Topology of compact solvmanifolds

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$M = G/\Gamma$: solvmanifold

G: real simply connected solvable Lie group

Γ: lattice (discrete cocompact subgroup)

Aims: find lattices, compute de Rham cohomology, study existence of symplectic structures, hard Lefschetz property, formality

We will explain:

• the difference with nilmanifolds

• known results on the computation of Rham cohomology in special cases (completely solvable, Mostow condition)

• a method to compute the de Rham cohomology in general (following results by Guan and Witte)

• applications: Nakamura manifold, almost abelian solvable Lie groups, hyperelliptic surface, three families of lattices on the oscillator group

Nilpotent and solvable Lie groups

G is k-step nilpotent $\iff$
the descending chain of normal subgroups

$$G_0 = G \supset G_1 = [G, G] \supset \cdots \supset G_{i+1} = [G_i, G] \supset \ldots .$$

degenerates, i.e. $G_i = \{e\}\ \forall i \geq k$, ($e$ is the identity element).

G is k-step solvable $\iff$
the derived series of normal subgroups

$$G_{(0)} = G \supset G_{(1)} = [G, G] \supset \cdots \supset G_{(i+1)} = [G_{(i)}, G_{(i)}] \supset \ldots .$$

degenerates.

In particular a solvable Lie group is completely solvable if every eigenvalue λ of every operator Ad_g, $g \in G$, is real.

Note that *a nilpotent Lie group is completely solvable*.

Nilmanifolds

G simply connected nilpotent Lie group

Recall: $\exp : \mathfrak{g} \to G$ is a diffeomorphism

For nilpotent Lie groups there is a simple criterion for the existence of lattices:

Theorem (Malčev)

G simply connected nilpotent Lie group
$\exists\, \Gamma$ lattice on G $\iff$
the Lie algebra $\mathfrak{g}$ of G has a basis such that the structure constants in this basis are rational $\iff$
$\exists\, \mathfrak{g}_{\mathbb{Q}}$ such that $\mathfrak{g} = \mathfrak{g}_{\mathbb{Q}} \otimes \mathbb{R}$

If $\mathfrak{g} = \mathfrak{g}_{\mathbb{Q}} \otimes \mathbb{R}$, one also says that $\mathfrak{g}$ has a rational structure

A nilmanifold is the a quotient $M = G/\Gamma$, where G is a real simply connected nilpotent Lie group and Γ is a lattice.

Solvmanifolds

There is no simple criterion for the existence of a lattice in a connected and simply-connected solvable Lie group G.

Here are some necessary criteria.

Proposition (Milnor)

If G admits a lattice then it is unimodular. $[\mathrm{tr}\, \mathrm{ad}_X = 0, \ \forall X \in \mathfrak{g}]$

The Mostow bundle

Let G/Γ be a solvmanifold that is not a nilmanifold.
$N =$ **nilradical** of G =largest connected nilpotent normal subgroup of G.
Then $\Gamma_N := \Gamma \cap N$ is a lattice in N, $\Gamma N = N\Gamma$ is closed in G and $G/(N\Gamma) =: \mathbb{T}^k$ is a torus.
$\Longrightarrow$ we have the fibration:

$$N/\Gamma_N = (N\Gamma)/\Gamma \hookrightarrow G/\Gamma \longrightarrow G/(N\Gamma) = \mathbb{T}^k$$

Much of the rich structure of solvmanifolds is encoded in this bundle.
The nilradical has an important rôle in the study of solvmanifolds.

Solvmanifolds

G connected and simply connected solvable Lie group

$$\Longrightarrow \qquad G \overset{\text{diffeo}}{\simeq} \mathbb{R}^n$$

(*BUT* $\exp : \mathfrak{g} \to G$ *is not necessarily injective or surjective.*)

$\Longrightarrow$ solvmanifolds G/Γ are aspherical and $\pi_1(G/\Gamma) \cong \Gamma$.

The fundamental group plays an important rôle:

Diffeomorphism Theorem

G_1/Γ_1 and G_2/Γ_2 solvmanifolds and
$\varphi : \Gamma_1 \to \Gamma_2$ isomorphism.
$\Longrightarrow \exists$ diffeomorphism Φ $G_1 \to G_2$ such that

(i) $\Phi|_{\Gamma_1} = \varphi$,

(ii) $\forall_{\gamma \in \Gamma_1} \forall_{p \in G_1}$ $\Phi(p\gamma) = \Phi(p)\varphi(\gamma)$.

Corollary

Two solvmanifolds with isomorphic π_1 *are diffeomorphic.*

de Rham Cohomology

G/Γ solvmanifold (nilmanifold)

General question: can one compute Dolbeault cohomology of M by invariant forms,
i.e., using the Chevalley-Eilenberg complex:

$$\cdots \to \Lambda^{k-1}\mathfrak{g}^* \xrightarrow{d} \Lambda^k\mathfrak{g}^* \xrightarrow{d} \Lambda^{k+1}\mathfrak{g}^* \to \dots$$

$$d\alpha(x_1, \dots, x_{k+1}) = \sum_{i<j} (-1)^{i+j} \alpha([x_i, x_j], x_1, \dots \hat{x}_i, \dots, \hat{x}_j, \dots, x_{k+1})$$

So, when is $H^*_{dR}(G/\Gamma) \cong H^*(\mathfrak{g})$?

de Rham Cohomology of nilmanifolds

G/Γ nilmanifold

Theorem (Nomizu)

$H^*_{dR}(G/\Gamma) \cong H^*(\mathfrak{g})$.

