Robust Control Design with Guaranteed State Ultimate Bound

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***Abstract*— We present a new control design method which guarantees any prespecified *componentwise* ultimate bound on the state of multiple-input systems with matched perturbations. The method is based on eigenstructure assignment by state feedback and utilises a componentwise bound computation procedure recently proposed by the authors. This procedure exploits both the system and perturbation structures by performing componentwise analysis, thus avoiding the need for bounds on the norm of the perturbation. We present examples which illustrate the simplicity and generality of the method.**

I. Introduction

Control systems are invariably subject to the effect of perturbations, which may arise from unknown disturbance signals, model uncertainty, unknown time delays, component aging, etc. Typically, the exact value of a perturbation variable is unknown but assumed to be bounded. In the presence of bounded perturbations that do not vanish as the state approaches an equilibrium point, asymptotic stability is in general not possible. However, under certain conditions, the *ultimate boundedness* of the system state trajectories can be guaranteed [1], [2]. A guaranteed ultimate bound on the system trajectories can be regarded as a measure of performance in steady state. Moreover, a small (in some sense) ultimate bound can be associated with good perturbation attenuation, which is a desirable feature of any controller.

When perturbations span the same subspace spanned by the control input, commonly referred to as *matched* perturbations, then any prespecified ultimate bound can be guaranteed by designing a state feedback controller appropriately [3]. Several control design methods that can achieve an arbitrarily small ultimate bound for linear uncertain systems have been reported in the robust control literature (see, for example, [4], [5], [6], [7]). These methods are based on Lyapunov analysis, a typical tool for the computation of ultimate bounds [2].

The present paper proposes a new controller design method whose aim is to ensure any prespecified *componentwise* ultimate bound on the system state trajectories. The proposed method can be applied to multiple-input perturbed linear continuous-time systems and its development involves results on componentwise ultimate bound computation that were presented in [8] and extended in [9] and [10]. The mentioned results directly derive componentwise ultimate bounds, exploiting the system geometry as well as the perturbation structure requiring neither the computation of a Lyapunov function for the system nor bounding the norm of the perturbation vector. In many cases, as is reported in [8], [9], [10], these facts permit to obtain estimations of the ultimate bounds that are less conservative than those obtained using Lyapunov based approaches.

In this context, the first contribution of the current paper is to further extend the componentwise approach of [8], [9], [10] to linear systems with perturbations bounded by a delayed function of the state. In addition, by specialising to the case when the perturbation bound is an affine function of the state, we are able to derive ultimate bounds that are valid globally, as opposed to the results in [9] and [10], where the corresponding ultimate bounds for perturbations bounded by functions of the state are guaranteed only when the initial state lies in a bounded set.

The main contribution of the paper consists in exploiting the structural dependency of the derived componentwise ultimate bound expressions on the system eigenstructure in order to develop the aforementioned control design method for systems with matched perturbations, using techniques of eigenvalue and eigenvector assignment by state feedback [11]. We show that a state feedback gain can be designed such that the ultimate bound expression decreases to zero as a "scaling" parameter, associated with the magnitude of the closed-loop eigenvalues, increases. The proposed design procedure is systematic in the sense that once a desired (stable but otherwise arbitrary) normalised configuration is chosen for the closed-loop eigenvalues, a suitably high value of the scaling parameter can be selected to achieve the desired ultimate bound.

The remainder of the paper proceeds as follows. Below we introduce notation and definitions used throughout the paper. In Section II we present the problem formulation. Section III derives an ultimate bound expression that depends on the system eigenstructure. In Section IV we present the proposed controller design method. Section V presents examples of application and Section VI concludes the paper.

A. Notation and preliminaries

In the sequel, $|M|$, $\mathbb{R}\mathrm{e}(M)$ and $\mathbb{I}\mathrm{m}(M)$ denote the elementwise magnitude, real part and imaginary part, respectively, of a (possibly complex) matrix or vector M. Also, $x \le y$ ($x < y$) denotes the set of componentwise (strict) inequalities between the components of the real vectors x and y, and similarly for $x \ge y$ ($x > y$). According to these definitions, it is easy to show that (see, for example [12, §8])

$$|x+y| \le |x| + |y|, \quad |M\,x| \le |M| \cdot |x|, \tag{1}$$

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whenever $x, y \in \mathbb{C}^n$ and $M \in \mathbb{C}^{m\times n}$. $\mathbb{Z}_{+,0}$ denotes the nonnegative integers. Similarly, $\mathbb{R}_+$ and $\mathbb{R}_{+,0}$ denote the positive and nonnegative real numbers, respectively. Consequently, if $x \in \mathbb{R}^n$ then $x \in \mathbb{R}^n_+ \Leftrightarrow x > 0$ and $x \in \mathbb{R}^n_{+,0} \Leftrightarrow x \geq 0$. For $c \in \mathbb{C}$, $\overline{c}$ denotes its complex conjugate.

II. Problem formulation

Consider the multiple-input linear perturbed system

$$\dot{x}(t) = Ax(t) + Bu(t) + Hw(t), \tag{2}$$

where $x(t) \in \mathbb{R}^n$ is the system state, $u(t) \in \mathbb{R}^m$ is the control input, $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $H \in \mathbb{R}^{n\times k}$ are constant matrices, (A, B) is a controllable pair, B has full column rank, and the perturbation variable $w(t) \in \mathbb{R}^k$ satisfies the componentwise bound

$$|w(t)| \leq F\theta(t) + \bar{w} \quad \text{for all } t \geq 0, \tag{3}$$

with $F \in \mathbb{R}^{k\times n}_{+,0}$, $\bar{w} \in \mathbb{R}^k_{+,0}$, and

$$\theta(t) \triangleq \max_{t-\bar{\tau}\leq\tau\leq t} |x(\tau)|, \tag{4}$$

where the maximum is taken componentwise and $\bar{\tau} \geq 0$.

