Analysis of Dynamic System Identification using Rational Orthonormal Bases

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I hereby certify that the work embodied in this thesis is the result of original research and has not been submitted for a higher degree to any other University or Institution.

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Contents

Abstract

This thesis illustrates the use of a particular class of rational orthonormal bases for the purposes of analyzing the performance of least-squares dynamic system estimates that involve model structures which are linear in the parameters. The genesis of this work comes from seminal results on approximating variance error of estimated frequency responses, and work on the use of restricted classes of orthonormal bases (Laguerre and two-parameter Kautz for example) as a model structure parameterization option. A key original perspective of this thesis is that by generalizing the bases involved to the case of arbitrary pole locations, these bases can be viewed as more than an implementational option, but also as an analysis tool of great utility, since it can be applied regardless of whether the bases are used for model structure parameterization or not. This utility is illustrated by deriving approximations of estimate variability that are extensions of pre-existing ones, in that for scenarios where poles are not all fixed at the origin, they can provide improved accuracy. The key tools in this analysis involve the development of new results of generalized Fourier series convergence and generalization of the asymptotic properties of Toeplitz matrices. Initially these results are derived for time invariant and single-input, single-output scenarios, but subsequently extensions to multiple-input, multiple-output and time varying situations are also provided.

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Introduction

This thesis deals with the use of rational orthonormal bases in identification of discrete-time linear systems from input-output data.

System Identification can be defined as the area of System Theory that deals with the study of methodologies for constructing a mathematical model of a dynamical system based on measurements from the system. The area has become an important tool in many branches of engineering (and in other fields such as biology and economy) that rely on accurate models of the systems for the purposes of analysis and design. In the Control and Signal Processing Areas, mathematical models are important for several reasons. Among them we can mention the following:

- Most of control design methodologies are based on the assumption that a model for the system is available.
- For the purposes of simulation and prediction, a parameterized model of the system is required.
- Mathematical models are required for the software and hardware implementation of digital filters.

A typical identification procedure can be summarized as follows:

- Design of the experiments to be performed on the system, and the way the data has to be collected.
- Selection of the model set. This is usually done based on physical laws, which allows the incorporation of prior information, but can also be done in a *black-box* fashion [Lju87]. Here the parameters to be estimated do not have necessarily a physical meaning.
- Selection of the model in the set that best match with the measured data.

- Model validation. This usually implies testing the model quality with a different set of data.

For the case of time invariant systems (that is, systems whose parameters do not change in time) the identification can be performed *off-line*, which means that the parameter estimation is performed once a whole set of data is collected from the system. Most of this thesis will deal with time-invariant systems, the exception being Chapter 6 where time-varying systems are considered. In these last cases the identification can not be performed off-line but has to be carried out *on-line* or *recursively*, which means that the parameter estimates are updated each time a new data is collected from the system.

A unified perspective of System Identification Theory is given in the textbooks [Lju87, SS89].

In this thesis, systems are modeled using rational orthonormal basis functions with fixed poles. These basis functions are generalizations of the well known trigonometric bases of classical Fourier analysis, but with arbitrary poles not restricted to be all at the origin. The transfer function $G(z)$ describing the input-output properties of the system is represented as a series expansion in terms of the basis functions. This leads to a model which is linear in the unknown parameters. The estimation can then be performed in the framework of Prediction Error Methods [Lju87] that for this case provide estimates in closed form.

A dominating theme of this thesis will be the analysis of the accuracy of obtained estimates. For systems modeled with arbitrary fixed denominator structures, the quantification of the estimation error, consisting of both bias error and variance error, has proven to be a very difficult problem. In this thesis it is shown that the analysis can be made more tractable by re-parameterizing the system using orthonormal model structures with the same fixed poles. In consideration of this, orthonormal structures are also viewed in this thesis as an analysis tool that can be used to facilitate the study of estimation accuracy for a class of models. In fact, this perspective that orthonormal bases are equally (perhaps more) useful as an analysis tool, than as an implementational option in the formation of a model structure, is one of the original contributions. Expanding on this a little more, work on orthonormal bases pre-dating that of this thesis (which will be reviewed in the sequel) concentrated solely on the utility of orthonormal bases as an implementational tool for parameterising a fixed denominator model structure. In this thesis we establish a different perspective, that regardless of how the fixed denominator model is implemented, there is great utility for analysis purposes of re-parameterising the structure in orthonormal form, whether or not it was actually implemented in this form. Key to this strategy is that by virtue of linearity, the estimated frequency response is *invariant* to this parameterisation, so the most convenient one may as well be chosen for analysis purposes.

