Symplectic structures on nilmanifolds

An obstruction for its existence

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We support our study in an important property of nilmanifolds: its de Rham cohomology is isomorphic to the Lie algebra cohomology of the Lie algebra $\mathfrak{n}$ of N ([Nom54])

$$H^p_{dR}(\Gamma\backslash N) \simeq H^p(\mathfrak{n}, \mathbb{R}), \qquad p \geq 0.$$

A consequence of this result is that any symplectic structure on $\Gamma\backslash N$ is cohomologous to an invariant one. Then

$$\exists\, \tilde{\omega} \text{ symplectic structure on } \Gamma\backslash N \iff \begin{array}{c} \exists\, \omega \text{ closed 2-form in } \mathfrak{n} \\ \text{such that } \omega^n \neq 0 \\ (\dim \mathfrak{n} = 2n). \end{array}$$

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Lets start....

Given a nilpotent Lie algebra $\mathfrak{n}$, its Chevalley-Eilenberg complex is

$$C^* : \qquad 0 \to \mathbb{R} \to \mathfrak{n}^* \xrightarrow{d} \Lambda^2\mathfrak{n}^* \to \ldots \to \Lambda^m\mathfrak{n}^* \to 0.$$

Here $\Lambda^p\mathfrak{n}^*$ is space the skew-symmetric p-forms on $\mathfrak{n}$ and the differential is the extension by derivation of $d : \mathfrak{n}^* \longrightarrow \Lambda^2\mathfrak{n}^*$, the dual mapping of the Lie bracket of $\mathfrak{n}$.

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- and in that case: $0 = V_0 \subsetneq V_1 \subsetneq \ldots \subsetneq V_{k-1} \subsetneq V_k = \mathfrak{n}^*$.

This yields into a filtration of $\Lambda^q \mathfrak{n}^*$ for each q,

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In addition, for each p, $d(\Lambda^q V_{k-p}) \subseteq \Lambda^{q+1} V_{k-p}$. Thus

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The groups $E_\infty^{p,q}(\mathfrak{n})$ with $p+q=i$ are the *intermediate cohomology groups of* $\mathfrak{n}$ *of degree* i.

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A very well known cohomological obstruction for the existence of symplectic structures on compact manifolds is

$$(*) \qquad M \text{ compact symplectic manifold} \Rightarrow H^2_{dR}(M) \neq 0.$$

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Theorem

If $\Gamma\backslash N$ is a symplectic nilmanifold and $\mathfrak{n} = Lie(N)$ then $E_\infty^{0,2}(\mathfrak{n}) \neq 0$.

Proof. Suppose there exists a symplectic form ω in the k-step nilpotent Lie algebra $\mathfrak{n}$. Consider the central extension $\tilde{\mathfrak{n}}$ of $\mathfrak{n}$ throughout ω; $\tilde{\mathfrak{n}}$ is a $k+1$-step with center of dimension 1.

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Which leads into a close relation between the intermediate cohomology. For example

$$E_\infty^{1,1}(\tilde{\mathfrak{n}}) \cong \frac{\{x \in \Lambda^2\mathfrak{n}^* : dx = 0\}}{\mathbb{R}\omega + \{x \in \Lambda^2 V_{k-1} : dx = 0\}} \quad E_\infty^{0,2}(\mathfrak{n}) \cong \frac{\{x \in \Lambda^2\mathfrak{n}^* : dx = 0\}}{\{x \in \Lambda^2 V_{k-1} : dx = 0\}}.$$

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$$\dim\ E_\infty^{0,2}(\mathfrak{n}) = \dim\ E_\infty^{1,1}(\tilde{\mathfrak{n}}) + 1,$$

which implies $\dim E_\infty^{0,2}(\mathfrak{n}) \geq 1$. ∎

Let $\mathfrak{n}$ be an even dimensional nilpotent Lie algebra.

