Control design with guaranteed ultimate bound for feedback linearizable systems

Ernesto Kofman [*,1] **Fernando Fontenla** [*] **Hernan Haimovich** [*]
María M. Seron [**]

[*] *CONICET; Depto. de Electrónica, Fac. de Cs. Exactas, Ing. y Agrim., Universidad Nacional de Rosario, Riobamba 245bis, 2000 Rosario, Argentina.*
[**] *Centre for Complex Dynamic Systems and Control, The University of Newcastle, Callaghan, NSW 2308, Australia.*

Abstract: For a class of perturbed feedback linearizable nonlinear systems, we consider the computation and assignment of prescribed ultimate bounds on the system states. We employ a recently proposed componentwise bound computation procedure, which directly takes into account both the system and perturbation structures by performing componentwise analysis. We first derive sufficient conditions to ensure that the trajectories originating from initial conditions in an appropriate set are ultimately bounded. Secondly, and most importantly, for state-feedback-linearizable nonlinear systems with matched perturbations, we provide a systematic design procedure to compute a state feedback control that ensures a prescribed ultimate bound for the closed-loop system states. The procedure combines nonlinear state-feedback-linearizing control with a state-feedback matrix computed via an eigenstructure assignment method previously reported by the authors. A simulation example illustrates the simplicity and systematicity of the proposed design method.

1. INTRODUCTION

This paper considers the computation and assignment via state feedback of ultimate bounds on the closed-loop trajectories of perturbed state-feedback-linearizable nonlinear systems. Perturbations in dynamic systems may arise from unknown disturbance signals, model uncertainty, component ageing, etc. Bounds on the perturbation variables are typically known and may be used to obtain, under certain conditions, ultimate bounds on the perturbed system trajectories.

A standard approach for the computation of ultimate bounds is the use of level sets of suitable Lyapunov functions [see, for example, Section 9.2 of Khalil (2002)]. An alternative approach was proposed in Kofman (2005), Kofman et al. (2007a) and Haimovich (2006), where componentwise ultimate bound formulae were derived exploiting the system geometry in modal coordinates as well as the perturbation structure, without requiring the computation of a Lyapunov function for the system. For some system and perturbation structures, this componentwise approach was shown to provide bounds that are much tighter than those obtained via standard Lyapunov analysis.

For linear systems with matched perturbations (that is, perturbations that span the same space spanned by the control input), an ultimate bound on the closed-loop trajectories can be arbitrarily assigned by state feedback control (Schmitendorf and Barmish, 1986). Several control design methods, based on a Lyapunov approach, which can achieve an arbitrarily small ultimate bound for linear uncertain systems have been reported in the robust control literature (see, for example, Barmish et al., 1983; Trinh and Aldeen, 1996; Cao and Sun, 1998; Oucheriah, 1999).

Taking a departure from Lyapunov analysis and exploiting the aforementioned componentwise approach for ultimate bound computation, we proposed in Kofman et al. (2008, 2007b) a new controller design method for linear, continous-time systems that guarantees a prespecified ultimate bound on the closed-loop system trajectories. The method takes advantage of the dependency of the componentwise ultimate bound expressions on the system eigenstructure. Using techniques of eigenvalue and eigenvector assignment by state feedback, we showed that a state feedback gain can be designed such that the ultimate bound expression decreases to zero as a "scaling" parameter, associated with the magnitude of the closed-loop eigenvalues, increases. The proposed design procedure is systematic in the sense that, once a desired "normalized" configuration is chosen for the closed-loop eigenvalues, it only requires to increase the scaling parameter at most once for the desired ultimate bound on each component of the state to be achieved.

In the current paper we extend the design method of Kofman et al. (2008, 2007b) to perturbed state-feedback-linearizable nonlinear systems (Isidori, 1995). The latter are systems which, under suitable nonlinear feedback and coordinate transformation, can be expressed as a linear asymptotically stable system with nonlinear perturbation terms. For these systems, we derive sufficient conditions to ensure that the trajectories originating from initial conditions in an appropriate set are ultimately bounded. Moreover, for state-feedback-linearizable nonlinear systems with matched perturbations, we provide an algorithm

[1] email: kofman@fceia.unr.edu.ar

to systematically design a state feedback control that ensures a prescribed ultimate bound for the closed-loop system states. The proposed algorithm combines standard nonlinear feedback linearizing control [see, for example, Isidori (1995)] with a state-feedback matrix designed based on the method of Kofman et al. (2008, 2007b). An example of a synchronous generator illustrates the simplicity and systematicity of the approach. Simulation results for this example under persistent perturbations satisfying appropriate bounds also demonstrate the potential of the method to obtain relatively tight bounds on the closed-loop trajectories.

