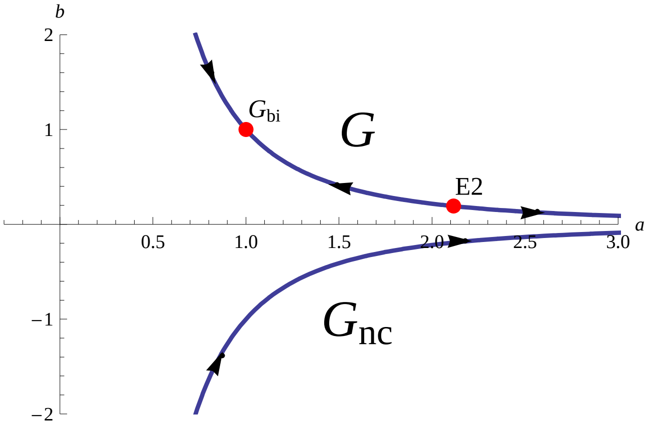

Figure: Volume-normalized bracket flow

Example in simple Lie groups

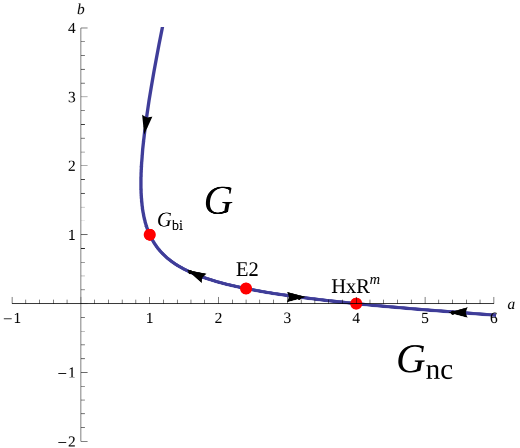

Figure: R-normalized bracket flow: $R \equiv 2$

Example in simple Lie groups

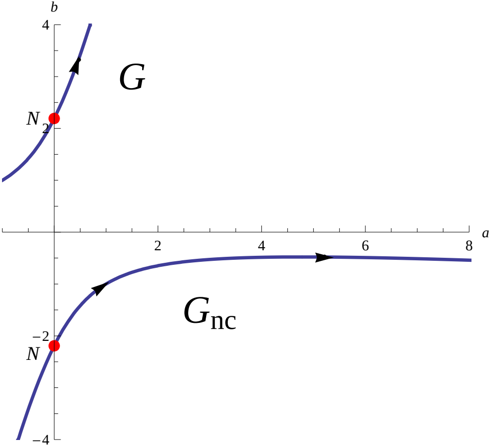

Figure: R-normalized bracket flow: $R \equiv -3$

Ricci solitons

Ricci solitons

(M, g) complete Riemannian manifold,

Ricci solitons

(M, g) complete Riemannian manifold, Ricci soliton:

Ricci solitons

(M, g) complete Riemannian manifold, Ricci soliton:

$$\mathrm{Rc}(g) = cg + \mathcal{L}_X(g), \qquad c \in \mathbb{R}, \qquad X \in \chi(M) \text{ (complete)}$$

Ricci solitons

(M, g) complete Riemannian manifold, Ricci soliton:

$$\mathrm{Rc}(g) = cg + \mathcal{L}_X(g), \qquad c \in \mathbb{R}, \qquad X \in \chi(M) \text{ (complete)}$$

$$\Leftrightarrow \qquad g(t) = (-2ct + 1)\varphi(t)^* g, \qquad \varphi(t) \in \mathrm{Diff}(M),$$

solution to Ricci flow with $g(0) = g$.

Ricci solitons

(M, g) complete Riemannian manifold, Ricci soliton:

$$\mathrm{Rc}(g) = cg + \mathcal{L}_X(g), \qquad c \in \mathbb{R}, \qquad X \in \chi(M) \text{ (complete)}$$

$$\Leftrightarrow \qquad g(t) = (-2ct + 1)\varphi(t)^* g, \qquad \varphi(t) \in \mathrm{Diff}(M),$$

solution to Ricci flow with $g(0) = g$.