The quantification of the estimation accuracy for the proposed identification schemes is performed by establishing new results on the convergence of (block) Toeplitz-like matrices. These matrices appear in the computation of the component of the frequency response estimation error that is induced by the measurement noise. The Toeplitz-like convergence results allow the generalization to the arbitrary fixed denominator case, of corresponding results that exist in the literature specific to FIR (Finite Impulse Response) model structures.

A second theme we want to emphasize in this thesis is the fact that, in contradiction to what one would expect by appealing to existent frequency domain results on the variability of the estimates for FIR structures [LY85] and on linear filtering, the variability of the estimates with fixed denominator model structures is affected by the location of the poles in the model. The way the variability is affected by the pole locations is also an important issue studied in this thesis.

We now present a brief overview of the rest of the thesis.

Chapter 2: In this chapter, most of our notation is introduced. The classes of signals and systems this thesis deal with, and the associated Hilbert spaces in which these signals and systems can be embedded are presented. A succinct description of the identification methods employed in this thesis, viz. Prediction Error Methods, is also given.

Chapter 3: The construction of several families of (scalar) rational orthonormal bases on the unit circle is reviewed in this chapter. The concept of reproducing kernel associated with the bases is introduced, and its importance for the analysis of the approximating properties of the bases is pointed out. We focus on a particular family of orthonormal bases, namely the Orthonormal Bases with Fixed Poles, which will be used in the context of system identification in the following chapters. For this family, a closed form expression for the associated reproducing kernel is derived, and a minimal state-space realization is presented. In addition, it is shown how families of matrix orthonormal bases for the space $H_2^{m\times n}(\mathbb{T})$ can be constructed from orthonormal bases on $H_2(\mathbb{T})$.

Chapter 4: This chapter deals with the Least Squares identification (from input-output data) of Discrete-Time (DT), Linear Time-Invariant (LTI), Single-Input Single-Output (SISO) systems represented using Orthonormal Bases with Fixed Poles. The accuracy of the estimation is analyzed and the two components of the estimation error (namely, the bias and variance errors) are quantified. The derivation of new results on convergence of some Toeplitz-like matrices will prove to be fundamental for the analysis in this chapter.

Chapter 5: In this chapter the idea of using orthonormal bases and PEM (least squares techniques) for the identification of discrete-time linear time-invariant systems is extended from the SISO to the MIMO (Multiple-Input Multiple-Output) setting. It will be shown how the rational orthonormal bases with fixed poles introduced in Section 3.5 (or the corresponding MIMO bases generated as in Section 3.6) can be used to linearly parameterize any multivariable fixed denominator model structure. As in the SISO case, the accuracy of the estimation is quantified by deriving expressions for an upper bound on the undermodelling error and for the asymptotic (in data-length and model order) covariance of the transfer matrix estimate (noise induced error). The obtained results generalize to the MIMO case and to general orthonormal bases with fixed poles the pre-existing FIR results in [Lju85, YL84]. Fundamental in this derivation are some new convergence properties of generalized block Toeplitz-like matrices. Subspace Identification methods are briefly reviewed for the purposes of comparison with the orthonormal basis-based methods proposed in this thesis.

Chapter 6: In this chapter, a frequency domain analysis of the tracking performance of several adaptive algorithms for the recursive identification of time-varying linear systems is carried out for the case in which the system is represented by a fixed denominator model structure. The focus is on the study of the trade-off between disturbance rejection and tracking ability, and how these properties are influenced by input and noise spectral densities, step size of the adaptive algorithms, and the choice of the fixed pole locations in the model structure.