Given an arbitrary positive vector $b^* \in \mathbb{R}^n_+$, our goal is to design a linear state-feedback control $u(t) = Kx(t)$, $K \in \mathbb{R}^{m\times n}$, such that the solutions to $\dot{x}(t) = (A + BK)x(t) + Hw(t)$, where $w(t)$ is componentwise bounded as in (3)–(4), are *ultimately bounded* with componentwise *ultimate bound* b^*, that is, they satisfy

$$\limsup_{t\to\infty} |x(t)| \leq b^*. \tag{5}$$

In the case of constant perturbation bounds [$F = 0$ in (3)], and provided the perturbations are matched, that is, provided $H = BG$ for some $G \in \mathbb{R}^{m\times k}$, it is well-known that the above goal is achievable [3]. However, additional assumptions must be made when the perturbation bound depends on either the current or a delayed version of the system state, as is the case when $F \neq 0$ in (3). In the current paper, we will provide a sufficient condition for achieving any desired componentwise ultimate bound and a method for computing the corresponding feedback matrix K.

Remark 2.1: Notice that the setting (2)–(4) can accommodate various types of uncertainty:

- Uncertainty in the system evolution matrix, where
$$\dot{x}(t) = (A + \Delta A(t))x(t) + Bu(t), \quad \text{and}$$
$$|\Delta A(t)| \leq \overline{\Delta A} \text{ for all } t \geq 0;$$
in this case, we can take $H = \mathrm{I}$ in (2), $F = \overline{\Delta A}$ and $\bar{w} = 0$ in (3), and $\bar{\tau} = 0$ in (4).
- Uncertainty in the system input matrix [assuming that $u(t) = Kx(t)$ in (2)], where
$$\dot{x}(t) = Ax(t) + (B + \Delta B(t))Kx(t), \quad \text{and}$$
$$|\Delta B(t)| \leq \overline{\Delta B} \text{ for all } t \geq 0;$$
in this case, we can take $H = \mathrm{I}$ in (2), $F = \overline{\Delta B}|K|$ and $\bar{w} = 0$ in (3), and $\bar{\tau} = 0$ in (4).
- Uncertain time delay, where
$$w(t) = A_d x(t-\tau), \quad \text{and} \quad 0 \leq \tau \leq \tau_{\max};$$
in this case, we can take $F = |A_d|$ and $\bar{w} = 0$ in (3), and $\bar{\tau} = \tau_{\max}$ in (4).
- Bounded disturbances, where
$$w(t) = q(t), \quad \text{and} \quad |q(t)| \leq \bar{q} \text{ for all } t \geq 0;$$
in this case, we can take $F = 0$ and $\bar{w} = \bar{q}$ in (3).

Note that we can also accommodate for combinations of the above cases. ◦

III. Componentwise ultimate bounds under perturbations

In this section we present a theorem that extends results of [8], [9] and [10]. The latter works proposed a new framework to obtain closed-form ultimate bound formulae based on the use of componentwise perturbation bounds and componentwise analysis of the system in modal coordinates. This componentwise framework allows the perturbation term to be bounded by constants or by a non-delayed function of the state. The following theorem extends the previous works by providing ultimate bounds on the components of the system state when the perturbations are bounded in the form (3)–(4).

Theorem 3.1: Consider the system

$$\dot{x}(t) = \bar{A}x(t) + Hw(t), \tag{6}$$

where $x(t) \in \mathbb{R}^n$, $w(t) \in \mathbb{R}^k$, $\bar{A} \in \mathbb{R}^{n\times n}$ and $H \in \mathbb{R}^{n\times k}$. Let $\bar{A}$ be a Hurwitz matrix with Jordan canonical form $\Lambda = V^{-1}\bar{A}V$. Suppose that (3) and (4) hold with $F \in \mathbb{R}^{k\times n}_{+,0}$, $\bar{w} \in \mathbb{R}^k_{+,0}$ and $\bar{\tau} \in \mathbb{R}_{+,0}$. Define the matrix

$$R \triangleq |V| \left| [\mathbb{R}\mathrm{e}(\Lambda)]^{-1} V^{-1} H \right|, \tag{7}$$

suppose that $\boldsymbol{\rho}(RF) < 1$ (where $\boldsymbol{\rho}(\cdot)$ denotes the spectral radius), and let

$$b \triangleq (\mathrm{I} - RF)^{-1} R\bar{w}. \tag{8}$$

Then, $\limsup_{t\to\infty} |x(t)| \leq b$.

Proof: For any $\gamma \in \mathbb{R}^n_{+,0}$ and $\ell \in \mathbb{Z}_{+,0}$ consider the maps $T : \mathbb{R}^n_{+,0} \to \mathbb{R}^n_{+,0}$ and $T^\ell_\gamma : \mathbb{R}^n_{+,0} \to \mathbb{R}^n_{+,0}$ defined by

$$\begin{aligned} &T(x) \triangleq R(Fx + \bar{w}), \\ &T^0_\gamma(x) \triangleq T(x) + |V|\gamma, \qquad T^{\ell+1}_\gamma(x) \triangleq T^0_\gamma(T^\ell_\gamma(x)). \end{aligned} \tag{9}$$

The proof requires the following claims, whose proofs are given in the Appendix.