[Ivey, Naber, Perelman, Petersen-Wylie]

Ricci solitons

(M, g) complete Riemannian manifold, Ricci soliton:

$$\mathrm{Rc}(g) = cg + \mathcal{L}_X(g), \qquad c \in \mathbb{R}, \qquad X \in \chi(M) \text{ (complete)}$$

$$\Leftrightarrow \qquad g(t) = (-2ct + 1)\varphi(t)^* g, \qquad \varphi(t) \in \mathrm{Diff}(M),$$

solution to Ricci flow with $g(0) = g$.

[Ivey, Naber, Perelman, Petersen-Wylie] $\rightsquigarrow$ Any nontrivial (i.e. non-Einstein and not the product of an Einstein homogeneous manifold with a euclidean space)

Ricci solitons

(M, g) complete Riemannian manifold, Ricci soliton:

$$\mathrm{Rc}(g) = cg + \mathcal{L}_X(g), \qquad c \in \mathbb{R}, \qquad X \in \chi(M) \text{ (complete)}$$

$$\Leftrightarrow \qquad g(t) = (-2ct + 1)\varphi(t)^* g, \qquad \varphi(t) \in \mathrm{Diff}(M),$$

solution to Ricci flow with $g(0) = g$.

[Ivey, Naber, Perelman, Petersen-Wylie] $\rightsquigarrow$ Any nontrivial (i.e. non-Einstein and not the product of an Einstein homogeneous manifold with a euclidean space) homogeneous Ricci soliton must be noncompact,

Ricci solitons

(M, g) complete Riemannian manifold, Ricci soliton:

$$\mathrm{Rc}(g) = cg + \mathcal{L}_X(g), \qquad c \in \mathbb{R}, \qquad X \in \chi(M) \text{ (complete)}$$

$$\Leftrightarrow \qquad g(t) = (-2ct + 1)\varphi(t)^* g, \qquad \varphi(t) \in \mathrm{Diff}(M),$$

solution to Ricci flow with $g(0) = g$.

[Ivey, Naber, Perelman, Petersen-Wylie] $\rightsquigarrow$ Any nontrivial (i.e. non-Einstein and not the product of an Einstein homogeneous manifold with a euclidean space) homogeneous Ricci soliton must be noncompact, expanding ($c < 0$),

Ricci solitons

(M, g) complete Riemannian manifold, Ricci soliton:

$$\mathrm{Rc}(g) = cg + \mathcal{L}_X(g), \qquad c \in \mathbb{R}, \qquad X \in \chi(M) \text{ (complete)}$$

$$\Leftrightarrow \qquad g(t) = (-2ct + 1)\varphi(t)^* g, \qquad \varphi(t) \in \mathrm{Diff}(M),$$

solution to Ricci flow with $g(0) = g$.

[Ivey, Naber, Perelman, Petersen-Wylie] $\rightsquigarrow$ Any nontrivial (i.e. non-Einstein and not the product of an Einstein homogeneous manifold with a euclidean space) homogeneous Ricci soliton must be noncompact, expanding ($c < 0$), non-gradient.

Algebraic solitons

Algebraic solitons

$(G/K, g)$: connected homogeneous space.

Algebraic solitons

$(G/K, g)$: connected homogeneous space.
semi-algebraic soliton: $\exists\, \varphi_t \in \mathrm{Aut}(G)$ with $\varphi_t(K) = K$ such that

$$g(t) = c(t)\varphi_t^* g_{\langle\cdot,\cdot\rangle}, \qquad g(0) = g_{\langle\cdot,\cdot\rangle}.$$

Algebraic solitons

$(G/K, g)$: connected homogeneous space.
semi-algebraic soliton: $\exists\, \varphi_t \in \mathrm{Aut}(G)$ with $\varphi_t(K) = K$ such that

$$g(t) = c(t)\varphi_t^* g_{\langle\cdot,\cdot\rangle}, \qquad g(0) = g_{\langle\cdot,\cdot\rangle}.$$

$\Rightarrow$ for any reductive decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, $\exists\, c \in \mathbb{R}$ and $D \in \mathrm{Der}(\mathfrak{g})$ such that $D\mathfrak{k} \subset \mathfrak{k}$