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Example: Let $\mathfrak{n}_{m,3}$ be the free 3-step nilpotent Lie algebra on m generators and consider $\mathfrak{n}_m = \mathbb{R}^s \oplus \mathfrak{n}_{m,3}$ where $s = 0$ or $s = 1$ depending on whether $\dim \mathfrak{n}_{m,3}$ is even or odd.

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Nevertheless there are some subfamilies of nilpotent Lie algebras where the answer for (**) is positive...

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Theorem (Classification of symplectic nilradicals)

If $\mathfrak{n}$ is as above, the Lie algebra of even dimension $\mathbb{R}^s \oplus \mathfrak{n}$, $s \geq 0$ admits symplectic structures if and only if $\mathfrak{g}$ is one of the followings:

$$\mathfrak{sl}(2,\mathbb{C}), \quad \mathfrak{sl}(3,\mathbb{C}), \quad \mathfrak{so}(5,\mathbb{C}).$$

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Lemma

Let $\mathfrak{n}$ be the nilradical of the Borel subalgebra corresponding to the complex classical simple Lie algebra $\mathfrak{g}$. Then

1. *if $\mathfrak{g}=\mathfrak{sl}(n+1,\mathbb{C})$ for some $n\geq 3$ then $E_\infty^{0,2}(\mathfrak{n})=0$,*
2. *if $\mathfrak{g}=\mathfrak{so}(2n+1,\mathbb{C})$ for some $n\geq 3$ then $E_\infty^{0,2}(\mathfrak{n})=0$,*
3. *if $\mathfrak{g}=\mathfrak{sp}(2n,\mathbb{C})$ for some $n\geq 3$ then $E_\infty^{0,2}(\mathfrak{n})=0$,*
4. *if $\mathfrak{g}=\mathfrak{so}(2n,\mathbb{C})$ for some $n\geq 4$ then $E_\infty^{0,2}(\mathfrak{n})=0$.*

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$$\mathfrak{n} = \bigoplus_{\alpha \in \triangle^+} \mathfrak{g}_\alpha.$$

- define the *length* of a positive root α as $\ell(\alpha) = \sum_{i=1}^r n_i$ if $\alpha = \sum_{i=1}^r n_i \alpha_i$, with α_i, $i = 1, \ldots, r$ the simple roots.
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Define for each $i \in \mathbb{N}$, $L_i = \bigoplus_{\alpha:\ell(\alpha)=i} \mathbb{R}X_\alpha$. Thus $\mathfrak{n}$ is graded:

$$\mathfrak{n} = \bigoplus_{j\geq 1} L_j \quad \text{and} \quad [L_j, L_i] = L_{i+j}.$$

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An insight into the space of closed 2-forms $\longrightarrow$ Let ω be a closed 2-form in $\mathfrak{n}$. In this context, a result of Benson and Gordon assures

$$\omega \in L_1^* \wedge L_k^* \oplus \bigoplus_{1\leq i<j\leq k-1} L_i^* \wedge L_j^* \quad \text{therefore} \quad \omega = \sigma + \tilde{\omega}.$$

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Root system of $\mathfrak{g}$ $\longrightarrow$ computation of $d\sigma$ and $-d\tilde{\omega}$,
In any case (family A,B,C,D) $\longrightarrow \sigma = 0 \longrightarrow E_\infty^{0,2}(\mathfrak{n}) = 0$.

And the Lemma follows.

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Corollary

Let $\mathfrak{n}$ be the Lie nilradical of a Borel subalgebra of a complex classical simple Lie algebra $\mathfrak{g}$ and let $s \in \mathbb{N}_0$ be such that $\mathbb{R}^s \oplus \mathfrak{n}$ has even dimension. If $\dim \mathfrak{n} \geq 2$ the following statements are equivalent.

- $\mathbb{R}^s \oplus \mathfrak{n}$ *is symplectic,*
- $E_\infty^{0,2}(\mathfrak{n}) \neq 0$,
- $\mathfrak{g} = \mathfrak{sl}(3,\mathbb{C})$ *or* $\mathfrak{g} = \mathfrak{so}(5,\mathbb{C})$.

Thanks for your attention.

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