The canonical contact structure on the space of oriented null geodesics

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Definition

Let M be a $2n+1$ dimensional manifold. A *contact structure* on M is a distribution $\mathcal{D} \subset TM$ of dimension $2n$ such that if locally $\mathcal{D} = \operatorname{Ker}\alpha$, then the 1-form α satisfies

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If there exists a 1-form α defined on M such that $\mathcal{D} = \operatorname{Ker} \alpha$, we call it *contact form*.

Example: On $\mathbb{R}^{2n+1}$ with coordinates $(x_1, ..., x_n, y_1, ..., y_n, z)$, the 1-form $\alpha = dz - \sum_{i=1}^{n} y_i dx_i$ is a contact form.

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Theorem (Darboux)

About of each point of a contact manifold $(M^{2n+1}, \mathcal{D})$ there exists a coordinate system $(U, (x_1, ..., x_n, y_1, ..., y_n, z))$ such that $\mathcal{D} = Ker\,\alpha$ on U, for

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So, two contact manifolds are locally indistinguishable.

Definition

Two contact manifolds $(M_1, \mathcal{D}_1)$ and $(M_2, \mathcal{D}_2)$ are said to be *contactomorphic* if there exists a diffeomorphism $F : M_1 \to M_2$ such that $dF(\mathcal{D}_1) = \mathcal{D}_2$.

Example:

Let M be a pseudo-Riemannian manifold and let

$$T^1M = \{u \in TM \,|\, \langle u, u\rangle = 1\}.$$

Notice that T^1M is not in general a round sphere.

Let $\pi : T^1M \to M$ be the canonical projection. For $u \in T^1M$ and $\xi \in T_u T^1M$, we consider

$$\alpha_u(\xi) = \langle u, d\pi_u \xi\rangle.$$

Then α is a contact form on T^1M.

Unit tangent bundles of pseudo-Riemannian manifolds are among the standard examples of contact manifolds.

The space of oriented null geodesics

- Let N be a complete pseudo-Riemannian manifold.
- For $u \in T_pN$ we denote $\|u\| = \langle u, u \rangle$. Let $T^0N = \{u \in TN \mid \|u\| = 0, u \neq 0\}$.
- Two null geodesics σ and γ are said to be equivalent if there exist $\lambda > 0$ and $b \in \mathbb{R}$ such $\sigma(s) = \gamma(\lambda s + b)$, for all $s \in \mathbb{R}$.
  That is, two null geodesics are equivalent if they have the same trajectory and orientation.
- We call $\mathcal{L}^0(N)$ the set of all equivalence classes of oriented null geodesics of N.
- Let γ_u denote the unique geodesic in N with initial velocity u.
- By abuse of notation, we say that $\mathcal{L}^0(N)$ is a manifold if it admits a differentiable structure such that the projection $\Pi : T^0N \to \mathcal{L}^0(N)$, $\Pi(u) = [\gamma_u]$, is a smooth submersion

The canonical contact distribution on $\mathcal{L}^0(N)$

B. Khesin and S. Tabachnikov introduce a canonical contact structure on $\mathcal{L}^0(N)$, provided that it is a manifold (generalizing the definition given in the Lorentz case by R. Low), and study it for the pseudo-Euclidean space. We continue in that direction for other spaces such as pseudospheres and some products.

Let $\pi : T^0N \to N$ be the canonical projection. Let θ be the canonical 1-form on T^0N, that is, for $u \in T^0N$ and $\xi \in T_u T^0N$,

$$\theta_u(\xi) = \langle u, d\pi_u \xi \rangle.$$

Definition (Low, Khesin)

Let N be a pseudo-Riemannian manifold such that $\mathcal{L}^0(N)$ is a manifold. The canonical contact distribution $\mathcal{D}$ on $\mathcal{L}^0(N)$ is well defined by

$$\mathcal{D}_{\Pi(u)} = d\,\Pi_u(\operatorname{Ker}\theta_u),$$

for each $u \in T^0N$.