Claim 1: There exists $\bar{x} \in \mathbb{R}^n_{+,0}$, which may depend on $\theta(0)$ [see (4)], such that $|x(t)| \leq \bar{x}$ for all $t \geq -\bar{\tau}$.

Claim 2: Suppose that $|x(t)| \leq \bar{x}$ for all $t \geq -\bar{\tau}$. Then, for any $\gamma \in \mathbb{R}^n_+$ and $\ell \in \mathbb{Z}_{+,0}$, there exists $\bar{t} = \bar{t}(\ell, \gamma) \geq 0$ such that $|x(t)| \leq T^{\ell+1}_\gamma(\bar{x})$, for all $t \geq \bar{t}$.

Claim 3: Suppose that $|x(t)| \leq \bar{x}$ for all $t \geq -\bar{\tau}$. Then, for any $\epsilon \in \mathbb{R}^n_+$, there exist $\ell = \ell(\epsilon) \in \mathbb{Z}_{+,0}$ and $\gamma = \gamma(\epsilon) \in \mathbb{R}^n_+$ such that $T^{\ell+1}_\gamma(\bar{x}) < b + \epsilon$.

From Claims 1 and 3, it straightforwardly follows that for any $\epsilon \in \mathbb{R}^n_+$, there exist $\ell = \ell(\epsilon) \in \mathbb{Z}_{+,0}$ and $\gamma = \gamma(\epsilon) \in \mathbb{R}^n_+$

such that $T_\gamma^{\ell+1}(\bar{x}) < b + \epsilon$. For these values of ℓ and γ, we have, from Claim 2, that there exists $\bar{t} = \bar{t}(\ell, \gamma) \geq 0$ such that $|x(t)| \leq T_\gamma^{\ell+1}(\bar{x}) < b + \epsilon$, for all $t \geq \bar{t}$. This implies that $\limsup_{t\to\infty} |x(t)| \leq b$, concluding the proof. ■

Remark 3.2: Theorem 3.1 can be directly employed to compute an ultimate bound for system (2)–(4) under a stabilising state feedback $u(t) = Kx(t)$. ∘

IV. CONTROL DESIGN WITH GUARANTEED COMPONENTWISE ULTIMATE BOUND

In the following theorem, which constitutes the main result of the paper, we combine the use of Theorem 3.1 with techniques of eigenstructure assignment by state feedback [11] to derive a new control design procedure that guarantees any desired componentwise ultimate bound for systems with matched perturbations. Specifically, we consider system (2)–(4), where $H = BG$ for some $G \in \mathbb{R}^{m\times k}$ and provide a feedback matrix $K \in \mathbb{R}^{m\times n}$ so that application of the control $u(t) = Kx(t)$ achieves any desired componentwise ultimate bound $b^* \in \mathbb{R}^n_+$.

Theorem 4.1: Consider the perturbed multiple-input system (2)–(4), where (A, B) is a controllable pair and $H = BG$ for some $G \in \mathbb{R}^{m\times k}$. Let $c_k \in \mathbb{C}$, $k = 1, \ldots, n$, satisfy $c_i \neq c_j$ for $i \neq j$, $\mathbb{R}\mathrm{e}(c_k) < 0$, and if $c_i \notin \mathbb{R}$, then either $c_{i-1} = \overline{c_i}$ or $c_{i+1} = \overline{c_i}$. Take $\mu > 0$ and let $\lambda_k \triangleq \mu c_k$ for $k = 1, \ldots, n$ and $\Lambda \triangleq \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$. Consider the matrix

$$V \triangleq \begin{bmatrix} V_1 \\ \vdots \\ V_m \end{bmatrix}, \quad \text{where, for } i = 1, \ldots, m,$$

$$V_i \triangleq \begin{bmatrix} e_{i,1}\lambda_1^{-(d_i-1)} & e_{i,2}\lambda_2^{-(d_i-1)} & \ldots & e_{i,n}\lambda_n^{-(d_i-1)} \\ \vdots & \vdots & \ddots & \vdots \\ e_{i,1}\lambda_1^{-1} & e_{i,2}\lambda_2^{-1} & \ldots & e_{i,n}\lambda_n^{-1} \\ e_{i,1} & e_{i,2} & \ldots & e_{i,n} \end{bmatrix}. \tag{10}$$

In (10), the integers d_i, $i = 1, \ldots, m$, are the controllability indices of the system[1] and $e_{i,j} \in \mathbb{C}$, $i = 1, \ldots, m$, $j = 1, \ldots, n$ are such that $e_{i,j+1} = \overline{e_{i,j}}$ whenever $c_{j+1} = \overline{c_j}$ and the matrix V has linearly independent columns. Construct matrices $\tilde{V}$ and $\tilde{\Lambda}$ from V and Λ in the following way. For every pair of columns v_i and v_{i+1} of V such that $v_{i+1} = \overline{v_i}$, we set the corresponding columns of $\tilde{V}$ as $\mathbb{R}\mathrm{e}(v_i)$ and $\mathbb{I}\mathrm{m}(v_i)$. Similarly, for every submatrix $\mathrm{diag}(\lambda_i, \overline{\lambda_i})$ of Λ we set the corresponding submatrix of $\tilde{\Lambda}$ as $\begin{bmatrix} \mathbb{R}\mathrm{e}(\lambda_i) & \mathbb{I}\mathrm{m}(\lambda_i) \\ -\mathbb{I}\mathrm{m}(\lambda_i) & \mathbb{R}\mathrm{e}(\lambda_i) \end{bmatrix}$. Let $x = Ux_c$ be the state variable transformation that brings system (2) into the controller canonical form[2] with matrices $A_c \triangleq U^{-1}AU$ and $B_c \triangleq U^{-1}B$, and define

$$R_\mu \triangleq |V| \left| [\mathbb{R}\mathrm{e}(\Lambda)]^{-1}(UV)^{-1}H \right|. \tag{11}$$