The remainder of the paper is organized as follows. Section 2 presents the class of nonlinear systems under consideration and outlines the goals of the paper. Section 3 reviews some preliminary results needed to achieve the desired goals. Section 4 presents the main contributions of the paper, namely, a systematic method to compute componentwise ultimate bounds for state-feedback-linearizable systems and a systematic control design methodology so that any desired ultimate bound is achieved in the case of matched perturbations. Section 5 illustrates the results with an example of a synchronous generator. Finally, Section 6 concludes the paper.

Notation. In the sequel, $\mathbb{R}$ and $\mathbb{C}$ denote the sets of real and complex numbers, respectively. $|M|$ and $\mathbb{R}\mathrm{e}(M)$ denote the elementwise magnitude and real part, respectively, of a (possibly complex) matrix or vector M. The expression $x \preceq y$ ($x \prec y$) denotes the set of componentwise (strict) inequalities between the elements of the real vectors (or matrices) x and y, and similarly for $x \succeq y$ ($x \succ y$). $\mathbb{R}_+$ and $\mathbb{R}_{+,0}$ denote the positive and nonnegative real numbers, respectively. For $c \in \mathbb{C}$, $\overline{c}$ denotes its complex conjugate.

2. PROBLEM STATEMENT

Consider the following continuous-time nonlinear system with n states, m inputs, and k disturbance variables:

$$\dot{x} = f(x) + \sum_{i=1}^{m} g_i(x)u_i + \sum_{j=1}^{k} h_j(x)w_j, \qquad (1)$$

where $x(t) \in \mathbb{R}^n$, $f(0) = 0$, and such that $f, g_1, \ldots, g_m$, $h_1, \ldots, h_k$ are smooth (C^∞) vector fields defined on an open set $U_x \subset \mathbb{R}^n$ containing the origin. The disturbance variables $w_1, \ldots, w_k$ are assumed to be bounded as follows:

$$|w_j(t)| \le \theta_j(x(t)), \quad \text{for all } t \ge 0, \quad \text{for } j = 1, \ldots, k, \quad (2)$$

where $\theta_j(x) \ge 0$ for all $x \in U_x$. We write (2) in condensed form as follows:

$$|w(t)| \preceq \theta(x(t)), \quad \text{for all } t \ge 0, \qquad (3)$$

with $w = \mathrm{col}(w_1, \ldots, w_k)$ and $\theta = \mathrm{col}(\theta_1, \ldots, \theta_k)$. Also, we hereafter denote $u = \mathrm{col}(u_1, \ldots, u_m)$, $g = [g_1|\ldots|g_m]$ and $h = [h_1|\ldots|h_k]$. Associated with (1) is the *nominal* system

$$\dot{x} = f(x) + \sum_{i=1}^{m} g_i(x)u_i = f(x) + g(x)u. \qquad (4)$$

The nominal system (4) is assumed to be state-feedback-linearizable in U_x (Isidori, 1995), that is, there exist a coordinate transformation (diffeomorphism) $z = \Phi(x)$ and a pair of feedback functions $\alpha(x)$ and $\beta(x)$, all defined on U_x, so that $\beta(x)$ is nonsingular for all $x \in U_x$ and

$$\left[\frac{\partial \Phi}{\partial x}(f(x) + g(x)\alpha(x))\right]_{x=\Phi^{-1}(z)} = A_0 z, \qquad (5)$$

$$\left[\frac{\partial \Phi}{\partial x}(g(x)\beta(x))\right]_{x=\Phi^{-1}(z)} = B_0, \qquad (6)$$

where

$$A_0 = \mathrm{diag}(A_1, \ldots, A_m), \quad B_0 = \mathrm{diag}(b_1, \ldots, b_m), \qquad (7)$$

$A_i \in \mathbb{R}^{d_i \times d_i}$, $b_i \in \mathbb{R}^{d_i \times 1}$,

$$A_i = \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 \\ \vdots & \ddots & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & 1 \\ 0 & \ldots & \ldots & \ldots & 0 \end{bmatrix}, \quad b_i = \begin{bmatrix} 0 \\ \vdots \\ \vdots \\ 0 \\ 1 \end{bmatrix}, \qquad (8)$$

for $i = 1, \ldots, m$, and $\sum_{i=1}^{m} d_i = n$. Therefore, in this case application of the state-feedback law $u = \alpha(x) + \beta(x)v$ to system (1), where $v = \mathrm{col}(v_1, \ldots, v_m)$ is a new input, jointly with the change of coordinates $z = \Phi(x)$, yields

$$\dot{z} = A_0 z + B_0 v + \left[\frac{\partial \Phi}{\partial x} h(x)\right]_{x=\Phi^{-1}(z)} w. \qquad (9)$$