Chapter 7: In this chapter we summarized the contributions of this thesis, and present some concluding remarks.

2

Background

In this chapter, some elementary definitions regarding the classes of signals and systems this thesis deals with, and the related Hilbert spaces in which these signals and systems can be embedded are given. We also give a succinct description of Prediction Error Methods for System Identification, since these are the identification methods we will employ throughout the thesis.

2.1 Signals, Systems, and related Hilbert Spaces

2.1.1 Signals and Systems

To begin with, some standard material concerning the representation of signals and systems will be presented. A more detailed and complete treatment of these topics can be found (for example) in the books [Lju87, SS89].

Most of this thesis will deal with discrete-time (DT), linear time-invariant (LTI), multiple-input–multiple-output (MIMO) systems whose input-output relationship can be described by

$$y_k = \sum_{\ell=0}^{\infty} g(\ell) u_{k-\ell} + \nu_k = G(q) u_k + \nu_k, \tag{2.1}$$

where $y_k \in \mathbb{R}^m$, $u_k \in \mathbb{R}^n$, and $\nu_k \in \mathbb{R}^m$ denote respectively the vectors of output, input, and disturbance (or measurement noise) signals at time k, where $\{g(\ell)\}$ are the so-called $(m \times n)$ 'Markov parameters', and where

$$G(q) \triangleq \sum_{\ell=0}^{\infty} g(\ell) q^{-\ell}$$

denotes the $(m \times n)$ input to output transfer matrix operator (provided the

infinite sum exists), with q standing for the 'forward shift operator' defined as[1]

$$q\ u_k \triangleq u_{k+1},$$

for any sequence $\{u_k\}$, and for all k.

Sometimes, a z-domain representation of system (2.1) will be preferred. This complex-plane representation can be obtained by taking the Z-transform on both sides of equation (2.1). The Z-transform of a signal $\{x_k\}$ is defined as

$$X(z) \triangleq \sum_{k=-\infty}^{\infty} x_k z^{-k}. \tag{2.2}$$

Assuming that the system is initially relaxed, the representation (2.1) can be written in the z-domain as

$$Y(z) = G(z)U(z) + V(z),$$

where $G(z)$ is the transfer matrix of the system, and $Y(z), U(z)$ and $V(z)$ are the Z-transforms of the sequences $\{y_k\}, \{u_k\}$ and $\{\nu_k\}$ respectively. The definition of the Z-transform in (2.2) requires that the signals be specified for the entire time range $-\infty < k < \infty$. In practical problems the signals are known for $k \geq 0$, but by no means are zero for $k < 0$. It is then useful to define the 'one-sided' or 'unilateral' Z-transform as

Definition 2.1.1. [OS89, PM92] The *one-sided* or *unilateral* Z-transform of a signal $\{x_k\}$ is defined by

$$X(z) \triangleq \sum_{k=0}^{\infty} x_k z^{-k}$$

♢

It will be assumed that the input sequence $\{u_k\}$ is a 'quasi-stationary' (n-dimensional) process, which means that the following limits defining the mean (m_u) and covariance ($R_u(\tau)$) functions exist [Lju87]

$$\begin{aligned} m_u &\triangleq \lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} \mathbf{E}\{u_k\}, \\ R_u(\tau) &\triangleq \lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} \mathbf{E}\left\{(u_k - m_u)(u_{k-\tau} - m_u)^T\right\}, \end{aligned}$$

[1] The 'backward shift operator' is defined similarly as

$$q^{-1}\ u_k \triangleq u_{k-1}.$$

where $\mathbf{E}\{\cdot\}$ stands for the statistical expectation operator; the expectation being over the probability space that any random components are defined on. When deterministic inputs are considered, the expectation operator can be omitted in the previous definitions, and of course, in these situations it makes no difference to retain it. We will sometimes denote the operator

$$\lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} \mathbf{E}\{\cdot\}$$

with the more compact notation $\overline{\mathbf{E}}\{\cdot\}$ introduced by Ljung in [Lju87].