Then:

a) The linear feedback

$$u = Kx, \quad K = (B_c^T B_c)^{-1} B_c^T (\tilde{V}\tilde{\Lambda}\tilde{V}^{-1} - A_c)U^{-1}, \tag{12}$$

is such that the eigenvalue and eigenvector matrices of $(A + BK)$ are Λ and UV, respectively.

b) There exists $\bar{\mu} \in \mathbb{R}_+$ such that $\boldsymbol{\rho}(R_\mu F|U|) < 1$ for all $\mu > \bar{\mu}$.

c) If $\mu > \bar{\mu}$, then the state of the closed-loop system (2)–(4) under the state feedback (12) satisfies $\limsup_{t\to\infty} |x(t)| \leq b_\mu$, where

$$b_\mu \triangleq |U|(\mathrm{I} - R_\mu F|U|)^{-1} R_\mu \bar{w}. \tag{13}$$

d) For $\mu > \bar{\mu}$, the ultimate bound b_μ in (13) is a componentwise nonincreasing function of μ and satisfies $\lim_{\mu\to\infty} b_\mu = 0$.

Proof: See [15]. ■

Theorem 4.1 requires the selection of a "normalised" configuration c_k, $k = 1, \ldots, n$, for the closed-loop eigenvalues. Once this selection has been made, the choice of the complex numbers $e_{i,j}$, $i = 1, \ldots, m$, $j = 1, \ldots, n$ in (10) represents an additional degree of freedom that could be exploited to yield different expressions for the ultimate bound formula (13) as functions of μ, all with the properties stated in Part d) of the theorem (see the example of Section V-B). In the following algorithm we present a design procedure, based on Theorem 4.1, that suggests a particular choice for the complex numbers $e_{i,j}$, $i = 1, \ldots, m$, $j = 1, \ldots, n$.

Algorithm 4.2: Given a desired componentwise ultimate bound $b^\star$:

1) Find the matrix U that brings system (2) to the controller canonical form and compute the transformed matrices $A_c \triangleq U^{-1}AU$ and $B_c \triangleq U^{-1}B$.
2) Choose a normalised configuration c_k, $k = 1, \ldots, n$, of n distinct stable eigenvalues, where complex eigenvalues appear in complex conjugate pairs (that is, $c_i \neq c_j$ for $i \neq j$, $\mathbb{R}\mathrm{e}(c_k) < 0$, and if $c_i \notin \mathbb{R}$, then either $c_{i-1} = \overline{c_i}$ or $c_{i+1} = \overline{c_i}$).
3) Find the controllability indices d_i, $i = 1, \ldots, m$, of system (2) and define $\sigma_i \triangleq \sum_{j=1}^{i} d_j$ for $i = 1, \ldots, m$. Set $e_{i,j} = c_j^{\sigma_i - n}$, $i = 1, \ldots, m$, $j = 1, \ldots, n$.[3]
4) Select $\mu \in \mathbb{R}_+$.
5) Compute V in (10) using the values of $e_{i,j}$, $i = 1, \ldots, m$, $j = 1, \ldots, n$, chosen in step 3 and $\lambda_k = \mu c_k$ for $k = 1, \ldots, n$. Set $\Lambda \triangleq \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$.
6) Compute R_μ in (11) and check the condition $\boldsymbol{\rho}(R_\mu F|U|) < 1$. If this condition is not satisfied, increase μ and go to step 5.
7) Compute b_μ in (13) and check the condition $b_\mu \leq b^\star$. If not satisfied, increase μ and go to step 5.
8) Form matrices $\tilde{V}$ and $\tilde{\Lambda}$ from V and Λ as described in the statement of Theorem 4.1.
9) Compute K using the formula (12).

[1] The controllability indices satisfy $d_i \geq 1$, for $i = 1, \ldots, m$ and $\sum_{i=1}^m d_i = n$. See [13] for an algorithm to compute these indices.

[2] See, for example, [14] for an algorithm to compute the multivariable controller canonical form.

[3] It is easy to check that this choice makes the columns of V in (10) linearly independent for all values of $\mu \in \mathbb{R}_+$.

In the following section we illustrate the use of Theorem 4.1 and Algorithm 4.2 by means of two examples taken from the literature.

