Identificación de SISTemas

Subspace State-Space System IDentification

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4SID Methods

- **Properties**
  - They combine tools of **System Theory**, **Numerical Linear Algebra** and **Geometry** (projections).
  - They have their origin in **Realization Theory** as developed in the 60/70s (Ho & Kalman, 1966).
  - They provide reliable state-space models of **multivariable** LTI systems **directly** from input-output data.
  - They don't require iterative optimization procedures → no problems with local minima, convergence and initialization.

- They don't require a particular (canonical) state-space realization → numerical conditioning improves.
- They require a modest computational load in comparison to traditional identification methods like PEM.
- The algorithms can be (they have been) efficiently implemented in software like **Matlab**.
- Main computational tools are QR and SVD.
- All subspace methods compute at some stage the **subspace** spanned by the columns of the extended observability matrix.
- The various algorithms (*e.g.,* N4SID, MOESP, CVA) differ in the way the extended observability matrix is estimated and also in the way it is used to compute the system matrices.

The system model

$$\begin{aligned} x_{k+1} &= Ax_k + Bu_k + Ke_k \\ y_k &= Cx_k + Du_k + e_k \end{aligned}$$

State-space model in innovation form

The identification problem

To estimate the system matrices *(A, B, C, D)* and *K* , and the model order *n*, from an *(N+α-1)*-point data set of input and output measurements

$$\{u_k, y_k\}_{k=1}^{N+\alpha-1}$$

❑ Realization-based 4SID Methods

For a LTI system, a **minimal** state-space realization *(A, B, C, D)* completely defines the input-output properties of the system through

$$\boxed{y_k = \sum_{\ell=0}^{\infty} h_\ell u_{k-\ell}}$$

convolution sum

where the impulse response coefficients h_ℓ are related to the system matrices by

$$h_\ell = \begin{cases} D & , \quad \ell = 0 \\ CA^{\ell-1}B & , \quad \ell > 0 \end{cases}$$

$$H_{ij} = \begin{bmatrix} h_1 & h_2 & \cdots & h_j \\ h_2 & h_3 & \cdots & h_{j+1} \\ \vdots & \vdots & \ddots & \vdots \\ h_i & h_{i+1} & \cdots & h_{i+j-1} \end{bmatrix} \Rightarrow \boxed{H_{ij} = \Gamma_i \mathbf{C}_j}$$

Impulse Response Hankel Matrix

Extended Observability Matrix *(I >n)*

Extended Controlability Matrix *(j >n)*

An estimate of the extended observability matrix can be computed by a **full rank** factorization of the impulse response Hankel matrix. This factorization is provided by the SVD of matrix H_{ij}.

$$H_{ij} = \begin{bmatrix} U_1 & U_2 \end{bmatrix} \begin{bmatrix} \Sigma_1 & 0 \\ 0 & \Sigma_2 \end{bmatrix} \begin{bmatrix} V_1^T \\ V_2^T \end{bmatrix} \approx U_1 \Sigma_1 V_1^T = \underbrace{\left(U_1 \Sigma_1^{1/2} \right)}_{\hat{\Gamma}_i} \underbrace{\left(\Sigma_1^{1/2} V_1^T \right)}_{\hat{\mathbf{C}}_j}$$

rank reduction

In the absence of noise, H_{ij} will be a rank n matrix, and Σ_1 will contain the n non-zero singular values $\rightarrow$ **model order is computed.** In the presence of noise, H_{ij} will have full rank and a rank reduction stage will be required for the model order determination.

Computation of the system matrices

Given estimates $\hat{\Gamma}_i$ and $\hat{\mathrm{C}}_j$ of the extended observability matrix, and the extended controllability matrix, respectively, estimates of the system matrices can be computed as:

- $\hat{C}$: first row block of $\hat{\Gamma}_i$
- $\hat{B}$: first column block of $\hat{\mathrm{C}}_j$
- $\hat{A}$: solving in the least squares sense

$$\overline{\overline{\Gamma}}_i = \underline{\underline{\Gamma}}_i \hat{A}$$

shift-invariance property

- $\hat{D} = h_0$

Problems: it is necessary to measure or to estimate (for example, via correlation analysis) the impulse response of the system $\rightarrow$ **not good**