$\bigwedge^* \mathfrak{g}$ is a minimal model of G/Γ (in the sense of Sullivan).

$\bigwedge^* \mathfrak{g}$ is formal $\iff$ G is abelian and G/Γ is a torus.

If a nilmanifold is Kählerian, then it is a torus.
[Benson & Gordon, Hasegawa]

Idea of the proof of Nomizu's Theorem

Suppose G is k-step nilpotent. $\mathfrak{g}_\ell$ Lie algebra of G_ℓ.

Let $H :=$ simply-connected Lie subgroup of G with Lie algebra $\mathfrak{h} = \mathfrak{g}_{k-1} \Longrightarrow H$ central and $H \cong \mathbb{R}^n$.

We have the fibration:

$$\mathbb{T}^n = H/H \cap \Gamma \hookrightarrow M = G/\Gamma \stackrel{\pi}{\to} \overline{M} = G/H\Gamma$$

$E_*^{p,q} :=$ Leray-Serre spectral sequence associated with the fibration:

$$E_2^{p,q} = H_{dR}^p(\overline{M}, H_{dR}^q(\mathbb{T}^n)) \cong H_{dR}^p(\overline{M}) \otimes \bigwedge^q \mathbb{R}^n ,$$

$$E_\infty^{p,q} \Rightarrow H_{dR}^{p+q}(M) .$$

Main idea: construct a second spectral sequence

$\tilde{E}_*^{p,q}$ = Leray-Serre spectral sequence for the complex of G-invariant forms $\bigwedge^* \mathfrak{g}^*$

$\bigwedge^* \mathfrak{g}^*$ subcomplex of $\bigwedge^* M \Longrightarrow \tilde{E}_*^{p,q} \subseteq E_*^{p,q}$ &

$$\tilde{E}_2^{p,q} = H^p(\mathfrak{g}/\mathfrak{h}) \otimes \bigwedge^q \mathbb{R}^n ,$$

$$\tilde{E}_\infty^{p,q} \Rightarrow H^{p+q}(\mathfrak{g}) .$$

$\dim \overline{M} < \dim M$, $\overset{\text{induction on dim}}{\Longrightarrow}$ $H^p_{dR}(\overline{M}) \cong H^p(\mathfrak{g}/\mathfrak{h})$ for any p.

$\Longrightarrow E_2 = \tilde{E}_2$ & $E_\infty = \tilde{E}_\infty$.

i.e., $H^\ell_{dR}(M) \cong H^\ell(\mathfrak{g})$ for any ℓ.

de Rham Cohomology of solvmanifolds

G/Γ solvmanifold

$\mathrm{Ad}_G(G) = \{e^{\mathrm{ad}_X} \mid X \in \mathfrak{g}\}$ solvable and $\mathrm{Aut}(G) \cong \mathrm{Aut}(\mathfrak{g})$.

$\mathcal{A}(\mathrm{Ad}_G(G))$ and $\mathcal{A}(\mathrm{Ad}_G(\Gamma))$: real algebraic closures of $\mathrm{Ad}_G(G)$ and $\mathrm{Ad}_G(\Gamma)$ (respectively)

Theorem (Borel density theorem)

Let Γ be a lattice of a simply connected solvable Lie group G, $\Longrightarrow \exists$ a maximal compact torus $\mathbb{T}_{cpt} \subset \mathcal{A}(\mathrm{Ad}_G(G))$, such that

$$\mathcal{A}(\mathrm{Ad}_G(G)) = \mathbb{T}_{cpt}\mathcal{A}(\mathrm{Ad}_G(\Gamma)).$$

When is the de Rham comology of a solvmanifold given by the Chevalley-Eilenberg complex?

There are 2 important cases:

1. **Hattori**: If G is completely solvable, i.e., if the linear map $\mathrm{ad}_X : \mathfrak{g} \to \mathfrak{g}$ has only real eigenvalues
2. **Mostow condition**: If $\mathcal{A}(\mathrm{Ad}_G(G)) = \mathcal{A}(\mathrm{Ad}_G(\Gamma))$

Remarks

- (1) is a particular case of (2).
  Indeed if all eigenvalues of Ad are real, then $\mathcal{A}(\mathrm{Ad}_G(G))$ has no non trivial conn. compact subgroups $\overset{\text{Borel density}}{\Longrightarrow}$ $\mathcal{A}(\mathrm{Ad}_G(G)) = \mathcal{A}(\mathrm{Ad}_G(\Gamma))$
- Recall that a nilpotent Lie group is completely solvable $\Longrightarrow$ (1) and (2) generalize Nomizu's Theorem.
- We will see that one can have the isomorphism $H^*(\mathfrak{g}) \cong H^*_{dR}(G/\Gamma)$ even if $\mathcal{A}(\mathrm{Ad}_G(\Gamma)) \neq \mathcal{A}(\mathrm{Ad}_G(G))$ (Example on hyperelliptic surface) hyperelliptic surface

Idea of the proof of (2)

We prove that if the Mostow condition holds, we still have a fibration of $M = G/\Gamma$ over a smaller dimensional solvmanifold with a torus as fibre. Then one can proceed as in the proof of Nomizu's Theorem.