Application of the additional state-feedback law $v = Kz$ to (9) or, equivalently, application of

$$u = \alpha(x) + \beta(x) K \Phi(x) \qquad (10)$$

to (1), yields

$$\dot{z} = (A_0 + B_0 K) z + \left[\frac{\partial \Phi}{\partial x} h(x)\right]_{x=\Phi^{-1}(z)} w. \qquad (11)$$

A particular important type of systems of the form (1) is given by the case of "matched perturbations", namely the case when $h(x) = g(x)\gamma(x)$, for some matrix $\gamma(x)$ of smooth functions defined on U_x. For this type of systems, application of (10) yields the closed-loop system

$$\dot{z} = (A_0 + B_0 K) z + B_0 \left[\beta^{-1}(x)\gamma(x)\right]_{x=\Phi^{-1}(z)} w. \qquad (12)$$

This work has the following goals:

G1) to provide sufficient conditions to ensure that the trajectories of system (1), under a feedback law of the form (10), are ultimately bounded.

G2) to estimate an ultimate bound for such closed-loop system, and

G3) to design matrix K in (10) so that the system state trajectories are ultimately bounded and satisfy a given ultimate bound in the case of matched perturbations.

Before proceeding with some preliminary results required to achieve the above goals, we observe that the setting (1)-(3) can accommodate any combination of the following types of uncertainty:

- Uncertainty in the system evolution function, where $\dot{x}(t) = (f(x)+\Delta f(x))+g(x)u(t)$, and $|\Delta f(x)| \preceq \phi(x)$, $\forall t \ge 0$; in this case, we can take $h(x) = \mathrm{I}_n$ in (1) and $\theta(x) = \phi(x)$ in (3).
- Uncertainty in the system input function [assuming a feedback $u = \kappa(x)$ in (1)], where $\dot{x}(t) = f(x)+(g(x)+\Delta g(x))\kappa(x)$, and $|\Delta g(x)| \preceq \gamma(x)$, $\forall\, t \ge 0$; in this case,

we can take $h(x) = \mathrm{I}_n$ in (1) and $\theta(x) = \gamma(x)|\kappa(x)|$ in (3).

- Bounded disturbances, where $|w(t)| \preceq \bar{w}$, $\forall t \geq 0$; in this case, we can take $\theta(x) = \bar{w}$ in (3).

3. PRELIMINARY RESULTS

In this section, we state previous results that are needed to achieve the goals outlined above. Section 3.1 presents a method for computing a componentwise ultimate bound for a linear system with an additive perturbation having a bound that may depend on the linear system state. Section 3.2 recalls a result that shows how a linear state feedback may be designed to assign closed-loop eigenvalues and eigenvectors for a linear system.

3.1 Ultimate Bound of a Perturbed Linear System

The following result computes a componentwise ultimate bound for a LTI system in presence of perturbations that are componentwise bounded by functions of the system state. This result is a modified version of Theorem 2 of Kofman et al. (2007a).

Theorem 1. Consider the system

$$\dot{x}(t) = Ax(t) + Hv(t), \tag{13}$$

where $x(t) \in \mathbb{R}^n$, $v(t) \in \mathbb{R}^k$, $H \in \mathbb{R}^{n\times k}$, and $A \in \mathbb{R}^{n\times n}$ is Hurwitz with (complex) Jordan canonical form $\Lambda = V^{-1}AV$. Suppose that

$$|v(t)| \preceq \delta(x(t)) \quad \forall t \geq 0, \tag{14}$$

where $\delta : \mathbb{R}^n \to \mathbb{R}^k_{+,0}$ is a continuous map verifying

$$|x_1| \preceq |x_2| \Rightarrow \delta(x_1) \preceq \delta(x_2). \tag{15}$$

Consider the map $T : \mathbb{R}^n \to \mathbb{R}^n_{+,0}$ defined by

$$T(x) \triangleq |V| S_H \delta(x), \tag{16}$$

where

$$S_H \triangleq \left|[\mathrm{Re}(\Lambda)]^{-1}\right| \left|V^{-1}H\right|. \tag{17}$$

Suppose that there exists $x_m \in \mathbb{R}^n$ satisfying, $T(x_m) \prec x_m$. Then,

1) $b \triangleq \lim_{r\to\infty} T^r(x_m)$ exists and satisfies $0 \preceq b \prec x_m$.
2) If $|V^{-1}x(0)| \preceq S_H\delta(x_m)$, then
   a) $|V^{-1}x(t)| \preceq S_H\delta(x_m)$ for all $t \geq 0$.
   b) Also, given a positive vector $\epsilon \in \mathbb{R}^n_+$, a finite time t_f exists so that for all $t \geq t_f$,
      i) $|V^{-1}x(t)| \preceq S_H\delta(b) + \epsilon$.
      ii) $|x(t)| \preceq b + |V|\,\epsilon$.