V. Examples

A. Application to robust tracking

This example was presented in [7], where a Lyapunov based method for robust tracking in uncertain linear time-delayed systems was developed. The system has the form

$$\begin{aligned}\dot{x}(t) &= [A + \Delta A(r(t))]x(t) + A_d(s(t))x(t-\tau) + Bu(t),\\ y(t) &= Cx(t),\\ x(t) &= \Phi(t), \quad t \in [-\tau, 0],\end{aligned}$$

where A, B, C are known matrices, $\Delta A(r(t))$ and $A_d(s(t))$ are matrices depending on bounded time-varying uncertain parameters $r(t)$ and $s(t)$, $\Phi(t)$ is the initial condition and τ is a time delay. The goal is that $y(t)$ follow the output of a reference model

$$\begin{aligned}\dot{x}_m(t) &= A_m x_m(t),\\ y_m(t) &= C_m x_m(t),\end{aligned} \tag{14}$$

within a predefined ultimately bounded error. The parameters for the example are

$$\begin{aligned}&A = \begin{bmatrix} 0 & 1 & 0\\ 0 & 1 & 2\\ -1 & -2 & 0\end{bmatrix},\ B = \begin{bmatrix} 0\\ 0.1\\ 1\end{bmatrix},\ C = \begin{bmatrix} 1\\ 1\\ 0\end{bmatrix}^T,\\ &A_m = \begin{bmatrix} 0 & 1\\ -1 & 0\end{bmatrix},\ C_m = \begin{bmatrix} 1\\ 1\end{bmatrix}^T,\end{aligned} \tag{15}$$

with matched uncertainties

$$\Delta A(r(t)) = BD(r(t)),\ D(r(t)) = [r_1(t)\ 0\ r_2(t)], \tag{16}$$

$$|r_1(t)| \leq 0.15, \quad |r_2(t)| \leq 0.2, \tag{17}$$

$$A_d(s(t)) = BE(s(t)),\ \ E(s(t)) = [0\ s_1(t)\ s_2(t)], \tag{18}$$

$$|s_1(t)| \leq 0.1, \quad |s_2(t)| \leq 0.15. \tag{19}$$

The time delay is $\tau = 0.1$ and the reference model initial condition is $x_m(0) = [1 \quad 0]^T$. The goal is to assign an ultimate bound $b^* = 0.15$ to the tracking error $e(t) \triangleq y(t) - y_m(t)$, so that $\limsup_{t\to\infty} |e(t)| \leq 0.15$.

To solve the tracking problem, the method of [7] requires the existence of matrices $\mathcal{G}$ and $\mathcal{H}$ satisfying

$$\begin{bmatrix} A & B\\ C & 0\end{bmatrix}\begin{bmatrix}\mathcal{G}\\ \mathcal{H}\end{bmatrix} = \begin{bmatrix}\mathcal{G}A_m\\ C_m\end{bmatrix}.$$

Solving the above equation using the data in (15) yields

$$\mathcal{G} = \begin{bmatrix} 1 & 0\\ 0 & 1\\ -0.5786 & -0.5711\end{bmatrix}, \quad \mathcal{H} = \begin{bmatrix} 1.5711 & 1.4214\end{bmatrix}.$$

Defining $z(t) \triangleq x(t) - \mathcal{G}\, x_m(t)$, the tracking error dynamics can be described as

$$\begin{aligned}\dot{z}(t) &= Az(t) + Bu(t) - B\mathcal{H}x_m(t) + \Delta A(r(t))x(t)\\ &\qquad + A_d(s(t))x(t-\tau),\\ e(t) &= C_m z(t).\end{aligned} \tag{20}$$

Using (16)–(19), the perturbation terms in (20) can be rewritten as

$$\Delta A(r(t))x(t) + A_d(s(t))x(t-\tau) = Bw(t), \tag{21}$$

where

$$\begin{aligned}w(t) \triangleq\ & D(r(t))z(t) + E(s(t))z(t-\tau) + D(r(t))\mathcal{G}x_m(t)\\ &+ E(s(t))\mathcal{G}x_m(t-\tau)\end{aligned} \tag{22}$$

can be bounded as in (3)–(4) with

$$F \triangleq \sup_t(D(r(t))) + \sup_t(E(s(t))) = \begin{bmatrix} 0.15 & 0.1 & 0.35\end{bmatrix}$$

and

$$\bar{w} \triangleq [\sup_t(D(r(t))\mathcal{G}) + \sup_t(E(s(t))\mathcal{G})] \sup_t(x_m(t)) = 0.2496.$$

In computing the value for $\bar{w}$ above we have used $|x_m(t)| \leq [1 \ \ 1]^T$ for all t, which follows from (14), (15) and the initial condition $x_m(0) = [1 \ \ 0]^T$.

We use the structure of the control law proposed in [7], namely

$$u(t) = Kx(t) + (\mathcal{H} - K\mathcal{G})x_m(t) = Kz(t) + \mathcal{H}x_m(t),$$

where the gain K will be computed here via the procedure proposed in Section IV. Substituting the above control law in (20)–(22) results in the closed-loop equation

$$\begin{aligned}\dot{z}(t) &= Az(t) + Bu(t) - B\mathcal{H}x_m(t) + Bw(t)\\ &= (A + BK)z(t) + Bw(t).\end{aligned}$$

Following Algorithm 4.2, we calculate matrix $U = \begin{bmatrix} 2 & 0.1 & 0\\ 0 & 2 & 0.1\\ -0.1 & -1.2 & 1\end{bmatrix}$, and we choose the complex numbers c_i with a Butterworth configuration $c_1 = -1$, $c_{2,3} = -1/2 \pm j\sqrt{3}/2$.

Being a controllable system with $m = 1$, step 3 of Algorithm 4.2 gives the coefficients $e_{1,j} = 1$ for $j = 1, \ldots, n$. Since the assigned eigenvalues are all different, this choice ensures that matrix V in (10) has linearly independent columns.