Algebraic solitons

$(G/K, g)$: connected homogeneous space.
semi-algebraic soliton: $\exists\, \varphi_t \in \mathrm{Aut}(G)$ with $\varphi_t(K) = K$ such that

$$g(t) = c(t)\varphi_t^* g_{\langle\cdot,\cdot\rangle}, \qquad g(0) = g_{\langle\cdot,\cdot\rangle}.$$

$\Rightarrow$ for any reductive decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, $\exists\, c \in \mathbb{R}$ and $D \in \mathrm{Der}(\mathfrak{g})$ such that $D\mathfrak{k} \subset \mathfrak{k}$ and

$$\mathrm{Ric} = cI + \tfrac{1}{2}(D_{\mathfrak{p}} + D_{\mathfrak{p}}^t).$$

Algebraic solitons

$(G/K, g)$: connected homogeneous space.
semi-algebraic soliton: $\exists\, \varphi_t \in \mathrm{Aut}(G)$ with $\varphi_t(K) = K$ such that

$$g(t) = c(t)\varphi_t^* g_{\langle\cdot,\cdot\rangle}, \qquad g(0) = g_{\langle\cdot,\cdot\rangle}.$$

$\Rightarrow$ for any reductive decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, $\exists\, c \in \mathbb{R}$ and $D \in \mathrm{Der}(\mathfrak{g})$ such that $D\mathfrak{k} \subset \mathfrak{k}$ and

$$\mathrm{Ric} = cI + \tfrac{1}{2}(D_{\mathfrak{p}} + D_{\mathfrak{p}}^t). \qquad \text{Conversely, if } G/K \text{ s.c. } \ldots$$

Algebraic solitons

$(G/K, g)$: connected homogeneous space.
semi-algebraic soliton: $\exists\ \varphi_t \in \mathrm{Aut}(G)$ with $\varphi_t(K) = K$ such that

$$g(t) = c(t)\varphi_t^* g_{\langle\cdot,\cdot\rangle}, \qquad g(0) = g_{\langle\cdot,\cdot\rangle}.$$

$\Rightarrow$ for any reductive decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, $\exists\ c \in \mathbb{R}$ and $D \in \mathrm{Der}(\mathfrak{g})$ such that $D\mathfrak{k} \subset \mathfrak{k}$ and

$$\mathrm{Ric} = cI + \tfrac{1}{2}(D_{\mathfrak{p}} + D_{\mathfrak{p}}^t). \qquad \text{Conversely, if } G/K \text{ s.c. } \ldots$$

$(G/K, g_{\langle\cdot,\cdot\rangle})$ with reductive decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$.

Algebraic solitons

$(G/K, g)$: connected homogeneous space.
semi-algebraic soliton: $\exists\, \varphi_t \in \text{Aut}(G)$ with $\varphi_t(K) = K$ such that

$$g(t) = c(t)\varphi_t^* g_{\langle\cdot,\cdot\rangle}, \qquad g(0) = g_{\langle\cdot,\cdot\rangle}.$$

$\Rightarrow$ for any reductive decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, $\exists\, c \in \mathbb{R}$ and $D \in \text{Der}(\mathfrak{g})$ such that $D\mathfrak{k} \subset \mathfrak{k}$ and

$$\text{Ric} = cI + \tfrac{1}{2}(D_{\mathfrak{p}} + D_{\mathfrak{p}}^t).$$ Conversely, if G/K s.c. ...

$(G/K, g_{\langle\cdot,\cdot\rangle})$ with reductive decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$.
algebraic soliton: $\exists c \in \mathbb{R}$ and $D \in \text{Der}(\mathfrak{g})$ such that $D\mathfrak{k} \subset \mathfrak{k}$

Algebraic solitons

$(G/K, g)$: connected homogeneous space.
semi-algebraic soliton: $\exists\, \varphi_t \in \mathrm{Aut}(G)$ with $\varphi_t(K) = K$ such that

$$g(t) = c(t)\varphi_t^* g_{\langle\cdot,\cdot\rangle}, \qquad g(0) = g_{\langle\cdot,\cdot\rangle}.$$

