Geometric inequalities for black holes

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Einstein equations (vacuum)

The spacetime is a four dimensional manifold M with a Lorenzian metric $g_{\mu\nu}$. Signature $-+++$:

- "$-$" is time.
- "$+++$" is space.

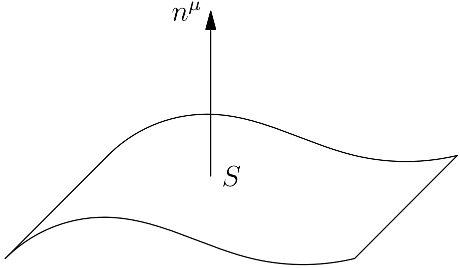
The metric $g_{\mu\nu}$ satisfies the Einstein **vacuum** equations

$$R_{\mu\nu} = 0,$$

where $R_{\mu\nu}$ is the Ricci tensor of the metric $g_{\mu\nu}$.

- These equations describe the dynamics of the pure gravitational field without matter.
- Although matter fields are of course necessary to describe the astrophysical phenomena (i.e. starts, galaxies, etc.) most of the fundamental properties of Einstein equations are already present in the pure vacuum case. This case has also the advantage that no phenomenological matter models are needed, the vacuum equations are expected to be fundamental equations.
- The two most spectacular predictions of Einstein equations (gravitational waves and black holes) can be studied in the vacuum case.

Einstein equations and Riemannian Geometry



- n^μ unit normal vector to the 3-dimensional hypersurface S in M.
- The vector n^μ is timelike, i.e. $n^\mu n^\nu g_{\mu\nu} = -1$.
- The hypersurface S is a 3-dimensional Riemannian manifold: h_{ij} the induced Riemannian metric, K_{ij} the extrinsic curvature (second fundamental form).
- S is a "$t = 0$" slice of the spacetime.

The Einstein vacuum equations implies that the following equations hold in S

$$R + K^2 - K_{ij}K^{ij} = 0, \quad (1)$$

$$D^i K_{ij} - D_j K = 0. \quad (2)$$

where $K = h^{ij}K_{ij}$ and R is the scalar curvature. The covariant derivatives are computed with respect to h_{ij}.

Definition

An initial data set for Einstein vacuum equations is a triple (S, h_{ij}, K_{ij}) where S is a 3-manifold, h_{ij} is a Riemannian metric, and K_{ij} is a symmetric tensor such that vacuum Einstein constraints equations (1)–(2) holds on S.

- Given an initial data the complete spacetime time is fixed: **initial value formulation**.
- The constraint equations are simpler to solve than the full Einstein equations.
- Many important questions about the Einstein equations can be analyzed in terms of the constraint equations. Typically, spacetime properties translate to statemets for arbitrary kind of initial data.

There exists an important particular case of initial data called **maximal initial data**:

$$K = 0$$

For these data the constraint equations reduce to

$$R - K_{ij}K^{ij} = 0, \tag{3}$$

$$D^i K_{ij} = 0. \tag{4}$$

In particular, equation (3) implies

$$R \geq 0.$$

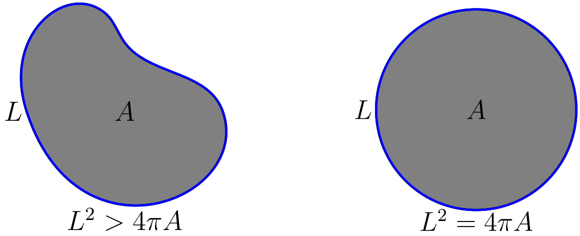
Riemannian metrics with non-negative scalar curvature are good initial data for Einstein equations.

Geometric inequalities

Geometric inequalities have an ancient history in Mathematics. A classical example is the **isoperimetric inequality** for closed plane curves given by

$$L^2 \geq 4\pi A,$$

where A is the **area** enclosed by a curve C of **length** L, and where **equality** holds if and only if C is a **circle**.



Geometrical inequalities in General Relativity

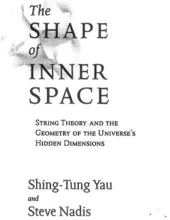
- General Relativity is a **geometric theory**, hence it is not surprising that geometric inequalities appear naturally in it. Many of these inequalities are similar in spirit as the isoperimetric inequality.
- However, General Relativity as a **physical theory** provides an important extra ingredient. It is often the case that the quantities involved have a clear physical interpretation and the expected behavior of the gravitational and matter fields often suggest geometric inequalities which can be highly non-trivial from the mathematical point of view.
- The **interplay between geometry and physics** gives to geometric inequalities in General Relativity their distinguished character.
- Many geometrical inequalities in General Relativity can be studied in **Riemannian manifolds**.

