The Ricci flow in a class of solvmanifolds

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vary Lie brackets $\leftrightsquigarrow$ vary inner products.

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Theorem ([L3],2012)

There exist time-dependent diffeomorphisms $\varphi(t) : G \to G_{\mu(t)}$ *such that* $g(t) = \varphi(t)^* g_{\mu(t)}$, $\forall t \in (a, b)$.

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$$\mu(e_0, e_i) = Ae_i, \;\; i = 1, \dots, n, \quad A \in \mathfrak{gl}_n(\mathbb{R}),$$
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$$\mathrm{Ric}_{\mu_A} = \begin{pmatrix} -\operatorname{tr}(S(A)^2) & 0 \\ 0 & \frac{1}{2}[A, A^t] - \operatorname{tr}(A)S(A) \end{pmatrix}. \tag{5}$$

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$$\begin{array}{ll} x' = x(x+y)(-\frac{3}{2}x + \frac{1}{2}y), & x(0) = x_0, \\ y' = y(x+y)(-\frac{3}{2}y + \frac{1}{2}x), & y(0) = y_0. \end{array} \tag{7}$$

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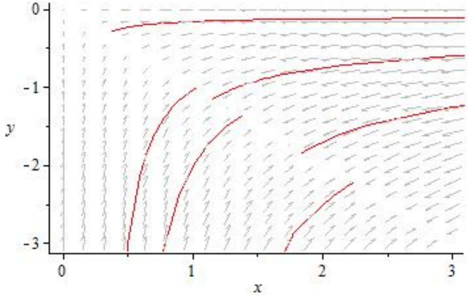

Flujo de corchetes de μ_A con $A=\begin{bmatrix} 0 & x \\ y & 0 \end{bmatrix}$, con $xy < 0$.

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- If $\mathrm{tr}(A) = 0$, we consider $F(A) = \frac{\|[A,A^t]\|^2}{\|A\|^4}$ for $A = A(t)$, and the negative gradient flow of F, $\bar{A}(t)$. Then if A is not nilpotent $\lim_{t\to\infty} \frac{\bar{A}(t)}{\|\bar{A}(t)\|} = \lim_{t\to\infty} \frac{A(t)}{\|A(t)\|}$.

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- If $\operatorname{tr}(A) = 0$, we consider $F(A) = \frac{\|[A,A^t]\|^2}{\|A\|^4}$ for $A = A(t)$, and the negative gradient flow of F, $\bar{A}(t)$. Then if A is not nilpotent $\lim_{t\to\infty} \frac{\bar{A}(t)}{\|\bar{A}(t)\|} = \lim_{t\to\infty} \frac{A(t)}{\|A(t)\|}$.
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The bracket flow in a class of solvmanifolds

Question: Limits of solutions?

Lemma

Let $A \in \mathfrak{gl}_n(\mathbb{R})$ *and consider the bracket flow* $\mu_{A(t)}$ *starting at* μ_A*. Then:*

- *If* $\mathrm{tr}(A) = 0$*, then* $\lim_{t\to\infty} \frac{A(t)}{\|A(t)\|} = A^1_\infty (\rightsquigarrow A(t) \to A_\infty)$,
- *If* $\mathrm{tr}(A) \neq 0$*, then* $A(t) \to 0$.

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- If $\mathrm{tr}(A) \neq 0$, it is easy to see that $A(t) \to 0$ using the spectra of A and $A(t)$.

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- $A(t)$ *is defined* $\forall t \in [0, \infty)$.
- $\operatorname{tr}(S(A(t))^2)$ *is strictly decreasing if A is not skew-symmetric. Moreover,* $\operatorname{tr}(S(A(t))^2) \to 0$ *as* $t \to \infty$.