We observe that the form θ on T^0N does not descend to $\mathcal{L}^0(N)$, but the contact distribution $\operatorname{Ker}\theta$ does.

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B. Khesin and S. Tabachnikov, Pseudo-Riemannian geodesics and billiards. *Adv. Math.* **221** (2009), 1364–1396.
R. Low, The space of null geodesics. *Nonlinear Anal.* **47** (2001), 3005–3017.

In order to give $\mathcal{L}^0(N)$ a differentiable structure, we identify it with a quotient of T^0N.

Given two null geodesics γ and σ on N,

$\gamma \sim \sigma$ if and only if there exist $\lambda > 0$ and $b \in \mathbb{R}$ such that $\dot{\sigma}(0) = \lambda\dot{\gamma}(b)$.

On the other hand, let $\mathcal{A} = \text{Aff}_+(\mathbb{R})$ be the Lie group of orientation preserving affine transformations of $\mathbb{R}$ and consider the right action from $\mathcal{A}$ on T^0N given as follows: if $u \in T^0N$ and $g \in \mathcal{A}$,

$$u \cdot g := \left.\frac{d}{dt}\right|_0 \gamma_u(g(t)).$$

Then, $\mathcal{L}^0(N) \simeq T^0N/\mathcal{A}$.

In addition, if this action is free and proper, $\mathcal{L}^0(N)$ is a Hausdorff differentiable manifold such that the canonical projection $\Pi : T^0N \to \mathcal{L}^0(N)$ is a submersion.

The canonical contact structure on $\mathcal{L}^0(S^{k,m})$

- Let $\mathbb{R}^{k+1,m}$ be the pseudo-Euclidean space of signature $(k+1, m)$, that is, $\mathbb{R}^{k+1} \times \mathbb{R}^m$ endowed with the inner product whose norm is given by $\|(u, v)\| = |u|^2 - |v|^2$
- The pseudosphere of radius 1 in $\mathbb{R}^{k+1,m}$ is the hyperquadric

$$S^{k,m} = \{(u, v) \in \mathbb{R}^{k+1,m} \mid \|(u, v)\| = 1\},$$

  which is a hypersurface of $\mathbb{R}^{k+1,m}$ with induced metric of signature (k, m) and diffeomorphic to $S^k \times \mathbb{R}^m$.
- The null geodesics of $S^{k,m}$ are straight lines in $\mathbb{R}^{k+1,m}$.

- Given two Riemannian manifolds M and N, let $M_+ \times N_-$ be the manifold $M \times N$ with the pseudo-Riemannian metric whose norm is defined by $\|(u,v)\| = |u|_M^2 - |v|_N^2$, for each $(u,v) \in T_{(p,q)}(M \times N)$ and $(p,q) \in M \times N$.
- Let S^n denote the sphere in $\mathbb{R}^{n+1}$.

Theorem

The set $\mathcal{L}^0(S^{k,m})$ is a manifold, and if one considers on $\mathcal{L}^0(S^{k,m})$ and $T^1(S^k_+ \times S^{m-1}_-)$ the canonical contact structures, then the map

$$F : T^1(S^k_+ \times S^{m-1}_-) \to \mathcal{L}^0(S^{k,m}), \;\; F((u,v),(x,y)) = [\gamma],$$

with $\gamma(t) = (x,y) + t(u,v)$, is a contactomorphism.