$\Rightarrow$ for any reductive decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, $\exists\, c \in \mathbb{R}$ and $D \in \mathrm{Der}(\mathfrak{g})$ such that $D\mathfrak{k} \subset \mathfrak{k}$ and

$$\mathrm{Ric} = cI + \tfrac{1}{2}(D_{\mathfrak{p}} + D_{\mathfrak{p}}^t). \qquad \text{Conversely, if } G/K \text{ s.c. } \ldots$$

$(G/K, g_{\langle\cdot,\cdot\rangle})$ with reductive decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$.
algebraic soliton: $\exists c \in \mathbb{R}$ and $D \in \mathrm{Der}(\mathfrak{g})$ such that $D\mathfrak{k} \subset \mathfrak{k}$ and

$$\mathrm{Ric} = cI + D_{\mathfrak{p}}$$

Algebraic solitons

$(G/K, g)$: connected homogeneous space.
semi-algebraic soliton: $\exists\, \varphi_t \in \text{Aut}(G)$ with $\varphi_t(K) = K$ such that

$$g(t) = c(t)\varphi_t^* g_{\langle\cdot,\cdot\rangle}, \qquad g(0) = g_{\langle\cdot,\cdot\rangle}.$$

$\Rightarrow$ for any reductive decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, $\exists\, c \in \mathbb{R}$ and $D \in \text{Der}(\mathfrak{g})$ such that $D\mathfrak{k} \subset \mathfrak{k}$ and

$$\text{Ric} = cI + \tfrac{1}{2}(D_{\mathfrak{p}} + D_{\mathfrak{p}}^t). \qquad \text{Conversely, if } G/K \text{ s.c. } \ldots$$

$(G/K, g_{\langle\cdot,\cdot\rangle})$ with reductive decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$.
algebraic soliton: $\exists c \in \mathbb{R}$ and $D \in \text{Der}(\mathfrak{g})$ such that $D\mathfrak{k} \subset \mathfrak{k}$ and

$$\text{Ric} = cI + D_{\mathfrak{p}} \Leftrightarrow \text{ bracket flow } \mu(t) = c_t \cdot [\cdot,\cdot], \quad c_t > 0.$$

Algebraic solitons

$(G/K, g)$: connected homogeneous space.
semi-algebraic soliton: $\exists\, \varphi_t \in \mathrm{Aut}(G)$ with $\varphi_t(K) = K$ such that

$$g(t) = c(t)\varphi_t^* g_{\langle\cdot,\cdot\rangle}, \qquad g(0) = g_{\langle\cdot,\cdot\rangle}.$$

$\Rightarrow$ for any reductive decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, $\exists\, c \in \mathbb{R}$ and $D \in \mathrm{Der}(\mathfrak{g})$ such that $D\mathfrak{k} \subset \mathfrak{k}$ and

$$\mathrm{Ric} = cI + \tfrac{1}{2}(D_{\mathfrak{p}} + D_{\mathfrak{p}}^t).$$ Conversely, if G/K s.c. ...

$(G/K, g_{\langle\cdot,\cdot\rangle})$ with reductive decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$.
algebraic soliton: $\exists c \in \mathbb{R}$ and $D \in \mathrm{Der}(\mathfrak{g})$ such that $D\mathfrak{k} \subset \mathfrak{k}$ and

$$\mathrm{Ric} = cI + D_{\mathfrak{p}} \Leftrightarrow \text{ bracket flow } \mu(t) = c_t \cdot [\cdot,\cdot], \quad c_t > 0.$$

[Lafuente-JL, 2012] Bracket flow evolution of semi-algebraic solitons:

Algebraic solitons

$(G/K, g)$: connected homogeneous space.
semi-algebraic soliton: $\exists\, \varphi_t \in \mathrm{Aut}(G)$ with $\varphi_t(K) = K$ such that

$$g(t) = c(t)\varphi_t^* g_{\langle\cdot,\cdot\rangle}, \qquad g(0) = g_{\langle\cdot,\cdot\rangle}.$$