❑ Direct 4SID Methods

$$\mathbf{Y}_\alpha = \Gamma_\alpha \mathbf{X} + H_\alpha \mathbf{U}_\alpha + \mathbf{N}_\alpha$$

fundamental equation (1)

$$\mathbf{Y}_\alpha = \begin{bmatrix} y_1 & y_2 & \cdots & y_N \\ y_2 & y_3 & \cdots & y_{N+1} \\ \vdots & \vdots & \vdots & \vdots \\ y_\alpha & y_{\alpha+1} & \cdots & y_{N+\alpha-1} \end{bmatrix}$$

Output block Hankel matrix

(In a similar way are defined the Input block Hankel matrix $\mathbf{U}_\alpha$ and the Noise block Hankel matrix $\mathbf{N}_\alpha$.)

$$\Gamma_\alpha = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{\alpha-1} \end{bmatrix}$$

Extended ($\alpha > n$) **Observability Matrix**

$$\mathbf{X} = [x_1, x_2, \cdots, x_N]$$

State Sequence Matrix

$$H_\alpha = \begin{bmatrix} D & 0 & 0 & \cdots & 0 \\ CB & D & 0 & \cdots & 0 \\ CAB & CB & D & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ CA^{\alpha-2}B & CA^{\alpha-3}B & CA^{\alpha-4}B & \cdots & D \end{bmatrix}$$

Lower triangular block Toeplitz matrix of impulse responses (unknown).

To derive equation (1), let us consider the case $\alpha = 3$ and $N = 3$. Then we have

$$\begin{array}{c|c} & \begin{bmatrix} x_1, & x_2, & x_3 \end{bmatrix} \\ \hline \begin{bmatrix} C \\ CA \\ CA^2 \end{bmatrix} & \begin{bmatrix} Cx_1 & Cx_2 & Cx_3 \\ CAx_1 & CAx_2 & CAx_3 \\ CA^2x_1 & CA^2x_2 & CA^2x_3 \end{bmatrix} = \Gamma_\alpha X \end{array}$$

$$\begin{array}{c|c} & \begin{bmatrix} u_1 & u_2 & u_3 \\ u_2 & u_3 & u_4 \\ u_3 & u_4 & u_5 \end{bmatrix} \\ \hline \begin{bmatrix} D & 0 & 0 \\ CB & D & 0 \\ CAB & CB & D \end{bmatrix} & \begin{bmatrix} Du_1 & Du_2 & Du_3 \\ CBu_1 + Du_2 & CBu_2 + Du_3 & CBu_3 + Du_4 \\ CABu_1 + CBu_2 + Du_3 & CABu_2 + CBu_3 + Du_4 & CABu_3 + CBu_4 + Du_5 \end{bmatrix} = H_\alpha U_\alpha \end{array}$$

Then, we can see that

$$\Gamma_\alpha X + H_\alpha U_\alpha =$$
$$= \begin{bmatrix} Cx_1 + Du_1 & Cx_2 + Du_2 & Cx_3 + Du_3 \\ CAx_1 + CBu_1 + Du_2 & CAx_2 + CBu_2 + Du_3 & CAx_3 + CBu_3 + Du_4 \\ CA^2x_1 + CABu_1 + CBu_2 + Du_3 & CA^2x_2 + CABu_2 + CBu_3 + Du_4 & CA^2x_3 + CABu_3 + CBu_4 + Du_5 \end{bmatrix}$$

$$\Gamma_\alpha X + H_\alpha U_\alpha = \begin{bmatrix} y_1 & y_2 & y_3 \\ y_2 & y_3 & y_4 \\ y_3 & y_4 & y_5 \end{bmatrix} = Y_\alpha$$

❑ The main idea of Direct 4SID methods

In the absence of noise ($\boldsymbol{N}_\alpha = 0$), eq. (1) becomes

$$\boxed{\mathbf{Y}_\alpha = \Gamma_\alpha \mathbf{X} + H_\alpha \mathbf{U}_\alpha} \qquad (2)$$

and the part of the output which does not emanate from the state can be removed by multiplying (from the right) both sides of eq. (2) by the **orthogonal projection onto the null space of $\mathbf{U}_\alpha$**, *i.e.* by

$$\Pi^{\perp}_{\mathbf{U}_\alpha^T} \stackrel{\Delta}{=} I - \mathbf{U}_\alpha^T\left(\mathbf{U}_\alpha \mathbf{U}_\alpha^T\right)^{-1}\mathbf{U}_\alpha \stackrel{\Delta}{=} \mathbf{U}_\alpha^{\perp}$$

orthogonal projection

such that $\mathbf{U}_\alpha \mathbf{U}_\alpha^{\perp} = 0$

This yields

$$\boxed{\mathbf{Y}_\alpha \mathbf{U}_\alpha^{\perp} = \Gamma_\alpha \mathbf{X} \mathbf{U}_\alpha^{\perp}} \qquad (3)$$