Example: Positive mass theorem

- The mass of the spacetime measures the total amount of energy and hence it should be positive from the physical point of view.
- The mass m in General Relativity is represented by a pure geometrical quantity on a Riemannian manifold.

$$0 \leq m,$$

with equality if and only if the spacetime is flat.
From the geometrical mass definition, without the physical picture, it would be very hard to conjecture that this quantity should be positive. In fact the proof turn out to be very subtle (**Schoen-Yau 79**, **Witten 81**).



niques we'd recently developed might be applied to the positive mass conjecture. The first thing we did, employing a strategy commonly used on big problems, was to break the problem up into smaller pieces, which could then be taken on one at a time. We proved a couple of special cases first, before tackling the full conjecture, which is difficult for a geometer even to comprehend, let alone attempt to prove. Moreover, we didn't believe it was true from a pure geometry standpoint, because it seemed to be too strong a statement.

We weren't alone. Misha Gromov, a famous geometer now at New York University and the Institut des Hautes Études Scientifiques in France, told us that

based on his geometric intuition, the general case was clearly wrong, and many geometers agreed. On the other hand, most physicists thought it was true (as they kept bringing it up at their conferences, year after year). That was enough to inspire us to take a closer look at the idea and see if it made any sense.

Asymptotically flat Riemannian manifolds

The manifold S is called Euclidean at infinity, if there exists a compact subset $\mathcal{K}$ of S such that $S \setminus \mathcal{K}$ is the disjoint union of a finite number of open sets U_k, and each U_k is isometric to the exterior of a ball in $\mathbb{R}^3$. Each open set U_k is called an end of S.

Consider one end U and the canonical coordinates x^i in $\mathbb{R}^3$ which contains the exterior of the ball to which U is diffeomorphic to. Set $r = \left(\sum (x^i)^2\right)^{1/2}$. The metric h_{ij} is called asymptotically flat if it tends to the euclidean metric

$$h_{ij} = \delta_{ij} + o(r^{-1/2})$$

as $r \to \infty$.

Examples of asymptotically flat manifolds

- $(\mathbb{R}^3, \delta_{ij})$ Euclidean space, trivial example. **One asymptotic end.**
- $(\mathbb{R}^3, \psi^4 \delta_{ij})$, where ψ is any smooth functions on $\mathbb{R}^3$ such that at infinity has the fall off behaviour

$$\psi = 1 + \frac{m}{2r} + O(r^{-2}),$$

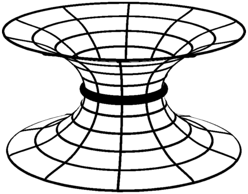
where m is a constant. **One asymptotic end.**

Examples of asymptotically flat manifolds: Schwarzschild black hole initial data

- $(\mathbb{R}^3 \setminus \{0\}, \psi^4\delta)$, where ψ is given by

$$\psi = 1 + \frac{m}{2r}.$$

Note that this function is singular at the origin. **Two asymptotic ends.**



The total mass m defined at each end U by

$$m = \frac{1}{16\pi} \lim_{r\to\infty} \oint_{\partial B_r} \left(\partial_j h_{ij} - \partial_i h_{jj}\right) n^i \, ds, \tag{5}$$

where ∂ denotes partial derivatives with respect to x^i, B_r is the euclidean sphere $r = constant$ in U, n^i is its exterior unit normal and ds is the surface element with respect to the euclidean metric.

Theorem (Riemannian positive mass)

Let (S, h) be any asymptotically flat, complete Riemannian manifold with nonnegative scalar curvature. Then the total mass defined by (5) satisfies $m \geq 0$, with equality if and only if (S, h) is isometric to $(\mathbb{R}^3, \delta)$.

Black holes

- Black holes represent a unique class of macroscopic object that plays, in some sense, the role of ‘elementary particle’ in the theory.
- The black hole uniqueness theorem ensures that stationary black holes in vacuum are characterized by two parameters: the area A of the black hole, the angular momentum J.
- Black holes are the most simple macroscopic objects: they are build out of pure geometry.
- They are ideal candidates for geometrical inequalities

Black hole and minimal surfaces

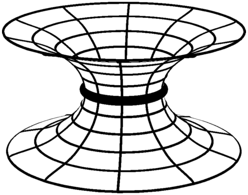
- A black hole is a region in (M, g) of 'no escape'. The boundary of this region is called the event horizon.
- In order to compute the event horizon we need to know the complete spacetime (M, g).
- However, for initial data with $K_{ij} = 0$ it can be proved that a minimal surface on the data is always inside the black hole.
- Minimal surfaces on Riemannian manifolds with non-negative scalar curvature represent black holes boundaries.

Example: Schwarzschild black hole

- $(\mathbb{R}^3 \setminus \{0\}, \psi^4 \delta_{ij})$, where ψ is given by

$$\psi = 1 + \frac{m}{2r}.$$

The surface $r = m/2$ is a minimal surface.



Riemannian Penrose inequality

Theorem (Huisken-Ilmanen 01, Bray 01)

Let (S, h) be a complete, asymptotically flat 3-manifold with nonnegative scalar curvature, with total mass m, and with an outermost minimal surface of area A. Then

$$m \geq \sqrt{\frac{A}{16\pi}},$$

and equality holds if and only if (S, h) is isometric to the Schwarzschild metric $\left(\mathbb{R}^3 \setminus \{0\}, \left(1 + \frac{m}{2r}\right)^4 \delta\right)$ outside the respective outermost minimal surfaces.