$G_{(k)} = [G_{(k-1)}, G_{(k-1)}]$ & $\Gamma_{(k)} = [\Gamma_{(k-1)}, \Gamma_{(k-1)}]$: derived series

Remark that $G_{(k)}$ is nilpotent for any $k \geq 1$

Mostow condition + a gen. result on lattices in nilpotent Lie groups
$\Longrightarrow G_{(k)}/\Gamma_{(k)}$ is compact for any k.
$\Longrightarrow G_{(k)}/\Gamma \cap G_{(k)}$ is compact for any k.

Let r be the last non-zero term in the derived series of G. Namely $G_{(r+1)} = (e)$ and $G_{(r)} =: A \neq (e)$.

A is abelian $\Longrightarrow A/A \cap \Gamma := \mathbb{T}^m$ is a compact torus.

Thus, $\overline{M} := G/A\Gamma$ is a compact solvmanifold with dimension smaller than $M := G/\Gamma$ and $\mathbb{T}^m \hookrightarrow M \stackrel{\pi}{\to} \overline{M}$ is a fibration.

If $\mathcal{A}(\mathrm{Ad}_G(G)) \neq \mathcal{A}(\mathrm{Ad}_G(\Gamma))$ it is more difficult to compute the de Rham cohomology.

We explain a method deriving from results of Guan and Witte

Main Theorem [– , A. Fino]

Let $M = G/\Gamma$ be a compact solvmanifold and let $\mathbb{T}_{cpt}$ be a compact torus such that

$$\mathbb{T}_{cpt}\mathcal{A}(\mathrm{Ad}_G(\Gamma)) = \mathcal{A}(\mathrm{Ad}_G(G)).$$

Then there exists a subgroup $\tilde{\Gamma}$ of finite index in Γ and a simply connected normal subgroup $\tilde{G}$ of $\mathbb{T}_{cpt} \ltimes G$ such that

$$\mathcal{A}(\mathrm{Ad}_{\tilde{G}}(\tilde{\Gamma})) = \mathcal{A}(\mathrm{Ad}_{\tilde{G}}(\tilde{G})).$$

$\Longrightarrow$ $\tilde{G}/\tilde{\Gamma}$ is diffeomorphic to $G/\tilde{\Gamma}$ and $H^*_{dR}(G/\tilde{\Gamma}) \cong H^*(\tilde{\mathfrak{g}})$.

Observe that $H^*_{dR}(G/\Gamma) \cong H^*_{dR}(G/\tilde{\Gamma})^{\Gamma/\tilde{\Gamma}}$
(the invariants by the action of the finite group $\Gamma/\tilde{\Gamma}$).

Proof of the Main Theorem

It is not restrictive to suppose that $\mathcal{A}(Ad_G(\Gamma))$ is connected. Otherwise we pass from Γ to a finite index subgroup $\tilde{\Gamma}$ [equivalently $M = G/\Gamma \rightsquigarrow G/\tilde{\Gamma}$ finite-sheeted covering of M]

Let $\mathbb{T}_{cpt}$ be a maximal compact torus of $\mathcal{A}(\mathrm{Ad}_G G)$ which contains a maximal compact torus $\overline{\mathbb{S}}_{cpt}$ of $\mathcal{A}(\mathrm{Ad}_G(\tilde{\Gamma}))$.

Let $\mathbb{S}_{cpt}$ be a subtorus of $\mathbb{T}_{cpt}$ complementary to $\overline{\mathbb{S}}_{cpt}$ so that $\mathbb{T}_{cpt} = \mathbb{S}_{cpt} \times \overline{\mathbb{S}}_{cpt}$.

Let σ be the composition of the homomorphisms:

$$\sigma : G \xrightarrow{\mathrm{Ad}} \mathcal{A}(\mathrm{Ad}_G(G)) \xrightarrow{\text{proj}} \mathbb{T}_{cpt} \xrightarrow{\text{proj}} \mathbb{S}_{cpt} \xrightarrow{x \to x^{-1}} \mathbb{S}_{cpt}.$$

The point now is to get rid of $\mathbb{S}_{cpt}$

$\rightsquigarrow$ nilshadow map:

$$\Delta : G \to \mathbb{S}_{cpt} \ltimes G, g \mapsto (\sigma(g), g),$$

[not a homomorphism (unless $\mathbb{S}_{cpt} = \{0\} \implies \sigma = 0$] One has:

$$\Delta(ab) = \Delta(\sigma(b^{-1})a\sigma(b))\,\Delta(b), \quad \forall a, b \in G$$

and $\Delta(\gamma g) = \gamma\Delta(g)$, for every $\gamma \in \tilde{\Gamma}, g \in G$.

Δ is a diffeomorphism onto it image $\implies$ $\Delta(G)$ is simply connected

The product in $\Delta(G)$ is given by:

$$\Delta(a)\Delta(b) = (\sigma(a), a)\,(\sigma(b), b) = (\sigma(a)\sigma(b), \sigma(b^{-1})a\sigma(b)\,b),$$

for any $a, b \in G$.

By construction , $\mathcal{A}(\mathrm{Ad}_G(\tilde{\Gamma}))$ projects trivially on $\mathbb{S}_{cpt}$ and $\sigma(\tilde{\Gamma}) = \{e\}$. $\implies$

$$\tilde{\Gamma} = \Delta(\tilde{\Gamma}) \subset \Delta(G).$$

Let $\tilde{G} = \Delta(G)$.
[Witte]: $\overline{\mathbb{S}}_{cpt}$ is a maximal compact subgroup of $\mathcal{A}(\mathrm{Ad}_{\tilde{G}}(\tilde{G}))$ and $\overline{\mathbb{S}}_{cpt} \subset \mathcal{A}(\mathrm{Ad}_{\tilde{G}}(\tilde{\Gamma})) \Longrightarrow \mathcal{A}(\mathrm{Ad}_{\tilde{G}}(\tilde{G})) = \mathcal{A}(\mathrm{Ad}_{\tilde{G}}(\tilde{\Gamma}))$

$\overset{\text{Diffeomorphism Theorem}}{\Longrightarrow}$ $G/\tilde{\Gamma}$ is diffeomorphic to $\tilde{G}/\tilde{\Gamma}$.