The proof of this theorem is almost identical to that of Theorem 2 of Kofman et al. (2007a), the only difference being the presence of matrix H. In the sequel, we will equivalently express Theorem 1 part 2)b)ii) as follows:

- If $|V^{-1}x(0)| \preceq S_H\delta(x_m)$, then x is ultimately bounded to the region $\{x : |x| \preceq b\}$.

3.2 Eigenvalue and Eigenvector Assignment by Feedback

The following result is part of Theorem 4.1 of Kofman et al. (2007b) and also part of Theorems 4.2 and 4.3 of Kofman et al. (2008). This result shows how a state-feedback matrix can be computed so that desired eigenvalues and eigenvectors are assigned to the closed-loop linear system. In addition, the result establishes how some matrices change as the desired eigenvalues are scaled according to a scaling factor μ.

Theorem 2. Take $\mu > 0$ and select an eigenvalue matrix $\Lambda_\mu = \mu\tilde{\Lambda} = \mu\mathrm{diag}(\tilde{\lambda}_1, \ldots, \tilde{\lambda}_n)$, where $\tilde{\lambda}_i \in \mathbb{C}$, $i = 1, \ldots, n$, satisfy $\tilde{\lambda}_i \neq \tilde{\lambda}_j$ whenever $i \neq j$, $\mathrm{Re}(\tilde{\lambda}_i) < 0$, and if $\tilde{\lambda}_i \notin \mathbb{R}$, then either $\tilde{\lambda}_{i-1} = \overline{\tilde{\lambda}_i}$ or $\tilde{\lambda}_{i+1} = \overline{\tilde{\lambda}_i}$. Select complex numbers $e_{i,j} \in \mathbb{C}$, $i = 1, \ldots, m$, $j = 1, \ldots, n$ so that $e_{i,j+1} = \overline{e_{i,j}}$ whenever $\tilde{\lambda}_{j+1} = \overline{\tilde{\lambda}_j}$ and such that the matrix V defined as

$$V \triangleq \begin{bmatrix} V_{1,1} & \ldots & V_{1,n} \\ \vdots & \ddots & \vdots \\ V_{m,1} & \ldots & V_{m,n} \end{bmatrix}, \quad V_{i,j} \triangleq \begin{bmatrix} e_{i,j}(\mu\tilde{\lambda}_j)^{-(d_i-1)} \\ \vdots \\ e_{i,j}(\mu\tilde{\lambda}_j)^{-1} \\ e_{i,j} \end{bmatrix}, \tag{18}$$

has linearly independent columns, where d_i, $i = 1, \ldots, m$, are the dimensions of A_i and b_i in (7)–(8). Define

$$R_\mu \triangleq |V| \left|[\mathrm{Re}(\Lambda_\mu)]^{-1}\right| \left|V^{-1}B_0\right|, \tag{19}$$

and,

$$K_\mu = (B_0^T B_0)^{-1} B_0^T (V\Lambda_\mu V^{-1} - A_0). \tag{20}$$

Then,

i) The entries of the matrix μR_μ are nonincreasing functions of μ.
ii) $A_0 + B_0K_\mu = V\Lambda_\mu V^{-1}$.

4. MAIN RESULTS

In this section, we present the main contribution of the paper. Namely, we provide a systematic method to compute componentwise ultimate bounds for the state-feedback-linearizable system (1),(3) and a systematic control design methodology so that any desired ultimate bound is achieved in the case of matched perturbations, that is, for a closed-loop system of the form (12).

4.1 Componentwise Ultimate Bound Analysis

The following theorem presents sufficient conditions to ensure the ultimate boundedness of the state trajectories of a perturbed system under state-feedback linearization. This theorem also shows how to compute an ultimate bound.