The bracket flow in a class of solvmanifolds

Corollary

There exists a sequence $(G_{\mu_{A(t_k)}}, g_{\mu_{A(t_k)}})$ *which converges in the pointed (Cheeger - Gromov) sense to a manifold locally isometric to* $(G_{\mu_{A_\infty}}, g_{\mu_{A_\infty}})$, *which is flat.*

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Proposition

If $\mathrm{Spec}(A) \not\subseteq i\mathbb{R}$ *then* $g_{\mu_{A(t)}} \to g_{\mu_{A_\infty}}$ *smoothly on* $\mathbb{R}^n$*.*

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The bracket flow in a class of solvmanifolds

Theorem

Let $A \in \mathfrak{gl}_n(\mathbb{R})$. Consider $\mu_{A(t)}$ the bracket flow starting at μ_A and $g(t)$ the Ricci flow starting at g_{μ_A}. Then:

(i) *$g(t)$ is defined $\forall t \in [0,\infty)$.*

(ii) *$A(t) \to A_\infty$.*

(iii) *There exists a sequence $(G_{\mu_{A(t_k)}}, g_{\mu_{A(t_k)}})$ which converges in the pointed sense to a manifold locally isometric to $(G_{\mu_{A_\infty}}, g_{\mu_{A_\infty}})$, which is flat.*

(iv) *If $\operatorname{Spec}(A) \not\subseteq i\mathbb{R}$, then $g_{\mu_{A(t)}} \to g_{\mu_{A_\infty}}$ smoothly on $\mathbb{R}^n$.*

(v) *If $\operatorname{tr}(A^2) \geq 0$, then $g(t)$ is a type - III solution for some constant C_n that only depends on the dimension of V_n.*

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- As μ_A admits an inner product with $K < 0$, the $\mathrm{Re}(\mathrm{Spec}(A)) > 0$ or $\mathrm{Re}(\mathrm{Spec}(A)) < 0$ ([H]).

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- Then $\mathrm{Re}(\text{Spec}(A_\infty)) > 0$ or $\mathrm{Re}(\text{Spec}(A_\infty)) < 0$, and A_∞ is normal because it is an algebraic soliton. Then $K_{A_\infty} < 0$.
- Finally, it is easy to see that the theorem is true for the norm-normalized bracket flow and then for the bracket flow.

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If $0 < \lambda \leq 2-\sqrt{3}$, then $0 < 1-\lambda$ and so $\mathrm{Re}(\mathrm{Spec}(\mathrm{ad}(e_0))) > 0$. Then $\mu_{\lambda,\alpha}$ admits an inner product with negative curvature ([H]). Hence, as $(\mu_{\lambda,\alpha}, \langle\cdot,\cdot\rangle)$ is an algebraic soliton, if $\mu(t)$ is the bracket flow starting at $\mu_{\lambda,\alpha}$ then $(G_{\mu(t)}, g_{\mu(t)})$ has planes with curvature bigger than or equal to zero.

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For sufficiently large α, $\mu_{\alpha,\lambda}$ has a negative curvature ([H])

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Question: What happens with the Ricci flow when we start with a metric whose sectional curvatures are all negative? Let λ be fixed and we consider $\mu(t)$ starting at $\mu_{\alpha,\lambda}$.

$$\mu(t)(e_0, e_i) = \alpha(t) \begin{pmatrix} \lambda & & \\ & 1-\lambda & \\ & & 1 \end{pmatrix} e_i, \quad \mu(t) = h(t)e_3,$$

with $\alpha(t) = \frac{1}{\sqrt{2c_\lambda t + \alpha^{-2}}}$ and $h(t) = \frac{1}{\sqrt{3t+1}}$. For each t, we have that

$$K(e_1, e_3) = \frac{h^2}{4} - \lambda\alpha^2 = \frac{1}{4(3t+1)} - \frac{\lambda}{2c_\lambda t + \alpha_0^{-2}}$$

$$K(e_1, e_3) \geq 0 \quad \Leftrightarrow \quad (2c_\lambda - 12\lambda)t \geq 4\lambda - \alpha_0^{-2}.$$

For sufficiently large α, $\mu_{\alpha,\lambda}$ has a negative curvature ([H]) but if $0 < \lambda \leq 2 - \sqrt{3}$ then from some t_0, $K(e_1, e_3) \geq 0, \forall t \geq t_0$.

¡Thank you for your attention!

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