Mostow condition holds $\Longrightarrow$ $H^*(G/\tilde{\Gamma}) \cong H^*(\tilde{\mathfrak{g}})$.

By the diffeomorphism $\Delta : G \to \tilde{G}$, Δ^{-1} induces a finite sheeted covering map $\Delta^* : \tilde{G}/\tilde{\Gamma} \to G/\Gamma$.

Corollary

The Lie algebra $\tilde{\mathfrak{g}}$ of $\tilde{G}$ can be identified by

$$\tilde{\mathfrak{g}} = \{(X_{\mathfrak{s}}, X) \mid X \in \mathfrak{g}\}$$

with Lie bracket:

$$[(X_{\mathfrak{s}}, X), (Y_{\mathfrak{s}}, Y)] = (0, [X, Y] - \mathrm{ad}(X_{\mathfrak{s}})(Y) + \mathrm{ad}(Y_{\mathfrak{s}})(X)).$$

where $X_{\mathfrak{s}}$ the image $\sigma_*(X)$, for $X \in \mathfrak{g}$

Applications

Now we obtain some applications of the Main Theorem, by computing explicitly the Lie group $\tilde{G}$.

Example (Nakamura manifold – description)

Consider the simply connected complex solvable Lie group G:

$$G = \left\{ \begin{pmatrix} e^z & 0 & 0 & w_1 \\ 0 & e^{-z} & 0 & w_2 \\ 0 & 0 & 1 & z \\ 0 & 0 & 0 & 1 \end{pmatrix}, w_1, w_2, z \in \mathbb{C} \right\}.$$

$G \cong \mathbb{C} \ltimes_\varphi \mathbb{C}^2$, where

$$\varphi(z) = \begin{pmatrix} e^z & 0 \\ 0 & e^{-z} \end{pmatrix}.$$

Let

$$L_{1,2\pi} = \mathbb{Z}[t_0, 2\pi i] = \{t_0 k + 2\pi h i, h, k \in \mathbb{Z}\},$$

$$L_2 = \left\{ P \begin{pmatrix} \mu \\ \alpha \end{pmatrix}, \mu, \alpha \in \mathbb{Z}[i] \right\}.$$

Then, by Yamada $\Gamma = L_{1,2\pi} \ltimes_\varphi L_2$ is a lattice of G.

G/Γ: Nakamura manifold

Example (Nakamura manifold – computation of cohomology)

G has trivial center $\Longrightarrow$ $\mathrm{Ad}_G(G) \cong G \cong \mathbb{R}^2 \ltimes \mathbb{R}^4$. Moreover,

$$\mathcal{A}(\mathrm{Ad}_G G) = (\mathbb{R}^\# \times S^1) \ltimes \mathbb{R}^4,$$
$$\mathcal{A}(\mathrm{Ad}_G \Gamma) = \mathbb{R}^\# \ltimes \mathbb{R}^4,$$

where the split torus $\mathbb{R}^\#$ corresponds to the action of $e^{\frac{1}{2}(z+\bar{z})}$ and the compact torus S^1 to the one of $e^{\frac{1}{2}(z-\bar{z})}$.

$\Longrightarrow$ $\mathcal{A}(\mathrm{Ad}_G(G)) = S^1 \mathcal{A}(\mathrm{Ad}_G(\Gamma))$ and $\mathcal{A}(\mathrm{Ad}_G(\Gamma))$ is connected.

Main Theorem $\Longrightarrow$ $\exists$ a simply connected normal subgroup $\tilde{G} = \Delta(G)$ of $S^1 \ltimes G$.

The new Lie group $\tilde{G}$ is obtained by **killing the action of** $e^{\frac{1}{2}(z-\bar{z})}$:

$$\tilde{G} \cong \left\{ \begin{pmatrix} e^{\frac{1}{2}(z+\bar{z})} & 0 & 0 & w_1 \\ 0 & e^{-\frac{1}{2}(z+\bar{z})} & 0 & w_2 \\ 0 & 0 & 1 & z \\ 0 & 0 & 0 & 1 \end{pmatrix}, w_1, w_2, z \in \mathbb{C} \right\}.$$

$G/\Gamma \overset{\text{diffeo}}{\simeq} \tilde{G}/\Gamma$ was already shown by Yamada. $\Longrightarrow$
$H^*_{dR}(G/\Gamma) \cong H^*(\tilde{\mathfrak{g}})$, ($\tilde{\mathfrak{g}}$ Lie algebra of $\tilde{G}$) and $H_{dR}(G/\Gamma) \ncong H^*(\mathfrak{g})$.