Theorem 3. Consider system (1), where the perturbation $w(t)$ satisfies (3). Suppose that the associated nominal system (4) is state-feedback-linearizable, and let $z = \Phi(x)$ be the coordinate transformation, and $\alpha(x)$ and $\beta(x)$ be the feedback functions, so that (1) is transformed into (11) under application of the feedback law (10). Let K be chosen so that $A \triangleq A_0 + B_0K$ is Hurwitz, and let $\Lambda = V^{-1}AV$ be the (complex) Jordan canonical form of A. Consider S_{I}, defined in (17) with $H = \mathrm{I}$, and let $R \triangleq |V|S_{\mathrm{I}}$. Define

$$\delta(z) \triangleq \sup_{\zeta:|\zeta|\preceq|z|} \left[\left|\frac{\partial\Phi}{\partial x}h(x)\right|\theta(x)\right]_{x=\Phi^{-1}(\zeta)} \tag{21}$$

and let $T(z) \triangleq R\delta(z)$. Let $U_z \triangleq \Phi(U_x)$ and suppose that $T(z_m) \prec z_m$ for some $z_m \in U_z$. In addition, let $B_m \triangleq \{z : |V^{-1}z| \preceq S_{\mathrm{I}}\delta(z_m)\}$ and suppose that $B_m \subset U_z$. Then,

i) $b_z \triangleq \lim_{r\to\infty} T^r(z_m)$ exists and if $x(0) \in \Phi^{-1}(B_m)$ the trajectories of the closed-loop system (1),(10) are ultimately bounded to the region

$$|x| \preceq b \triangleq \sup_{z:|V^{-1}z|\preceq S_{\rm I}\delta(b_z)} |\Phi^{-1}(z)|. \qquad (22)$$

ii) If, in addition, $h(x) = g(x)\gamma(x)$ [matched perturbations, recall (12)], then i) above also holds if we replace $S_{\rm I}$ above with S_{B_0} [defined as in (17) with $H = B_0$ and B_0 as in (6)] and δ above by δ_m, defined as

$$\delta_m(z) \triangleq \sup_{\zeta:\,|\zeta|\preceq|z|} \Big[\big|\beta^{-1}(x)\gamma(x)\big|\,\theta(x)\Big]_{x=\Phi^{-1}(\zeta)}. \qquad (23)$$

Proof. Defining

$$v(t) \triangleq \left[\frac{\partial\Phi}{\partial x}h(x)\right]_{x=\Phi^{-1}(z(t))} w(t), \qquad (24)$$

system (11) can be rewritten as

$$\dot{z}(t) = Az(t) + v(t) \qquad (25)$$

From (3) and (21), it follows that $|v(t)| \preceq \delta(z(t))$ where $\delta(z)$ verifies (15). If, in addition, $h(x) = g(x)\gamma(x)$, then $v(t) = B_0 s(t)$ [see (12)], where

$$s(t) = \left[\beta^{-1}(x)\gamma(x)\right]_{x=\Phi^{-1}(z(t))} w(t) \qquad (26)$$

satisfies $|s(t)| \preceq \delta_m(z(t))$ [see (23)] and δ_m also verifies (15). Then, Theorem 1, part 1) establishes the existence of b_z. Next, applying Theorem 1, part 2)a) to system (25) we conclude that any trajectory starting from an initial condition $z(0) \in B_m$ does not leave the region $B_m \subset U_z$. Moreover, it follows from Theorem 1, part 2)b)i) that $z(t)$ is ultimately bounded to the region $B_z = \{z : |V^{-1}z| \preceq S_{\rm I}\delta(b_z)\} \subset B_m \subset U_z$. A similar argument applied to the case of matched perturbations yields that $z(t)$ is ultimately bounded to the region $B_z = \{z : |V^{-1}z| \preceq S_{B_0}\delta_m(b_z)\} \subset B_m \subset U_z$.

Taking into account that $x = \Phi^{-1}(z)$ (with Φ^{-1} being a diffeomorphism defined in U_z), and that $z(t)$ cannot abandon U_z, it follows that $x(t)$ is ultimately bounded to the region $B_x = \Phi^{-1}(B_z)$. Then, taking b as the supremum of $|x|$ in B_x we obtain the ultimate bound of Eq.(22). □

4.2 Robust control design

The next theorem establishes design conditions to achieve a desired ultimate bound in a perturbed system with state-feedback-linearization under the hypothesis of matched perturbations. The result is then translated into a design algorithm that constitutes the core of the proposed systematic control design methodology.