The disturbance sequence $\{\nu_k\}$ will usually be assumed to be a stationary[2] (m-dimensional) stochastic process with mean and covariance functions defined as

$$\begin{aligned} m_\nu &\triangleq \mathbf{E}\{\nu_k\}, \\ R_\nu(\tau) &\triangleq \mathbf{E}\left\{(\nu_k - m_\nu)(\nu_{k-\tau} - m_\nu)^T\right\}, \end{aligned}$$

respectively. At times, we will also assume that the disturbance $\{\nu_k\}$ is statistically independent of the input $\{u_k\}$.

Associated with a stochastic process $\{u_k\}$ (either stationary or quasi-stationary) is the '(Power) Spectral Density Function' defined as the Discrete Fourier Transform of the covariance function

$$\Phi_u(\omega) \triangleq \sum_{\tau=-\infty}^{\infty} R_u(\tau) e^{-\mathrm{j}\tau\omega}.$$

The covariance function can then be recovered from the spectral density by Inverse Discrete Fourier transformation:

$$R_u(\tau) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \Phi_u(\omega) e^{\mathrm{j}\tau\omega} \mathrm{d}\omega.$$

In the case of a quasi-stationary process $\{u_k\}$, the last expression leads to

$$\overline{\mathbf{E}}\left\{u_k u_k^T\right\} = \frac{1}{2\pi} \int_{-\pi}^{\pi} \Phi_u(\omega) \mathrm{d}\omega, \tag{2.3}$$

which is a form of *Parseval's identity* or *Parseval's Theorem* (see next subsection). We will frequently use this identity in the derivation of frequency domain expressions of estimation accuracy.

[2] By 'stationary' we actually mean 'weakly stationary'. A stochastic process $\{\nu_k\}$ is said to be weakly stationary if [Doo53, Ros85]

$$\begin{aligned} \mathbf{E}\{\nu_k\} &= m_\nu = \text{constant} \\ \mathrm{Cov}\{\nu_k, \nu_s\} &= R_\nu(k-s), \qquad \text{depends only on } (k-s). \end{aligned}$$

Sometimes we will be interested in the statistics of the output y_k of a linear system $G(q)$ as a function of the statistics of the input u_k, in the absence of noise. In this case we have a linear filtering of the input u_k

$$y_k = G(q)\ u_k,$$

where the filter is the transfer matrix operator $G(q)$. As already mentioned, assuming that the filter (the system) is stable and causal, it can be represented as

$$G(q) = \sum_{\ell=0}^{\infty} g(\ell)\ q^{-\ell}.$$

Stability of $G(q)$ implies that $\|g(\ell)\| \to 0$ as $\ell \to \infty$, where $\|\cdot\|$ denotes a matrix norm (for instance the 2-norm). Causality means that the impulse response sequence $\{g(\ell)\}$ is one-sided, which implies that the output at time k doesn't depend on future values of the input $u_{k+1}, u_{k+2}, \cdots$. In this situation it is not difficult to prove (see [Lju87, SS89]) that the mean value of y_k is given by

$$m_y \triangleq \mathsf{E}\{y_k\} = \sum_{\ell=0}^{\infty} g(\ell)\ m_u = G(1)\ m_u,$$

and that the deviations from the mean value of inputs and outputs are related according to

$$y_k - m_y = G(q)\ (u_k - m_u).$$

Furthermore, the spectral density of the output signal is given by

$$\Phi_y(\omega) = G(e^{\mathrm{j}\omega})\ \Phi_u(\omega)\ G^{\star}(e^{\mathrm{j}\omega}),$$

where $(\cdot)^{\star}$ denotes the complex conjugate transposed.

2.1.1.1 Convergence of random variables

At several points in this thesis, different concepts of convergence of random variables will be used. These concepts are summarized in the following definition (see [Doo53, Pap84] for a more detailed treatment of these topics).