Example (A three dimensional example)

$G = \mathbb{R} \ltimes \mathbb{R}^2$ with structure equations $\begin{cases} de^1 = 0, \\ de^2 = 2\pi e^1 \wedge e^3 \\ de^3 = -2\pi e^1 \wedge e^2 \end{cases}$ is non-completely solvable and admits a lattice $\Gamma = \mathbb{Z} \ltimes \mathbb{Z}^2$. Indeed,

$$\mathbb{R} \ltimes \mathbb{R}^2 = \left\{ \begin{pmatrix} \cos(2\pi t) & \sin(2\pi t) & 0 & x \\ -\sin(2\pi t) & \cos(2\pi t) & 0 & y \\ 0 & 0 & 1 & t \\ 0 & 0 & 0 & 1 \end{pmatrix} \right\}$$

and Γ is generated by 1 in $\mathbb{R}$ and the standard lattice $\mathbb{Z}^2$.

$\mathcal{A}(\mathrm{Ad}_G(G)) = S^1 \ltimes \mathbb{R}^2$ and $\mathcal{A}(\mathrm{Ad}_G(\Gamma)) = \mathbb{R}^2 \overset{\text{Main Theorem}}{\Longrightarrow}$
$$\tilde{G} \cong \mathbb{R}^3 \subset S^1 \ltimes G.$$

Indeed, it is well known that G/Γ is diffeomorphic to a torus.

The previous example $\mathbb{R} \ltimes \mathbb{R}^2$ is an *almost abelian* Lie group:

A Lie algebra $\mathfrak{g}$ is called ***almost abelian*** *if it has an abelian ideal of codimension* 1*,*
i.e. $\mathfrak{g} \cong \mathbb{R} \ltimes \mathfrak{b}$*, where* $\mathfrak{b} \cong \mathbb{R}^n$ *is an abelian ideal of* $\mathfrak{g}$*.*

In this case the Mostow bundle is a torus bundle over S^1

The action φ of $\mathbb{R}$ on $\mathbb{R}^n$ is represented by a family of matrices $\varphi(t)$, which encode the monodromy or "twist" in the Mostow bundle.

A nice feature of almost abelian solvable groups is that there is a criterion on the existence of a lattice

Proposition (Bock)

Let $G = \mathbb{R} \ltimes_\varphi \mathbb{R}^n$ be almost abelian solvable Lie group. Then G admits a lattice if and only if there exists a $t_0 \neq 0$ for which $\varphi(t_0)$ can be conjugated to an integer matrix.

The Lie algebra $\mathfrak{g}$ of G has form

$$\mathbb{R} \ltimes_{\mathrm{ad}_{X_{n+1}}} \mathbb{R}^n ,$$

where we consider $\mathbb{R}^n$ generated by $\{X_1, \dots, X_n\}$ and $\mathbb{R}$ by X_{n+1}, and $\varphi(t) = e^{t\,\mathrm{ad}_{X_{n+1}}}$.

Moreover, a lattice can be always represented as $\Gamma = \mathbb{Z} \ltimes \mathbb{Z}^n$

For almost abelian solvmanifolds, Gorbatsevich found a criterion to decide whether the Mostow condition holds:

Proposition (Gorbatsevich)

The Mostow condition is satisfied if and only if πi can not be written as linear combination in $\mathbb{Q}$ of the eigenvalues of $t_0 \mathrm{ad}_{X_{n+1}}$, where Γ is generated by t_0.

• If $i\pi$ is not representable as a $\mathbb{Q}$-linear combination of the numbers λ_k, $\overset{\text{Mostow condition}}{\Longrightarrow}$ $H^*_{dR}(G/\Gamma) \cong H^*(\mathfrak{g})$.

• Otherwise the only known result on cohomology [Bock]

$$b_1(G/\Gamma) = n + 1 - \text{rank}(\varphi(1) - \mathrm{id}).$$

By applying the Main Theorem one obtains a method to compute the de Rham cohomology of G/Γ.

[– , M. Macrì] construct lattices on six dimensional not completely solvable almost abelian Lie groups, for which the Mostow condition does not hold. We compute

- cohomology (does not agree with the one of $\mathfrak{g}$)
- minimal model
- show that some of these solvmanifolds admit not invariant symplectic structures and we study formality and Lefschetz properties

Example (6-dimensional indecomposable almost abelian solvmanifolds not satisfying the Mostow condition)

G	$\Gamma_{\bar t}$	$H^*(\mathfrak{g})$	$H^*(G/\Gamma_{\bar t})$	**F**	**IS**	**S**	**HL**
$G^{p=0}_{6.8}$	$\bar t = 2\pi$	$b_1 = 1, b_2 = 1, b_3 = 2$	$b_1 = 3, b_2 = 3, b_3 = 2$	Yes	No	No	\
	$\bar t = \pi, \frac{\pi}{2}, \frac{\pi}{3}$		$b_1 = 1, b_2 = 1, b_3 = 2$	Yes	No	\	\
$G^{a=0}_{6.10}$	$\bar t = 2\pi$	$b_1 = 2, b_2 = 3, b_3 = 4$	$b_1 = 4, b_2 = 7, b_3 = 8$	No	Yes	Yes	No$^\times$
	$\bar t = \pi, \frac{\pi}{2}, \frac{\pi}{3}$		$b_1 = 2, b_2 = 3, b_3 = 4$	No	Yes	\	No*
$G^{p=0}_{6.11}$	$\bar t = 2\pi$	$b_1 = 1, b_2 = 1, b_3 = 1$	$b_1 = 3, b_2 = 4, b_3 = 4$	Yes	No	No	\
	$\bar t = \pi, \frac{\pi}{2}, \frac{\pi}{3}$		$b_1 = 1, b_2 = 1, b_3 = 1$	Yes	No	\	\

$^\times$ = for both the invariant and not invariant symplectic structures considered.