Selecting $\mu > 2.31$, the closed-loop system satisfies the stability condition $\boldsymbol{\rho}(R_\mu F|U|) < 1$, where R_μ is defined in (11), and when $\mu = 4.978$ we obtain an ultimate bound $b_\mu = \begin{bmatrix} 0.0251 & 0.1249 & 0.3098\end{bmatrix}^T$ for $z(t)$ [see (13)]. Therefore, an ultimate bound for the tracking error $e(t) = Cz(t)$ is $|C|b_\mu \approx 0.15$, and the tracking goal is ensured. The controller gain for this last configuration, given by (12), is $K = \begin{bmatrix} -61.1024 & -24.8103 & -8.4750\end{bmatrix}$. The direct application of Theorem 3.1 on the resulting closed-loop system gives, in this case, a less conservative estimation of the ultimate bound. For the controller gain K calculated before, we have $\tilde{b} = \begin{bmatrix} 0.0164 & 0.0818 & 0.2782\end{bmatrix}^T$ which ensures that the tracking error is ultimately bounded by 0.0982.

B. Application to axisymmetric spacecraft rotational motion

We consider an example presented in [1], which analyses the rotational motion of a rigid spacecraft subject to bounded disturbances. Under the assumption that the spacecraft is axisymmetric about the third body-fixed axis and there are no torques about the symmetry axis, the system can be described by (2), where $x(t) = [\omega_1(t) \quad \omega_2(t)]^T \in \mathbb{R}^2$ are the components of the inertial angular velocity of the spacecraft with respect to the first two axes and $u(t) \in \mathbb{R}^2$, $w(t) \in \mathbb{R}^2$ are, respectively, control input and unknown bounded disturbance torques about the first two axes. The matrices in (2) take the form

$$A = \begin{bmatrix} 0 & \dfrac{(J_{11}-J_{33})*\omega_3}{J_{11}} \\ -\dfrac{(J_{11}-J_{33})*\omega_3}{J_{11}} & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} \dfrac{1}{J_{11}} & 0 \\ 0 & \dfrac{1}{J_{11}} \end{bmatrix}, \; H = B, \tag{23}$$

where J_{11} and J_{33} are the inertia coefficients with respect to the first and third axes, respectively, and ω_3 is the component of the inertial angular velocity of the spacecraft with respect to the third axis, which is constant under the above assumptions. The values for the parameters are $J_{11} = 100$, $J_{33} = 150$ and $\omega_3 = 0.1$. The disturbance torques are assumed bounded as $|w(t)| \le [1 \;\; 1]^T$ for all $t \ge 0$.

In [1] the control law $u(t) = K_0 x(t)$, with $K_0 = -5\,\mathrm{I}_2$, where I_2 is the 2×2 identity matrix, was used in closed loop with (2), (23). Ultimate bounds were then computed on the states of the closed-loop system via a method based on parameterised quadratic Lyapunov functions, yielding the value of 0.2 for the second state $\omega_2(t)$. We note that the method proposed in [1] can be used to compute ultimate bounds once a linear controller is given but it is not evident how the method can be employed in a controller design procedure to assign a desired ultimate bound. Our goal here is to design a state feedback gain such that an ultimate bound of 0.01 can be guaranteed for $\omega_2(t)$. We use the procedure proposed in Section IV. The matrix that transforms the system to the canonical controller form is $U = 0.01\,\mathrm{I}_2$. The closed-loop eigenvalues are chosen as $\lambda_{1,2} = \mu c_{1,2}$ with $c_{1,2} = -1/\sqrt{2} \pm j1/\sqrt{2}$ and where μ will be selected to achieve the desired ultimate bound. For this example, the matrix V in (10) is independent of $\lambda_{1,2}$ and all its entries can be freely chosen provided its columns are linearly independent. An ultimate bound on the closed-loop states is b_μ given in (13), computed with $F = 0$, $\bar{w} = [1 \;\; 1]^T$ and R_μ defined in (11). Note that R_μ changes with different choices of V, a degree of freedom that will be exploited next in the design. Figure 1 shows the ultimate bound on the second state $\omega_2(t)$ as a function of μ for three different values of the matrix V, denoted as V_a, V_b and V_c. The choice $V = V_a$ corresponds to the one suggested in Algorithm 4.2. Using this choice, the desired ultimate bound 0.01 for $\omega_2(t)$ can be achieved with $\mu = 4$. However, choosing $V = \begin{bmatrix} 0.0439 + j0.3127 & 0.0439 - j0.3127 \\ 0.0272 + j0.0129 & 0.0272 - j0.0129 \end{bmatrix}$, Figure 1 shows that the desired ultimate bound can be achieved with a lower value of μ, namely $\mu = 2$. The controller gain for this case, computed using (12), is $K = \begin{bmatrix} -48.29 & -1771.13 \\ 11.14 & -234.55 \end{bmatrix}$.

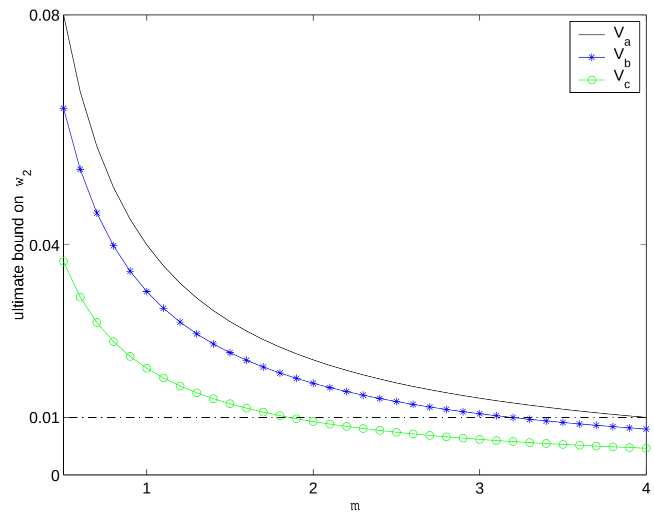

Fig. 1. Ultimate bound on $\omega_2(t)$ as a function of μ for three different closed-loop eigenvector matrices.