Theorem 4. Consider system (1), where the perturbation $w(t)$ satisfies (3). Suppose that the associated nominal system (4) is state-feedback-linearizable, and let $z = \Phi(x)$ be the coordinate transformation, and $\alpha(x)$ and $\beta(x)$ be the feedback functions, so that (5)–(8) hold. Assume also that $h(x) = g(x)\gamma(x)$ (matched perturbations). Select V, Λ_μ according to Theorem 2, calculate R_μ from (19), and consider δ_m as defined in (23) and

$$T_\mu(z) \triangleq R_\mu\delta_m(z) \qquad (27)$$

Let $z_c \in U_z$, $z_c \succ 0$, be such that $\{z : |z| \preceq z_c\} \subset U_z$. Then,

i) The components of the vector $\mu T_\mu(z_c)$ are non-increasing functions of μ for $\mu > 0$.

ii) Let $\mu > 0$ be such that $T_\mu(z_c) \prec z_c$. Then, the feedback law (10) with $K = K_\mu$ given by (20) ensures that the closed-loop system (12) is ultimately bounded to the region

$$|x| \preceq b \triangleq \sup_{z:|V^{-1}z|\preceq S_{B_0}\delta_m(b_z)} |\Phi^{-1}(z)| \qquad (28)$$

where S_{B_0} and δ_m are as defined in Theorem 3 ii) and $b_z = \lim_{r\to\infty} T^r_\mu(z_c) \prec z_c$.

Proof. i): the proof follows straightforwardly from Theorem 2 i) and the fact that $\delta_m(z)$ does not depend on μ.

ii): Defining $v(z,w) \triangleq \beta^{-1}(\Phi^{-1}(z))\gamma(\Phi^{-1}(z))w$, the closed-loop system (11) under matched perturbations and with $K = K_\mu$ takes the form

$$\dot{z}(t) = (A_0 + B_0K_\mu)z(t) + B_0 v(z(t), w(t)). \qquad (29)$$

According to Theorem 2 ii), $A_\mu \triangleq A_0 + B_0K_\mu = V\Lambda_\mu V^{-1}$, which, since $\Lambda_\mu = \mu\text{diag}(\tilde{\lambda}_1,\ldots,\tilde{\lambda}_n)$ and $\mathbb{R}\text{e}(\tilde{\lambda}_i) < 0$ ensures that A_μ is Hurwitz.

From (3) and (23), it follows that $|v(z,w)| \preceq \delta_m(z)$. Also, it can be straightforwardly verified that δ_m satisfies (15). Then, taking $H = B_0$ and applying Theorem 1 ii) to system (29) we conclude that any trajectory such that $|V^{-1}z(0)| \preceq S_{B_0}\delta(z_c)$ cannot leave the region $B_c = \{z : |V^{-1}z| \preceq S_{B_0}\delta(z_c)\}$. Notice that B_c is entirely contained in the region $\{z : |z| \preceq z_c\} \subset U_z$ and then $B_c \subset U_z$. Then, application of Theorem 3 yields that x is ultimately bounded to the region (28). □

Theorem 4 i) states that the components of the vector $T_\mu(z_c)$ decrease (at least inverse-linearly) with μ. Therefore, the condition $T_\mu(z_c) \prec z_c$ can always be achieved through the choice of a sufficiently large value of μ. Theorem 4 ii) can be applied in order to design matrix K in (10) so that the closed-loop system exhibits an arbitrary ultimate bound. The following algorithm implements this idea.

Algorithm 1. Given a desired componentwise ultimate bound $b^* \in \mathbb{R}^n_{+,0}$ for the state x,

(i) Find the change of coordinates $z = \Phi(x)$, jointly with the feedback functions $\alpha(x)$ and $\beta(x)$, so that (5)–(8) are satisfied.
(ii) Take $z_c \succ 0$ so that $|z| \preceq z_c \Rightarrow z \in U_z$, and,

$$\sup_{z:|z|\preceq z_c} |\Phi^{-1}(z)| \preceq b^* \qquad (30)$$

(iii) Calculate $\delta_m(z)$ according to (23).
(iv) Select an arbitrary $\mu > 0$, and Λ_μ and V as specified in Theorem 2.
(v) Compute R_μ from (19).
(vi) Evaluate $T_\mu(z_c)$ according to (27). If $T_\mu(z_c) \prec z_c$, go to step (viii).
(vii) Compute[2] $\max_i((T_\mu(z_c))_i/z_{c_i})$ and set the new μ equal to this value. Reevaluate V according to (18).
(viii) Compute $K = K_\mu$ according to (20).

Algorithm 1, according to Theorem 4, .finds the feedback matrix K_μ of the feedback law (10) that guarantees an ultimate bound b given by (28). This bound b can be proven to be less than or equal to b^* (the desired ultimate bound) as follows.