Definition 2.1.2. Let $\{x_k\}$ be an indexed sequence of random vectors, and let $x_\star$ be a random vector of the same dimension. Then

- x_k is said to converge to $x_\star$ with probability 1, or almost surely, and denoted

  $$x_k \xrightarrow{\text{a.s.}} x_\star, \qquad \text{as} \qquad k \to \infty,$$

  if $P\{x_k \to x_\star\} = 1$. Here P is the underlying measure on the probability space $\{\Omega, \mathcal{F}, P\}$, that the random variables $\{x_k\}$ are defined on.

- x_k is said to converge to $x_\star$ in probability, denoted

$$x_k \xrightarrow{\mathcal{P}} x_\star, \qquad \text{as} \qquad k \to \infty,$$

if for any $\epsilon > 0$,

$$P\{|x_k - x_\star| > \epsilon\} \to 0, \qquad \text{as} \qquad k \to \infty,$$

where $|\cdot|$ denotes the Euclidean norm of a vector.

- x_k is said to weakly converge to $x_\star$ if the distribution function $F_{x_k}(x)$ of x_k converges to that of $x_\star$, say $F_{x_\star}(x)$, as $k \to \infty$. We also say that x_k converges in distribution to $x_\star$, and we denote

$$x_k \xrightarrow{\text{dist}} x_\star, \qquad \text{as} \qquad k \to \infty.$$

- x_k is said to converge to $x_\star$ in the mean square sense if

$$\mathbf{E}\left\{|x_k - x_\star|^2\right\} \to 0, \qquad \text{as} \qquad k \to \infty.$$

$\diamondsuit$

It can be proved that a.s. convergence implies convergence in probability, which in turn implies weak convergence; also that convergence in mean square sense implies convergence in probability [Doo53, Pap84].

2.1.1.2 Persistency of Excitation

In Chapter 6, the convergence properties of adaptive identification algorithms are analyzed. A fundamental concept in this analysis is that of 'persistency of excitation' of a signal. This property is required of the input signal to ensure the stability of the algorithms. Persistency of excitation essentially means that the signal is rich enough to excite all the modes of the system. A more precise meaning of this concept is given in the following definition (see for instance [And82, GS84, SS89]).

Definition 2.1.3. The sequence $\{u_k\}$ (with $u_k \in \mathbb{R}^n$) is said to be 'persistently exciting' of order M if there exists some integer p, and positive constants $\alpha > 0$ and $\beta > 0$ such that[3]

$$0 < \alpha I < \sum_{k=\ell}^{\ell+p} \varphi_k \varphi_k^T < \beta I < \infty, \qquad \text{for all } \ell, \tag{2.4}$$

[3] Given a symmetric matrix R, the notation $R > 0$ means that R is positive definite, i.e.

$$x^T R x > 0, \qquad \text{for all } x \neq 0,$$

where x is a column vector of compatible dimension. Analogously, given R_1 and R_2 symmetric matrices, the expression $R_1 > R_2$ (respectively, $R_1 \geq R_2$) indicates that $R_1 - R_2$ is positive definite (respectively, positive semidefinite).

where I is the identity matrix (of dimensions $nM \times nM$ in this case), and the vector φ_k is defined as

$$\varphi_k \triangleq [u_{k-1}^T, u_{k-2}^T, \cdots, u_{k-M}^T]^T.$$

$\diamondsuit$

In the case that $\{u_k\}$ is a stationary stochastic process, condition (2.4) can be simplified to

$$\lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} \varphi_k \varphi_k^T > 0.$$

If in addition $\{u_k\}$ is ergodic[4], the time average operator

$$\lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} (\cdot)$$

can be substituted by the ensemble average operator $\mathbf{E}\{\cdot\}$, so that the condition for the signal $\{u_k\}$ to be persistently exciting of order M becomes

$$\mathbf{E}\left\{\varphi_k \varphi_k^T\right\} > 0.$$

Sometimes, a frequency domain interpretation of this concept will be more useful. In the frequency domain, persistent excitation of a signal requires the positive definiteness of its spectral density matrix for a number of frequencies. The precise statement is given in the following proposition (see for instance [SS89]).