Note that the matrix on the left depends exclusively on the input-output data. Then, a **full rank factorization** of this matrix will provide an estimate $\hat{\Gamma}_\alpha$ of the extended observability matrix. Estimates of the corresponding system matrices can be obtained by resorting to the **shift invariance property** of the extended observability matrix, and by solving a system of linear equations in the least squares sense.

The factorization is provided by the SVD of the matrix on the left side

$$\mathbf{Y}_\alpha \mathbf{U}_\alpha^{\perp} = \begin{bmatrix} U_1 & U_2 \end{bmatrix} \begin{bmatrix} \Sigma_1 & 0 \\ 0 & \Sigma_2 \end{bmatrix} \begin{bmatrix} V_1^T \\ V_2^T \end{bmatrix} \approx U_1 \Sigma_1 V_1^T = \underbrace{\left(U_1 \Sigma_1^{1/2}\right)}_{\hat{\Gamma}_\alpha}\left(\Sigma_1^{1/2} V_1^T\right) \qquad (4)$$

rank reduction
(model order estimation)

(In the absence of noise $\Sigma_2 = 0$)

Computation of the system matrices

Given an estimate $\hat{\Gamma}_\alpha$ of the extended observability matrix, estimates of the system matrices can be computed as:

- $\hat{C}$: first row block of $\hat{\Gamma}_\alpha$
- $\hat{A}$: solving in the least squares sense

$$\boxed{\overline{\overline{\Gamma}}_\alpha = \underline{\underline{\Gamma_\alpha}} \hat{A}}$$

shift-invariance property

- $\hat{B}$ and $\hat{D}$: solving a system of linear equations

Pre-multiply (2) by U_2^T and post-multiply it by

$$U_\alpha^\dagger = U_\alpha^T \left(U_\alpha U_\alpha^T \right)^{-1}$$

Equation (2) becomes

$$U_2^T Y_\alpha U_\alpha^\dagger = \underbrace{U_2^T \underbrace{\hat{\Gamma}_\alpha}_{=U_1}}_{=0} X U_\alpha^\dagger + U_2^T H_\alpha \underbrace{U_\alpha U_\alpha^\dagger}_{=I}$$

$$\boxed{U_2^T Y_\alpha U_\alpha^\dagger = U_2^T H_\alpha} \longrightarrow$$

Linear equations on B and D

❑ Presence of noise

In the presence of noise

$$\mathbf{Y}_\alpha = \Gamma_\alpha \mathbf{X} + H_\alpha \mathbf{U}_\alpha + \mathbf{N}_\alpha$$

and

$$\mathbf{Y}_\alpha \mathbf{U}_\alpha^\perp = \Gamma_\alpha \mathbf{X} \mathbf{U}_\alpha^\perp + \underbrace{\mathbf{N}_\alpha \mathbf{U}_\alpha^\perp}_{\text{noise term needs to be removed}}$$

The noise term can be removed by **correlating it away** with a suitable matrix (**Instrumental Variable**). This can be interpreted as an **oblique projection.**

For this to work, the noise variables in the output must be uncorrelated with the Instrumental Variables (IVs).

Let us partition the input and output Hankel matrices into the past and future parts (somewhat arbitrary names !!)

$$\mathbf{Y}_\alpha = \left[\begin{array}{cccc} y_1 & y_2 & \cdots & y_N \\ y_2 & y_3 & \cdots & y_{N+1} \\ \vdots & \vdots & \vdots & \vdots \\ y_\beta & y_{\beta+1} & & y_{N+\beta-1} \\ \hdashline y_{\beta+1} & y_{\beta+2} & & y_{N+\beta} \\ \vdots & \vdots & \cdots & \vdots \\ \vdots & \vdots & \cdots & \vdots \\ y_\alpha & y_{\alpha+1} & \cdots & y_{N+\alpha-1} \end{array}\right] = \begin{bmatrix} \mathbf{Y}_P \\ \mathbf{Y}_F \end{bmatrix}$$