[2] $(T_\mu(z_c))_i$ denotes the i-th component of $T_\mu(z_c)$. Similarly for z_{c_i}.

First, notice that the condition $|V^{-1}z| \preceq S_{B_0}\delta_m(b_z)$ in (28) implies that $|z| \prec z_c$ since $|z| = |VV^{-1}z| \preceq |V| \cdot |V^{-1}z| \preceq |V|S_{B_0}\delta_m(b_z) = T_\mu(b_z) = b_z \prec z_c$.

Thus, the supremum of $|\Phi^{-1}(\cdot)|$ on $z : |z| \preceq z_c$ is greater than or equal to the supremum on $z : |V^{-1}z| \preceq S_{B_0}\delta_m(b_z)$. Then, comparing (28) and (30) it follows that $b \preceq b^*$.

Remark 1. The design procedure begins by feedback linearizing the nonlinear plant. Then, eigenvalue/eigenvector assignment is performed, similarly to Kofman et al. (2008). We stress that the procedure is not just a straightforward application of the latter results since, even after feedback linearization, the perturbations exhibit nonlinear dependence on the state variables. The ultimate bound is hence estimated via the technique developed in Kofman et al. (2007a), where the ultimate bound estimate is given by the fixed point of a nonlinear map instead of the simple closed-form expression of Kofman et al. (2008). In addition, the nonlinear coordinate transformation required poses an additional problem, because relationships between bounds on the original and the linear coordinates must be computed. The contribution of this work lies precisely in providing a solution to all these problems.

5. EXAMPLES

The dynamics of a synchronous generator on an infinite bus can be expressed by Eq.(1), with

$$f(x) = \begin{bmatrix} x_2 \\ -p[(1+x_3)\sin(x_1+d) - \sin d] - qx_2 \\ -rx_3 + s[\cos(x_1+d) - \cos d] \end{bmatrix},$$

$$g(x) = g_1(x) = h(x) = h_1(x) = [0\ 0\ 1]^T,$$

where p, q, r, s, d are real parameters and where we consider a bounded perturbation term $|w(t)| \preceq w_m$. The associated nominal plant (4) is state-feedback-linearizable. The map $z = \Phi(x)$, where

$$\Phi(x) = \begin{bmatrix} x_1 \\ x_2 \\ -p[(1+x_3)\sin(x_1+d) - \sin d] - qx_2 \end{bmatrix},$$

jointly with the feedback functions

$$\alpha(x) = \frac{-px_2(1+x_3)\cos(x_1+d) - pq\sin d + q^2x_2}{p\sin(x_1+d)} + \\ +q(1+x_3) + rx_3 - s[\cos(x_1+d) - \cos d])$$

and

$$\beta(x) = -\frac{1}{p\sin(x_1+d)},$$

is such that (5)–(8) are satisfied with

$$A_0 = A_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad B_0 = b_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},$$

for all $x \in U_x = \{x : 0 < x_1 + d < \pi\}$. Also, $U_z = \Phi(U_x) = \{z : 0 < z_1 + d < \pi\}$.

The goal is to design the feedback matrix K in (10) so that the closed-loop system (1),(10) is ultimately bounded to the region $|x| \preceq b^* = [0.1\ 0.05\ 0.001]^T$. We consider the set of parameters $p = 136.0544, q = 4, r = 0.4091, s = 0.2576, d = \pi/4$, and the perturbation bound $w_m = 0.001$. We next follow Algorithm 1.

Step (i) of Algorithm 1 was performed above. At Step (ii), z_c can be chosen in many ways. A possible selection is

$$z_c = [0.0500\ 0.0025\ 5.0098]^T. \quad (31)$$

Note that $|z| \prec z_c \Rightarrow z \in U_z$. Step (iii) requires the calculation of $\delta_m(z)$, according to (23), where $\gamma(x) = 1$ and $\theta(x) = w_m = 0.001$. This computation yields

$$\delta_m(z) = \begin{cases} p\sin(|z_1| + d)w_m & \text{if } (|z_1| + d) < \pi/2, \\ pw_m & \text{otherwise.} \end{cases}$$

At Step (iv) we take $\mu = 1$, and propose the eigenvalue configuration $\Lambda_\mu = \mu\,\mathrm{diag}(-1, -5, -20)$. For V, we take $e_{1,j} = 1$ for $j = 1, 2, 3$ in (18). This choice yields

$$V = \begin{bmatrix} 1.0000 & 0.0400 & 0.0025 \\ -1.0000 & -0.2000 & -0.0500 \\ 1.0000 & 1.0000 & 1.0000 \end{bmatrix}. \quad (32)$$