VI. Conclusions

We have presented a new control design method for perturbed multiple-input systems, which guarantees a pre-specified componentwise ultimate bound on the closed-loop system's trajectories. The method uses techniques of eigenstructure assignment by state feedback, can be applied to systems where the perturbation is bounded by a delayed function of the state, and employs a componentwise bound computation procedure. The latter procedure exploits the system geometry as well as the perturbation structure, and requires neither Lyapunov analysis nor bounds on the norm of the perturbation vector. We have illustrated the utility of the method by means of examples taken from the literature.

Appendix

The proofs require the following additional claim and lemma.

Claim 4: For any $\theta_0 \in \mathbb{R}^n_{+,0}$, there exists $\beta > \theta_0$ such that $R(F\beta + \bar{w}) < \beta$.

Proof: The proof follows from the fact that $RF \ge 0$, $\boldsymbol{\rho}(RF) < 1$, and $\theta_0 \ge 0$ (see, for example [12, §8]). ∎

Lemma 1.1: Consider system (6), where $x(t) \in \mathbb{R}^n$, $w(t) \in \mathbb{R}^k$, $\bar{A} \in \mathbb{R}^{n\times n}$ and $H \in \mathbb{R}^{n\times k}$. Let $\bar{A}$ be a Hurwitz matrix with Jordan canonical form $\Lambda = V^{-1}\bar{A}V$ and consider the matrix R defined in (7). Let $\bar{w} \in \mathbb{R}^k_{+,0}$ and $t_c \in \mathbb{R}_{+,0}$.

a) Suppose that $|w(t)| \le \bar{w}$ for $0 \le t \le t_c$ and $|x(0)| \le R\bar{w}$. Then, $|x(t)| \le R\bar{w}$ for $0 \le t \le t_c$.
b) Suppose that $|w(t)| \le \bar{w}$ for all $t \ge 0$. Then, for any $\epsilon \in \mathbb{R}^n_+$, there exists a continuous function $\bar{t}_f(\epsilon, \cdot) : \mathbb{R}^n \to \mathbb{R}_{+,0}$ such that $|x(t)| \le R\bar{w} + |V|\epsilon$ for all $t \ge \bar{t}_f(\epsilon, x(0))$.

Proof: The proof combines minor modifications to the proofs of Theorem 3 of [8], Theorem 3.3 of [9], and Theorem 7.3 of [10], and is therefore omitted. ■

Proof of Claim 1. For a contradiction, suppose that $|x(t)|$ becomes unbounded. Note that $|x(t)| \leq \theta_0 \triangleq \theta(0)$ for $-\bar{\tau} \leq t \leq 0$ by (4). Also, note that if $|x(t)|$ becomes unbounded, then $\theta(t)$ in (4) must also become unbounded. Let β be given by Claim 4. Define

$$t_c \triangleq \inf t, \quad \text{subject to } t \geq 0 \text{ and } |\theta(t)| \not\leq \beta. \tag{24}$$

Note that, since $x(t)$ is continuous, then $\theta(t)$ also is, and since $\theta(0) = \theta_0 < \beta$, we have $0 < t_c < \infty$. By (24), $\theta(t) \leq \beta$ for all $0 \leq t \leq t_c$. From (3), then $|w(t)| \leq F\beta + \bar{w}$ for all $0 \leq t \leq t_c$. Applying Lemma 1.1 a) and using Claim 4, then $|x(t)| \leq R(F\beta + \bar{w}) < \beta$, for $0 \leq t \leq t_c$. Combining this with the fact that $\theta(0) = \theta_0 < \beta$, it follows that $\theta(t) < \beta$ for $0 \leq t \leq t_c$. From the continuity of $\theta(t)$, it then follows that there exists $\alpha \in \mathbb{R}_+$ such that $\theta(t) \leq \beta$ for $0 \leq t \leq t_c + \alpha$. This contradicts (24), concluding the proof. ◇

Proof of Claim 2. By induction on ℓ. By assumption, we have $|x(t)| \leq \bar{x}$ for all $t \geq -\bar{\tau}$. From (3) and (4), then $|w(t)| \leq F\bar{x} + \bar{w}$, for all $t \geq 0$. Applying Lemma 1.1 b), then given $\gamma \in \mathbb{R}^n_+$, there exists $\bar{t}_f$ such that $|x(t)| \leq R(F\bar{x} + \bar{w}) + |V|\gamma = T^0_\gamma(\bar{x})$ for all $t \geq \bar{t}_f$. The claim is then established for $\ell = 0$ by setting $\bar{t}(0, \gamma) = \bar{t}_f$.