Billards

- Let N be a complete pseudo-Riemannian manifold.
- Let R be a region in N with smooth nondegenerate boundary M.
- We require that any null geodesic γ intersecting the interior of R satisfies that $\gamma(\mathbb{R}) \cap R = \gamma([t_0, t_1])$.
- We call $\mathfrak{L} \subset \mathcal{L}^0(N)$ the set of all oriented null geodesics intersecting the interior of R.
- Let γ be a null geodesic of N such that $[\gamma] \in \mathfrak{L}$. Decompose $\dot{\gamma}(t_1) = u^T + u^\perp$ with $u^T \in T_{\gamma(t_1)}M$ and $u^\perp \in (T_{\gamma(t_1)}M)^\perp$.
- The null billiard operator B is well defined in the following way:

$$B : \mathfrak{L} \to \mathfrak{L}, \qquad B([\gamma]) = [\gamma_w], \quad \text{with} \quad w = u^T - u^\perp.$$

- The null billiard operator preserves the contact structure on $\mathcal{L}^0(N)$.

Khesin and Tabachnikov deal with the dynamics of the null billiard operator for the pseudo-Euclidean space. We study the null billiard operator for some simple regions in pseudospheres.

For $c > 0$, let

$$R_c = \{(u, v) \in S^{k,m} \,|\, |v| \leq c\},$$

with boundary $M_c = \{(u, v) \in S^{k,m} \,|\, |v| = c\}$, which is nondegenerate since $V(u, v) = (c^2 u, (1 + c^2)v)$ is an outside pointing normal time-like vector field.

We have the diagram

$$\begin{array}{ccccccc}
L & \subset & T^1(S^k_+ \times S^{m-1}_-) & \xrightarrow{F} & \mathcal{L}^0(S^{k,m}) & \supset & \mathfrak{L} \\
\bar{B}\downarrow & & & & & & \downarrow B \\
L & \subset & T^1(S^k_+ \times S^{m-1}_-) & \xrightarrow{F} & \mathcal{L}^0(S^{k,m}) & \supset & \mathfrak{L}
\end{array}$$

We express the null billiard operator B via F of the Theorem in terms of the geodesic flows of spheres.

The canonical contact structure on $\mathcal{L}^0(M_+ \times N_-)$

- Let M and N be complete Riemannian manifolds.
- Let $\mathcal{L}(M)$ be the space of oriented geodesics of M, that is, the quotient of T^1M by the action of $\mathbb{R}$ on it determined by the geodesic flow of M.
- We call p_1, p_2 the projections of $\mathcal{L}(M) \times T^1N$ onto the first and second factors, respectively, and let α_1 and α_2 be the canonical 1-forms on T^1M and T^1N.

Next, we find conditions on M and N for $\mathcal{L}^0(M_+ \times N_-)$ to be a manifold and contactomorphic to a more concrete contact manifold.

Theorem

Let M and N be complete Riemannian manifolds such that the geodesic flow of M is free and proper. Then, $\mathcal{L}^0(M_+ \times N_-)$ is a manifold.

Suppose additionally that there exists a smooth global section $S : \mathcal{L}(M) \to T^1M$. Then $\theta_S = p_1^* S^* \alpha_1 - p_2^* \alpha_2$ is a contact form on $\mathcal{L}(M) \times T^1N$ and the map

$$G : \mathcal{L}(M) \times T^1N \to \mathcal{L}^0(M_+ \times N_-), \qquad G(\ell, v) = [(\gamma_{S(\ell)}, \gamma_v)]$$

is a contactomorphism, where $\mathcal{L}^0(M_+ \times N_-)$ is endowed with its canonical contact structure.

Examples

1) If M is a Hadamard manifold, then M satisfies the hypotheses of the Theorem: The geodesic flow of M is proper and free. And, fixing $p \in M$, there exists a global section $S : \mathcal{L}(M) \to T^1M$ which assigns to each class of an oriented unit speed geodesic of M its velocity at the closest point to p.

2) The same for the paraboloid $M = \{(x, y, x^2 + y^2) \,|\, x, y \in \mathbb{R}\}$. (Notice that most geodesics of M have self-intersections.) The existence of a smooth global section is proved as above taking p the origin in $\mathbb{R}^3$.

3) For $M = S^2$, such sections do not exist (otherwise, there would be a nowhere zero vector field on S^2).