Axially symmetric initial data

- The initial data are axially symmetric if there exists a vector field η^i with complete periodic orbits such that

$$\mathcal{L}_\eta h_{ij} = 0 \quad \text{(Killing equation)}, \quad \mathcal{L}_\eta K_{ij} = 0,$$

where $\mathcal{L}$ denotes the Lie derivative.

- The Cauchy development of such initial data will be an **axially symmetric spacetime.**
- **The axis** is the set where the norm $\eta = \eta^i \eta_i$ vanished.

Angular momentum in axial symmetry

Define the vector field J^i by

$$J^i = (K^{ij} - Kh^{ij})\eta_j.$$

By the Killing equation and the momentum constraint we have that

$$D^i J_i = 0.$$

Consider a closed 2-dimensional surface Σ. The angular momentum of Σ is defined by

$$J(\Sigma) = \oint_{\Sigma} J^i n_i \, dS,$$

where n^i is the unit normal of Σ.

If Σ is the boundary of some compact domain $\Omega \subset S$, by the Gauss theorem, we have $J(\Sigma) = 0$.

To have non-zero angular momentum in vacuum the manifold S should have a non-trivial topology. For example, multiples asymptotic ends. That signals the presence of a black hole.

Angular momentum–mass inequality

Theorem (Dain 06)

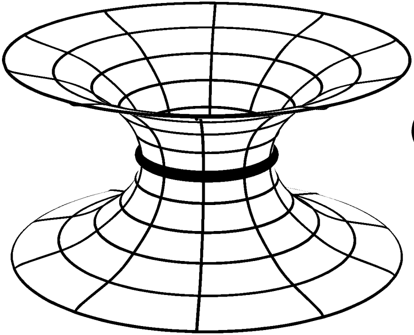
Consider an axially symmetric, vacuum asymptotically flat, maximal initial data set with two asymptotic ends. Then, the inequality

$$\sqrt{|J|} \leq m$$

holds, where m and J are the total mass and angular momentum at any asymptotic end. Moreover, the equality implies that the initial data set is the extreme Kerr initial data set.

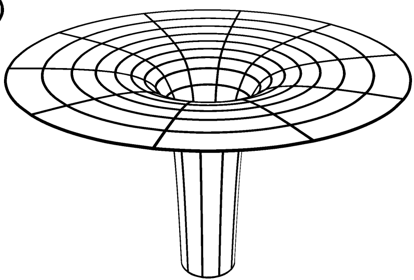
Rigidity

Two asymptotic flat ends



$m > \sqrt{|J|}$

Extreme Kerr initial data



$m = \sqrt{|J|}$

Area– angular momentum inequality

Theorem (Dain-Reiris 11)

Consider an axisymmetric, vacuum and maximal initial data. Assume that the initial data contain an orientable closed stable minimal axially symmetric surface Σ. Then

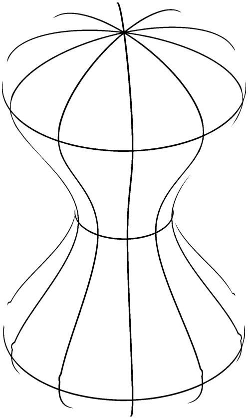
$$A \geq 8\pi|J|, \tag{6}$$

where A is the area and J the angular momentum of Σ. Moreover, if the equality in (6) holds then the local geometry of the surface Σ is an extreme Kerr throat sphere.

- A minimal surface is stable if it is a local minimum of the area.
- This inequality is a purely local inequality. There is global asumption on the manifold. In particular the manifold is not required to be asymptotically flat.

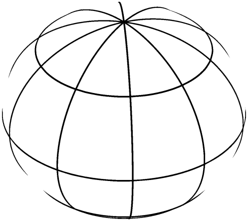
Rigidity

Generic minimal surface



$A > 8\pi|J|$

Extreme Kerr throat



$A = 8\pi|J|$

Extreme Kerr throat

This geometry is characterized by the concept of an extreme Kerr throat sphere, with angular momentum J, defined as follows. The sphere is embedded in an initial data with intrinsic metric given by

$$\gamma_0 = 4J^2 e^{-\sigma_0} d\theta^2 + e^{\sigma_0} \sin^2\theta d\phi^2,$$

where

$$\sigma_0 = \ln(4|J|) - \ln(1 + \cos^2\theta).$$

Moreover, the sphere must be totally geodesic, the twist potential evaluated at the surface must be given by

$$\omega_0 = -\frac{8J\cos\theta}{1 + \cos^2\theta},$$

and the components of the second fundamental

$$K_{ij}\xi^i = K_{ij}n^j n^i = K_{ij}\eta^j\eta^i = 0,$$

must vanish at the surface.

For more details see the review article:

Geometric inequalities for axially symmetric black holes
S. Dain

Class.Quant.Grav. 29 (2012) 073001 “Topical Review”
http://arxiv.org/abs/1111.3615