The model for the future outputs becomes

$$\mathbf{Y}_F = \Gamma_F \mathbf{X}_F + H_F \mathbf{U}_F + \mathbf{N}_F$$

The first step is to eliminate the input by post-multiplying by $\mathbf{U}_F^{\perp}$ (orthogonal projection)

$$\mathbf{Y}_F \mathbf{U}_F^{\perp} = \Gamma_F \mathbf{X}_F \mathbf{U}_F^{\perp} + \mathbf{N}_F \mathbf{U}_F^{\perp}$$

Noting that the noise is not correlated to the inputs, and the future inputs are not correlated to the past outputs, good candidates for the IVs are the past inputs and outputs.

Under the assumption of ergodicity, it can be proved that

$$\lim_{N\to\infty} \frac{1}{N} \mathbf{N}_F \mathbf{U}_F^{\perp} \mathbf{U}_P^T = 0$$
$$\lim_{N\to\infty} \frac{1}{N} \mathbf{N}_F \mathbf{U}_F^{\perp} \mathbf{Y}_P^T = 0$$

which imply that the IV matrix

$$\mathbf{P} = \begin{bmatrix} \mathbf{U}_P \\ \mathbf{Y}_P \end{bmatrix}$$

can be used to asymptotically decorrelate the noise from $\mathbf{Y}_F \mathbf{U}_F^{\perp}$

$$\lim_{N\to\infty}\frac{1}{N}\mathbf{Y}_F\mathbf{U}_F^{\perp}\mathbf{P}^T = \lim_{N\to\infty}\frac{1}{N}\Gamma_F\mathbf{X}_F\mathbf{U}_F^{\perp}\mathbf{P}^T + \underbrace{\lim_{N\to\infty}\frac{1}{N}\mathbf{N}_F\mathbf{U}_F^{\perp}\mathbf{P}^T}_{=0}$$

$$\lim_{N\to\infty}\frac{1}{N}\mathbf{Y}_F\mathbf{U}_F^{\perp}\mathbf{P}^T = \lim_{N\to\infty}\frac{1}{N}\Gamma_F\mathbf{X}_F\mathbf{U}_F^{\perp}\mathbf{P}^T \qquad (5)$$

The signal subspace can then be estimated consistently from the *n* principal left singular vectors of matrix

$$\frac{1}{N}\mathbf{Y}_F\mathbf{U}_F^{\perp}\mathbf{P}^T$$

❑ Weighting Matrices

Row and column weighting matrices can be introduced in (5) before performing the SVD of the matrix in the left hand side. Any choice of positive-definite weighting matrices W_r and W_c will result in consistent estimates of the extended observability matrix.

$$\frac{1}{N}W_r\mathbf{Y}_F\mathbf{U}_F^{\perp}\mathbf{P}^T W_c = \begin{bmatrix} U_1 & U_2 \end{bmatrix}\begin{bmatrix} \Sigma_1 & 0 \\ 0 & \Sigma_2 \end{bmatrix}\begin{bmatrix} V_1^T \\ V_2^T \end{bmatrix} \approx U_1\Sigma_1V_1^T = \underbrace{\left(U_1\Sigma_1^{1/2}\right)}_{\hat{\Gamma}_F}\left(\Sigma_1^{1/2}V_1^T\right)$$

change of coordinates in state-space

Existing algorithms employ different choices for matrices W_r and W_c,

- **MOESP** (Verhaegen, 1994):

$$W_r = I, \quad W_c = \left(\frac{1}{N}\mathbf{P}\mathbf{U}_F^{\perp}\mathbf{P}^T\right)^{-1/2}$$

- **CVA** (Larimore, 1990):

$$W_r = \left(\frac{1}{N}\mathbf{Y}_F\mathbf{U}_F^{\perp}\mathbf{Y}_F{}^T\right)^{-1/2}, \quad W_c = \left(\frac{1}{N}\mathbf{P}\mathbf{U}_F^{\perp}\mathbf{P}^T\right)^{-1/2}$$

- **N4SID** (Van Overschee and de Moor, 1994):

$$W_r = I, \quad W_c = \left(\frac{1}{N}\mathbf{P}\mathbf{U}_F^{\perp}\mathbf{P}^T\right)^{-1}\left(\frac{1}{N}\mathbf{P}\mathbf{P}^T\right)^{1/2}$$