Step (v) yields $R_\mu = [0.0167\ 0.0333\ 0.1667]^T$ and for Step (vi) we calculate $T_\mu(z_c) = R_\mu\delta_m(z_c) = [0.001682\ 0.003363\ 0.01682]^T$. Since the condition $T_\mu(z_c) \prec z_c$ is not satisfied, we proceed with Step (vii), computing $\max_i((T_\mu(z_c))_i/z_{c_i}) = 1.3452$, setting $\mu = 1.3452$ and recalculating V for this new value of μ. Finally, Step (viii) gives the feedback matrix $K_\mu = [-243.4\ -226.2\ -34.98]$.

Theorem 4 ensures that the ultimate bound of the closed-loop system (1),(10) with $K = K_\mu$ is at least as tight as b^*. We can also employ (28) to estimate a (possibly) tighter bound. The fixed point b_z of map T_μ iterated from z_c is $b_z = [0.6591\ 1.7733\ 11.93]^T 10^{-3}$, and then, using (28) we conclude that $|x(t)|$ is bounded to $b_x = [0.6591\ 1.7733\ 0.7101]^T 10^{-3}$.

Figures 1 and 2 show the region defined in the state space by the bound b_x and simulation results for the closed-loop trajectories with zero initial conditions and for different perturbations satisfying Eq.(2). For this simulation, several sinusoidal and pulse train perturbations of different frequencies were applied.

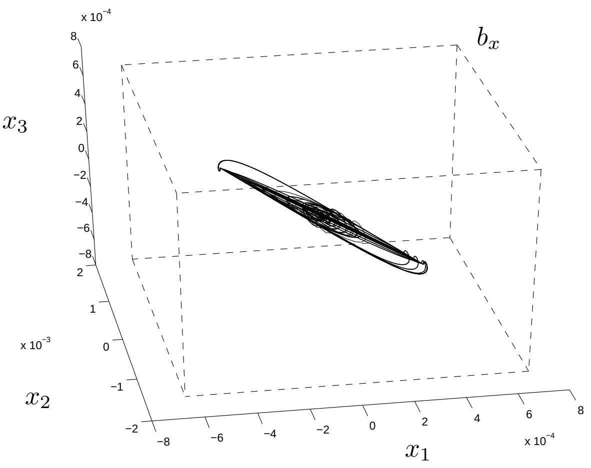

Fig. 1. Ultimate bound region (dashed box) and closed-loop trajectories for different perturbations.

We observe from the figures that the proposed design procedure achieves relatively tight bounds under a large range of persistent perturbations. Also note from the steps carried out in this example, the systematicity of the approach and the simplicity of the computations involved.

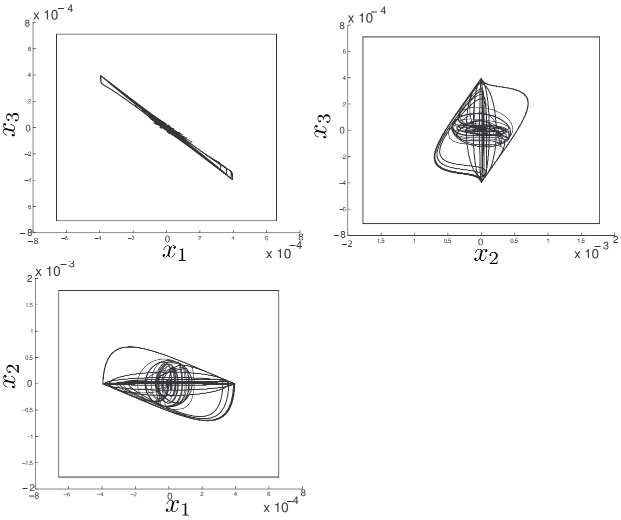

Fig. 2. Ultimate bound region (dashed box) and closed-loop trajectories for different perturbations.

6. CONCLUSIONS

For a class of perturbed feedback-linearizable nonlinear systems, we have presented a systematic design procedure to compute a state feedback control that ensures a prescribed ultimate bound for the closed-loop system states. The procedure utilizes a componentwise bound computation method previously introduced by the authors, and combines nonlinear state-feedback-linearizing control with linear state feedback computed via eigenstructure assignment. The proposed procedure was illustrated on an example of a synchronous generator.

Future work will consider the more practical output feedback case, both for linear and nonlinear systems

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