* = for the invariant symplectic structures.

F: *formality*
IS: *existence of invariant symplectic structures*
S: *existence of symplectic structures*
(induced by ones on the modified Lie alg)
HL: *Hard Lefschetz property*

Example (6-dimensional decomposable almost abelian solvmanifolds not satisfying the Mostow condition)

G	$\Gamma_{\bar t}$	$H^*(\mathfrak{g})$	$H^*(G/\Gamma_{\bar t})$	**F**	**IS**	**S**	**HL**
$G^0_{5.14}\times\mathbb{R}$	$\bar t=2\pi$	$b_1=3, b_2=5, b_3=6$	$b_1=5, b_2=11, b_3=14$	No	Yes	Yes	No$^\times$
	$\bar t=\pi,\frac{\pi}{2},\frac{\pi}{3}$		$b_1=3, b_2=5, b_3=6$	No	Yes	\	No*
$G^{p,-p,r}_{5.17}\times\mathbb{R}$ $r=\frac{r_1}{r_2}\in\mathbb{Q}$	$\bar t=2\pi r_2$	if $p\neq 0, r\neq\pm1$ $b_1=2, b_2=1, b_3=0$	$p\neq 0:\ b_1=6, b_2=15, b_3=20$ $p=0:\ b_1=2, b_2=5, b_3=8$	Yes	Yes	Yes	Yes$^\times$
	$\bar t=\pi$, r even	if $p=0, r\neq\pm1$ or $p\neq 0, r=\pm1$ $b_1=2, b_2=3, b_3=4$	$p\neq 0:\ b_1=2, b_2=1, b_3=0$ $p=0:\ b_1=4, b_2=7, b_3=8$	Yes	Yes	\	Yes*
	$\bar t=\pi$, r odd		$p\neq 0:\ b_1=2, b_2=5, b_3=8$ $p=0:\ b_1=2, b_2=7, b_3=12$	Yes	Yes	\	Yes*
	$\bar t=\frac{\pi}{2}, r\equiv_4 0$		$p=0:\ b_1=4, b_2=7, b_3=8$	Yes	Yes	\	Yes*
	$\bar t=\frac{\pi}{2}$, $r\equiv_4 1,3$	if $p=0, r=\pm1$ $b_1=2, b_2=5, b_3=8$	$p\neq 0:\ b_1=2, b_2=3, b_3=4$ $p=0:\ b_1=2, b_2=5, b_3=8$	Yes	Yes	\	Yes*
	$\bar t=\frac{\pi}{2}, r\equiv_4 2$		$p=0:\ b_1=2, b_2=3, b_3=4$	Yes	Yes	\	Yes*
$G^0_{5.18}\times\mathbb{R}$	$\bar t=2\pi$	$b_1=2, b_2=3, b_3=4$	$b_1=4, b_2=9, b_3=13$	No	Yes	Yes	No$^\times$
	$\bar t=\pi$,		$b_1=2, b_2=5, b_3=8$	No	Yes	\	No*
	$\bar t=\frac{\pi}{2},\frac{\pi}{3}$		$b_1=2, b_2=3, b_3=4$	No	Yes	\	No*
$G^0_{3.5}\times\mathbb{R}^3$	$\bar t=2\pi$	$b_1=4, b_2=7, b_3=8$	$b_1=6, b_2=15, b_3=20$	Yes	Yes	Yes	Yes$^\times$
	$\bar t=\pi,\frac{\pi}{2},\frac{\pi}{3}$		$b_1=4, b_2=7, b_3=8$	Yes	Yes	\	Yes*

$^\times$ = for both the invariant and the not invariant symplectic structures considered.

Example (6-dimensional decomposable almost abelian solvmanifolds not satisfying the Mostow condition)

G	$\Gamma_{\bar t}$	$H^*(\mathfrak{g})$	$H^*(G/\Gamma_{\bar t})$	F	IS	S	HL
$G^0_{5.14}\times\mathbb{R}$	$\bar t=2\pi$	$b_1=3, b_2=5, b_3=6$	$b_1=5, b_2=11, b_3=14$	No	Yes	Yes	No$^\times$
	$\bar t=\pi, \frac{\pi}{2}, \frac{\pi}{3}$		$b_1=3, b_2=5, b_3=6$	No	Yes	\	No*
$G^{p,-p,r}_{5.17}\times\mathbb{R}$	$\bar t=2\pi r_2$	if $p\neq 0, r\neq\pm 1$ $b_1=2, b_2=1, b_3=0$	$p\neq 0:\ b_1=6, b_2=15, b_3=20$ $p=0:\ b_1=2, b_2=5, b_3=8$	Yes	Yes	Yes	Yes$^\times$
	$\bar t=\pi$, r even	if $p=0, r\neq\pm 1$ or $p\neq 0, r=\pm 1$	$p\neq 0:\ b_1=2, b_2=1, b_3=0$ $p=0:\ b_1=4, b_2=7, b_3=8$	Yes	Yes	\	Yes*
	$\bar t=\pi$, r odd	$b_1=2, b_2=3, b_3=4$	$p\neq 0:\ b_1=2, b_2=5, b_3=8$ $p=0:\ b_1=2, b_2=7, b_3=12$	Yes	Yes	\	Yes*
$r=\frac{r_1}{r_2}\in\mathbb{Q}$	$\bar t=\frac{\pi}{2}, r\equiv_4 0$		$p=0: b_1=4, b_2=7, b_3=8$	Yes	Yes	\	Yes*
	$\bar t=\frac{\pi}{2}$, $r\equiv_4 1,3$	if $p=0, r=\pm 1$ $b_1=2, b_2=5, b_3=8$	$p\neq 0: b_1=2, b_2=3, b_3=4$ $p=0: b_1=2, b_2=5, b_3=8$	Yes	Yes	\	Yes*
	$\bar t=\frac{\pi}{2}, r\equiv_4 2$		$p=0: b_1=2, b_2=3, b_3=4$	Yes	Yes	\	Yes*
$G^0_{5.18}\times\mathbb{R}$	$\bar t=2\pi$	$b_1=2, b_2=3, b_3=4$	$b_1=4, b_2=9, b_3=13$	No	Yes	Yes	No$^\times$
	$\bar t=\pi$,		$b_1=2, b_2=5, b_3=8$	No	Yes	\	No*
	$\bar t=\frac{\pi}{2}, \frac{\pi}{3}$		$b_1=2, b_2=3, b_3=4$	No	Yes	\	No*
$G^0_{3.5}\times\mathbb{R}^3$	$\bar t=2\pi$	$b_1=4, b_2=7, b_3=8$	$b_1=6, b_2=15, b_3=20$	Yes	Yes	Yes	Yes$^\times$
	$\bar t=\pi, \frac{\pi}{2}, \frac{\pi}{3}$		$b_1=4, b_2=7, b_3=8$	Yes	Yes	\	Yes*