$\Rightarrow$ for any reductive decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, $\exists\, c \in \mathbb{R}$ and $D \in \mathrm{Der}(\mathfrak{g})$ such that $D\mathfrak{k} \subset \mathfrak{k}$ and

$$\mathrm{Ric} = cI + \tfrac{1}{2}(D_{\mathfrak{p}} + D_{\mathfrak{p}}^t). \qquad \text{Conversely, if } G/K \text{ s.c. } \ldots$$

$(G/K, g_{\langle\cdot,\cdot\rangle})$ with reductive decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$.
algebraic soliton: $\exists c \in \mathbb{R}$ and $D \in \mathrm{Der}(\mathfrak{g})$ such that $D\mathfrak{k} \subset \mathfrak{k}$ and

$$\mathrm{Ric} = cI + D_{\mathfrak{p}} \Leftrightarrow \text{ bracket flow } \mu(t) = c_t \cdot [\cdot,\cdot], \quad c_t > 0.$$

[Lafuente-JL, 2012] Bracket flow evolution of semi-algebraic solitons:
$A := \frac{1}{2}(D_{\mathfrak{p}} - D_{\mathfrak{p}}^t)$,

Algebraic solitons

$(G/K, g)$: connected homogeneous space.
semi-algebraic soliton: $\exists\ \varphi_t \in \mathrm{Aut}(G)$ with $\varphi_t(K) = K$ such that

$$g(t) = c(t)\varphi_t^* g_{\langle\cdot,\cdot\rangle}, \qquad g(0) = g_{\langle\cdot,\cdot\rangle}.$$

$\Rightarrow$ for any reductive decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, $\exists\ c \in \mathbb{R}$ and $D \in \mathrm{Der}(\mathfrak{g})$ such that $D\mathfrak{k} \subset \mathfrak{k}$ and

$$\mathrm{Ric} = cI + \tfrac{1}{2}(D_{\mathfrak{p}} + D_{\mathfrak{p}}^t). \qquad \text{Conversely, if } G/K \text{ s.c. } \ldots$$

$(G/K, g_{\langle\cdot,\cdot\rangle})$ with reductive decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$.
algebraic soliton: $\exists c \in \mathbb{R}$ and $D \in \mathrm{Der}(\mathfrak{g})$ such that $D\mathfrak{k} \subset \mathfrak{k}$ and

$$\mathrm{Ric} = cI + D_{\mathfrak{p}} \Leftrightarrow \text{ bracket flow } \mu(t) = c_t \cdot [\cdot,\cdot], \quad c_t > 0.$$

[Lafuente-JL, 2012] Bracket flow evolution of semi-algebraic solitons:
$A := \frac{1}{2}(D_{\mathfrak{p}} - D_{\mathfrak{p}}^t), \qquad \mu(t) = \left[\begin{smallmatrix} I & 0 \\ 0 & e^{tA} \end{smallmatrix}\right] \cdot \mu_0 \in \mathrm{O}(q+n) \cdot \mu_0.$

Example in dim = 3

$\mu = \mu_{a,b} \in \mathcal{H}_{1,3}$

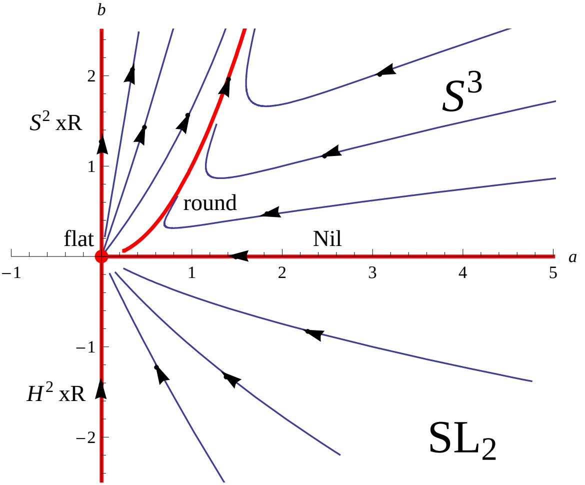

Figure: Phase plane for the ODE system

Example in simple Lie groups.

G/H irreducible compact symmetric space, $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$.