Proposition 2.1.1. A stationary, ergodic, stochastic process $\{u_k\}$ is persistently exciting of order M provided its spectral density matrix $\Phi_u(\omega)$ is positive definite for at least M distinct values of ω in the interval $(-\pi, \pi)$. $\square$

[4] A stationary stochastic process $\{u_k\}$ is said to be ergodic up to second order statistics if [Ros85]

$$\frac{1}{N} \sum_{k=1}^{N} u_k \quad \xrightarrow{\text{a.s.}} \quad \mathbf{E}\{u_k\}$$

$$\frac{1}{N} \sum_{k=1}^{N} u_{k+\tau} u_k \quad \xrightarrow{\text{a.s.}} \quad \mathbf{E}\{u_{k+\tau} u_k\}$$

as $N \to \infty$.

2.1.1.3 State-Space Descriptions

At times in this thesis we will consider internal representation of systems in the form of state-space realizations of the form

$$x_{k+1} = Ax_k + Bu_k + w_k, \tag{2.5}$$
$$y_k = Cx_k + Du_k + v_k, \tag{2.6}$$

$$\mathbf{E}\left\{\begin{bmatrix} w_k \\ v_k \end{bmatrix}\begin{bmatrix} w_s^T & v_s^T \end{bmatrix}\right\} = \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix}\delta_{ks} \tag{2.7}$$

where $x_k \in \mathbb{R}^{n_x}$, $y_k \in \mathbb{R}^m$, $u_k \in \mathbb{R}^n$, are respectively the state, the output and the input vectors at time k, and $A \in \mathbb{R}^{n_x \times n_x}$, $B \in \mathbb{R}^{n_x \times n}$, $C \in \mathbb{R}^{m \times n_x}$, and $D \in \mathbb{R}^{m \times n}$ are the state-feedback, the input, the output, and the input feedthrough matrices respectively [Kai80]. The vectors $w_k \in \mathbb{R}^{n_x}$, and $v_k \in \mathbb{R}^m$, represent the process and output measurement noise vectors at time k respectively, while the matrices $Q \in \mathbb{R}^{n_x \times n_x}$, $S \in \mathbb{R}^{n_x \times m}$, and $R \in \mathbb{R}^{m \times m}$ are the covariance matrices of these noise sequences. In most of the cases it will be assumed that w_k and v_k are zero mean, stationary, white noise vector sequences.

For obvious reasons, equations (2.5) and (2.6) are called *state equation* and *output equation*, respectively.

For deterministic systems (where both the process noise and the output measurement noise are identically zero), the state space description (2.5)-(2.7) reduces to

$$x_{k+1} = Ax_k + Bu_k, \tag{2.8}$$
$$y_k = Cx_k + Du_k. \tag{2.9}$$

An $m \times n$ rational transfer matrix $G(z)$ has an n_x-dimensional state-space realization (A, B, C, D), with $A \in \mathbb{R}^{n_x \times n_x}$, $B \in \mathbb{R}^{n_x \times n}$, $C \in \mathbb{R}^{m \times n_x}$, and $D \in \mathbb{R}^{m \times n}$, if

$$G(z) = C(zI - A)^{-1}B + D.$$

The above expression can be obtained from equations (2.8) and (2.9) by taking the Z-transform of both equations and eliminating the state variable.

The *observability* and *controllability* matrices associated with the system (2.8)-(2.9) are defined as

$$\mathcal{O} \triangleq [C^T, (CA)^T, \cdots, (CA^{n_x-1})^T]^T,$$
$$\mathcal{C} \triangleq [B, AB, \cdots, A^{n_x-1}B],$$

respectively. In a similar way the *extended* $(i, j > n_x)$ observability $\mathcal{O}_i$ and controllability $\mathcal{C}_j$ matrices are defined as

$$\mathcal{O}_i \triangleq [C^T, (CA)^T, \cdots, (CA^{i-1})^T]^T, \tag{2.10}$$
$$\mathcal{C}_j \triangleq [B, AB, \cdots, A^{j-1}B]. \tag{2.11}$$