$^\times$ = for both the invariant and the not invariant symplectic structures considered.

Example (6-dimensional decomposable almost abelian solvmanifolds not satisfying the Mostow condition)

$G^0_{5.18} \times \mathbb{R}$	$\bar{t} = 2\pi$	$b_1 = 2, b_2 = 3, b_3 = 4$	$b_1 = 4, b_2 = 9, b_3 = 13$	No	Yes	Yes	No×
	$\bar{t} = \pi,$		$b_1 = 2, b_2 = 5, b_3 = 8$	No	Yes	\	No*
	$\bar{t} = \frac{\pi}{2}, \frac{\pi}{3}$		$b_1 = 2, b_2 = 3, b_3 = 4$	No	Yes	\	No*

∃ examples where the cohomology depends strongly on the lattice:

$$H^*_{dR}(G/\Gamma_\pi) \ncong H^*_{dR}(G/\Gamma_{2\pi}) \ncong H^*(\mathfrak{g}), \qquad G = G^0_{5.18} \times \mathbb{R}.$$

Kähler structures on solvmanifolds

Benson-Gordon conjecture: a compact solvmanifold has a Kähler structure if and only if it is a complex torus

Hasegawa (2006):

A solvmanifold carries a Kähler metric if and only if it is covered by a finite quotient of a complex torus, which has the structure of a complex torus bundle over a complex torus.

An example is provided by the hyperelliptic surface

▸ hyperelliptic surface

in particular,

a compact solvmanifold of completely solvable type has a Kähler structure if and only if it is a complex torus

Half-flat symplectic structures on solvmanifolds

Six dimensional almost abelian solvmanifolds were consider in string backgrounds where the internal compactification manifold is a solvmanifold (see e.g. [Andriot, Goi, Minasian and Petrini]).

They are related to solutions of the supersymmetry (SUSY) equations.

By [Fino-Ugarte], solution of the SUSY equations IIA possess a symplectic half-flat structure, whereas solutions of the SUSY equations IIB admit a half-flat structure

An $SU(3)$ structure on a six-dimensional manifold M (i.e., an $SU(3)$ reduction of the frame bundle of M) $\rightsquigarrow$

- a non-degenerate 2-form Ω,
- an almost-complex structure J,
- a complex volume form Ψ.

The SU(3) structure is called half-flat if $\Omega \wedge \Omega$ and the real part of Ψ are closed [Chiossi-Salamon].

If in addition Ω is closed, the half-flat structure is called symplectic.

Half-flat symplectic structures on solvmanifolds

Proposition (– , M. Macrì)

We have the following behavior concerning half flatness of (invariant) symplectic structures for the above solvmanifolds:

- $G_{6.10}^{a=0}/\Gamma_{2\pi}$ and $G_{5.14}^{0} \times \mathbb{R}/\Gamma_{2\pi}$ admit (not) invariant symplectic forms which are not half flat.
- $G_{5.17}^{p,-p,r} \times \mathbb{R}/\Gamma_{2\pi r_2}$ ($r = \frac{r_1}{r_2} \in \mathbb{Q}$) admits an invariant symplectic form which is half flat only for $p \geq 0$ and $r = 1$ and it admits a not invariant symplectic form which is half flat.
- $G_{5.18}^{0} \times \mathbb{R}/\Gamma_{2\pi}$ and $G_{3.5}^{0} \times \mathbb{R}^3/\Gamma_{2\pi}$ admit (not) invariant symplectic forms which are half flat.