Suppose next that the claim holds for some $\ell \in \mathbb{Z}_{+,0}$, that is for any $\gamma \in \mathbb{R}^n_+$, there exists $\bar{t}(\ell, \gamma)$ such that $|x(t)| \leq T^{\ell+1}_\gamma(\bar{x})$ for all $t \geq \bar{t}(\ell, \gamma)$. Then, by (3), $\theta(t) \leq T^{\ell+1}_\gamma(\bar{x})$ and hence, by (4), $|w(t)| \leq FT^{\ell+1}_\gamma(\bar{x}) + \bar{w}$, for $t \geq \bar{t}(\ell, \gamma) + \bar{\tau}$. Applying Lemma 1.1 b) and considering the fact that the system is time-invariant, then given $\gamma \in \mathbb{R}^n_+$, there exists $\tilde{t}_f$ such that $|x(t)| \leq R(FT^{\ell+1}_\gamma(\bar{x}) + \bar{w}) + |V|\gamma = T^{\ell+2}_\gamma(\bar{x})$, for $t \geq \bar{t}(\ell, \gamma) + \bar{\tau} + \tilde{t}_f$. The claim is then established for $\ell + 1$ by setting $\bar{t}(\ell + 1, \gamma) = \bar{t}(\ell, \gamma) + \bar{\tau} + \tilde{t}_f$.

This concludes the proof of the claim. ◇

Proof of Claim 3. Note that $|T^{\ell+1}_\gamma(\bar{x}) - b| \leq |T^{\ell+1}_\gamma(\bar{x}) - T^{\ell+1}(\bar{x})| + |T^{\ell+1}(\bar{x}) - b|$. By (8), (9), and since $\boldsymbol{\rho}(RF) < 1$, then $\lim_{\ell\to\infty} T^\ell(\bar{x}) = b$. Also, since $\bar{x} \geq 0$ and $T : \mathbb{R}^n_{+,0} \to \mathbb{R}^n_{+,0}$, then $b \geq 0$. Therefore, given $\epsilon \in \mathbb{R}^n_+$, we can select $\ell = \ell(\epsilon)$ such that $|T^{\ell+1}(\bar{x}) - b| < \epsilon/2$. From (9), it follows straightforwardly that, for the selected value of ℓ, we may select $\gamma = \gamma(\epsilon) \in \mathbb{R}^n_+$ small enough so that $|T^{\ell+1}_\gamma(\bar{x}) - T^{\ell+1}(\bar{x})| < \epsilon/2$. Then, $|T^{\ell+1}_\gamma(\bar{x}) - b| < \epsilon$. Since $b \geq 0$ and $T^0_\gamma : \mathbb{R}^n_{+,0} \to \mathbb{R}^n_{+,0}$ for any $\gamma \in \mathbb{R}^n_{+,0}$, then $T^{\ell+1}_\gamma(\bar{x}) = |T^{\ell+1}_\gamma(\bar{x}) - b + b| \leq |T^{\ell+1}_\gamma(\bar{x}) - b| + b < b + \epsilon$, establishing the claim. ◇

References

[1] M. L. Brockman and M. Corless, "Quadratic boundedness of nominally linear systems," *International Journal of Control*, vol. 71, no. 6, pp. 1105–1117, 1998.

[2] H. Khalil, *Nonlinear Systems*, 3rd ed. New Jersey: Prentice-Hall, 2002.

[3] W. E. Schmitendorf and B. R. Barmish, "Robust asymptotic tracking for linear systems with unknown parameters," *Automatica*, vol. 22, pp. 355–360, 1986.

[4] B. R. Barmish, I. R. Petersen, and A. Feuer, "Linear ultimate boundedness control of uncertain dynamical systems," *Automatica*, vol. 19, no. 5, pp. 523–532, 1983.

[5] H. Trinh and M. Aldeen, "Output tracking for linear uncertain time-delay systems," *IEE Proceedings, Control Theory and Applications*, vol. 143, no. 6, pp. 481–488, 1996.

[6] Y.-Y. Cao and Y.-X. Sun, "Robust stabilization of uncertain systems with time-varying multistate delay," *IEEE Transactions on Automatic Control*, vol. 43, no. 10, pp. 1484–1488, 1998.

[7] S. Oucheriah, "Robust Tracking and Model Following of Uncertain Dynamic Delay Systems by Memoriless Linear Controllers," *IEEE Trans. Automat. Contr.*, vol. 44, no. 7, pp. 1473–1477, 1999.

[8] E. Kofman, "Non-conservative ultimate bound estimation in lti perturbed systems," *Automatica*, vol. 41, no. 10, pp. 1835–1838, October 2005.

[9] E. Kofman, H. Haimovich, and M. Seron, "A systematic method to obtain ultimate bounds for perturbed systems," *International Journal of Control*, vol. 80, no. 2, pp. 167–178, 2007.

[10] H. Haimovich, "Quantisation issues in feedback control," Ph.D. dissertation, School of Electrical Engineering and Computer Science, The University of Newcastle, Australia, Mar. 2006.

[11] V. Sinswat and F. Fallside, "Eigenvalue/eigenvector assignment by state-feedback," *International Journal of Control*, vol. 26, no. 3, pp. 389–403, 1977.

[12] R. A. Horn and C. R. Johnson, *Matrix Analysis*. Cambridge University Press, Cambridge, UK, 1985.

[13] W. A. Wolovich, *Linear multivariable systems*. New York: Springer-Verlag, 1974.

[14] D. Luenberger, "Canonical forms for linear multivariable systems," *IEEE Trans. Automat. Contr.*, vol. 12, pp. 290–293, 1967.

[15] E. Kofman, M. Seron, and H. Haimovich, "Control design with guaranteed ultimate bound in perturbed systems," School of Electrical Engineering and Computer Science, The University of Newcastle, Tech. Rep. EE06022, 2006, http://www.ee.newcastle.edu.au/reports/EE06022.pdf.