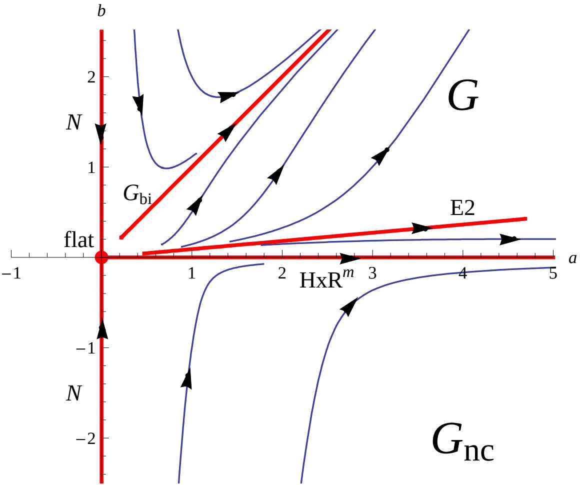

Figure: Phase plane for the ODE system

algebraic soliton: $\mathrm{Ric} = cI + D_{\mathfrak{p}}$.

algebraic soliton: $\mathrm{Ric} = cI + D_{\mathfrak{p}}$.

solvsoliton: algebraic soliton with $K = e$ and G solvable.

algebraic soliton: $\mathrm{Ric} = cI + D_{\mathfrak{p}}$.

solvsoliton: algebraic soliton with $K = e$ and G solvable.

nilsoliton: algebraic soliton with $K = e$ and G nilpotent.

algebraic soliton: $\mathrm{Ric} = cI + D_{\mathfrak{p}}$.

solvsoliton: algebraic soliton with $K = e$ and G solvable.

nilsoliton: algebraic soliton with $K = e$ and G nilpotent.

Classification and structure of nilsolitons ($\leftrightarrow$ GIT): [Arroyo, Fernández Culma, Jablonski, Nikolayevsky, Oscari, Payne, Will, JL, ...]

algebraic soliton: $\mathrm{Ric} = cI + D_{\mathfrak{p}}$.

solvsoliton: algebraic soliton with $K = e$ and G solvable.

nilsoliton: algebraic soliton with $K = e$ and G nilpotent.

Classification and structure of nilsolitons ($\leftrightarrow$ GIT): [Arroyo, Fernández Culma, Jablonski, Nikolayevsky, Oscari, Payne, Will, JL, ...]

Classification and structure of solvsolitons: [Lafuente, Will, Williams, JL]

Classification and structure of semi-algebraic solitons: [Jablonski, Lafuente-JL]

Classification and structure of semi-algebraic solitons: [Jablonski, Lafuente-JL]

Open questions:

Classification and structure of semi-algebraic solitons: [Jablonski, Lafuente-JL]

Open questions:

- Is any homogeneous Ricci soliton (or semi-algebraic soliton) isometric to an algebraic soliton ??

Classification and structure of semi-algebraic solitons: [Jablonski, Lafuente-JL]

Open questions:

- Is any homogeneous Ricci soliton (or semi-algebraic soliton) isometric to an algebraic soliton ??
- Is any algebraic soliton isometric to a solvsoliton ??

Classification and structure of semi-algebraic solitons: [Jablonski, Lafuente-JL]

Open questions:

- Is any homogeneous Ricci soliton (or semi-algebraic soliton) isometric to an algebraic soliton ??
- Is any algebraic soliton isometric to a solvsoliton ??

Theorem (Lafuente-JL 2012)

A homogeneous Ricci soliton is Ricci flow diagonal if and only if it is isometric to an algebraic soliton.

Classification and structure of semi-algebraic solitons: [Jablonski, Lafuente-JL]

Open questions:

- Is any homogeneous Ricci soliton (or semi-algebraic soliton) isometric to an algebraic soliton ??
- Is any algebraic soliton isometric to a solvsoliton ??

Theorem (Lafuente-JL 2012)

A homogeneous Ricci soliton is Ricci flow diagonal if and only if it is isometric to an algebraic soliton.

(M, g) Ricci flow diagonal: $\exists$ o.b. β of T_pM such that the Ricci flow solution $g(t)$ starting at g is diagonal with respect to β for all t.