Example (Hyperelliptic surface)

$G = \mathbb{R} \ltimes_{\varphi} (\mathbb{C} \times \mathbb{R})$, with $\varphi : \mathbb{R} \to \mathrm{Aut}\,(\mathbb{C} \times \mathbb{R})$ defined by

$$\varphi(t)(z,s) = (e^{i\eta t}z, s), \qquad \text{where } \eta = \pi, \frac{2}{3}\pi, \frac{1}{2}\pi \text{ or } \frac{1}{3}\pi$$

Hasegawa: G has 7 isomorphism classes of lattices $\Gamma = \mathbb{Z} \ltimes_{\varphi} \mathbb{Z}^3$, where $\varphi : \mathbb{Z} \to \mathrm{Aut}\,(\mathbb{Z}^3)$ has matrix $\varphi(1)$ with eigenvalues 1, $e^{i\eta}$, $e^{-i\eta}$.

$\varphi(1)$ has a pair of cx conj roots $\overset{\text{Gorbatsevich}}{\Longrightarrow}$ $\mathcal{A}(\mathrm{Ad}_G(G)) \neq \mathcal{A}(\mathrm{Ad}_G(\Gamma))$.

In this case $\mathcal{A}(\mathrm{Ad}_G(\Gamma))$ is not connected, but
Γ contains as a finite index subgroup $\tilde{\Gamma} \cong \mathbb{Z}^4$
$\Longrightarrow$ G/Γ is a finite covering of a torus

Note: $H^1_{dR}(G/\Gamma) \cong H^1(\mathfrak{g})$ even if $\mathcal{A}(\mathrm{Ad}_G(G)) \neq \mathcal{A}(\mathrm{Ad}_G(\Gamma))$
Indeed, G has structure equations:

$$\begin{cases} de^1 = e^2 \wedge e^4 \\ de^2 = -e^1 \wedge e^4 \\ de^3 = 0 \\ de^4 = 0 \end{cases} \quad \text{and} \quad H^1(\mathfrak{g}) = \text{span} < e^3, e^4 > .$$

Example (Three families of lattices in the oscillator group [– , G. Ovando, M. Subils])

Oscillator group: $G = \mathbb{R} \ltimes_\alpha \mathrm{H}_3(\mathbb{R})$
$\mathrm{H}_3(\mathbb{R})$ (real) three dimensional Heisenberg group

$$\alpha : \mathbb{R} \to \mathrm{Aut}\,(\mathfrak{h}_3)\,, \qquad t \mapsto \begin{pmatrix} \cos(t) & \sin(t) & 0 \\ -\sin(t) & \cos(t) & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The oscillator group is an almost nilpotent solvable Lie group

If we regard $\mathrm{H}_3(\mathbb{R})$ as $\mathbb{R}^3$ endowed with the operation

$$(x, y, z) \cdot (x', y', z') = (x + x', y + y', z + z' + \frac{1}{2}(xy' - x'y))$$

$\Longrightarrow \mathrm{H}_3(\mathbb{R})$ admists the co-compact subgroups $\Gamma_k \subset \mathrm{H}_3(\mathbb{R})$ given by

$$\Gamma_k = \mathbb{Z} \times \mathbb{Z} \times \frac{1}{2k}\mathbb{Z}\,.$$

Example (The oscillator group)

The lattice Γ_k (for any k) is invariant under the subgroups generated by $\alpha(0) = \alpha(2\pi)$, $\alpha(\pi)$ and $\alpha(\frac{\pi}{2}) \implies$
we have three families of lattices in $G = \mathbb{R} \ltimes_\alpha \mathrm{H}_3(\mathbb{R})$:

$$\Lambda_{k,0} = 2\pi\mathbb{Z} \ltimes \Gamma_k \subset G\,,$$
$$\Lambda_{k,\pi} = \pi\mathbb{Z} \ltimes \Gamma_k \subset G\,,$$
$$\Lambda_{k,\pi/2} = \tfrac{\pi}{2}\mathbb{Z} \ltimes \Gamma_k \subset G\,.$$

$\implies \Lambda_{k,0} \triangleright \Lambda_{k,\pi} \triangleright \Lambda_{k,\pi/2}$ ($\triangleright$: "contains as a normal subgroup"),
$\rightsquigarrow$ we have the solvmanifolds

$$M_{k,0} = G/\Lambda_{k,0}\,,$$
$$M_{k,\pi} = G/\Lambda_{k,\pi}\,,$$
$$M_{k,\pi/2} = G/\Lambda_{k,\pi/2}\,.$$

All subgroups of the families $\Lambda_{k,i}$ are not pairwise isomorphic $\implies$ determine non-diffeomorphic solvmanifolds.

Example (The oscillator group)

The action of $\alpha(0)$ is trivial, so $\Lambda_{k,0} = 2\pi\mathbb{Z} \times \Gamma_k$

Diffeomorphism Theorem $\Longrightarrow$

$$M_{k,0} = G/\Lambda_{k,0} \cong S^1 \times \mathrm{H}_3(\mathbb{R})/\Gamma_k,$$

a *Kodaira–Thurston manifold*.

Moreover, for any fixed k, we have the finite coverings

$$p_\pi : M_{k,0} \to M_{k,\pi}\,, \qquad p_{\pi/2} : M_{k,0} \to M_{k,\pi/2}\,,$$

[– , G. Ovando, M. Subils]

The Betti numbers b_i of the solvmanifolds $M_{k,*}$ are given by

	b_0	b_1	b_2
$M_{k,0}$	1	3	4
$M_{k,\pi}$	1	1	0
$M_{k,\pi/2}$	1	1	0

(clearly $b_3 = b_1$ and $b_4 = b_0$, by Poincaré duality).

There are many symplectic structures on $M_{k,0}$ which are invariant by the group $\mathbb{R} \times \mathrm{H}_3(\mathbb{R})$ but not under the oscillator group G.