Definition 2.1.4. Controllability: [Kai80] The pair (A, B) is said to be *controllable* if there exists an input sequence $\{u_k\}$ that takes the state of the system from any initial state $x_0 = x^0$ to any desired final state $x_k = x^f$ in a finite number of steps (k is finite). This concept should more properly be called *Controllability from the origin* or *Reachability* [Kai80], but with some abuse of terminology we will still refer to it as *Controllability*. A necessary and sufficient condition for the controllability of the pair (A, B) is that the controllability matrix $\mathcal{C}$ has full rank n_x. $\diamondsuit$

Definition 2.1.5. Observability: [Kai80] The pair (C, A) is said to be *observable* if for any initial state $x_0 = x^0$ there is a finite number of steps ℓ such that x_0 can be determined from the sequences u_k and y_k for $0 \leq k \leq \ell$. A necessary and sufficient condition for the observability of the pair (C, A) is that the observability matrix $\mathcal{O}$ has full rank n_x. $\diamondsuit$

Definition 2.1.6. Minimal Realization: A realization is said to be *minimal* if it has minimal dimension, that is if there exists no other realization with lower dimension. The dimension of a minimal realization is called the *McMillan degree* of the MIMO system. A necessary and sufficient condition for the minimality of a realization is that the pair (A, B) is controllable and the pair (C, A) is observable [Kai80]. The minimality of the realization (A, B, C, D) then implies that the matrices $\mathcal{O}_i$, $\mathcal{C}_j$ (with $i, j > n_x$), $\mathcal{O}$ and $\mathcal{C}$ have all rank n_x [Kai80]. $\diamondsuit$

The realization (A, B, C, D) in (2.8)-(2.9) uniquely defines the input-output properties of the system via

$$y_k = \sum_{\ell=0}^{\infty} g(\ell)\, u_{k-\ell},$$

where, as already mentioned, $g(\ell) \in \mathbb{R}^{m \times n}$ are the impulse response coefficients or Markov (matrix) parameters given by

$$g(\ell) = \begin{cases} D, & \ell = 0 \\ CA^{\ell-1}B, & \ell > 0 \end{cases}.$$

Based on the impulse response coefficients, the following *Impulse Response Block Hankel Matrix* can be constructed [HK66]

$$\mathcal{H}_{ij} \triangleq \begin{bmatrix} g(1) & g(2) & \cdots & g(j) \\ g(2) & g(3) & \cdots & g(j+1) \\ \vdots & \vdots & \ddots & \vdots \\ g(i) & g(i+1) & \cdots & g(i+j-1) \end{bmatrix}, \tag{2.12}$$

where $\mathcal{H}_{ij} \in \mathbb{R}^{im \times jn}$. It is not difficult to see that $\mathcal{H}_{ij}$ can be factorized as [Kai80]

$$\mathcal{H}_{ij} = \mathcal{O}_i \mathcal{C}_j. \tag{2.13}$$

Definition 2.1.7. Internal Stability:[5] [Kai80] We say that a realization (A, B, C)

[5] Internal stability is often called asymptotic stability.

is *(internally) stable* or *stable in the sense of Lyapunov*, if the solution of

$$x_{k+1} = Ax_k,$$

with initial condition x_0, tends to zero as $k \to \infty$, for arbitrary x_0. A necessary and sufficient condition for internal stability is that all the eigenvalues of A must lie strictly inside the unit circle, that is

$$|\lambda_i(A)| < 1,$$

where $\lambda_i(A)$ are the eigenvalues of A. ♢

If the system (2.8)-(2.9) is stable, the associated controllability Gramian $\mathcal{P}$ and observability Gramian $\mathcal{Q}$ are defined as the solution of the Lyapunov equations

$$\begin{aligned} A\mathcal{P}A^T + BB^T &= \mathcal{P}, \\ A^T\mathcal{Q}A + C^TC &= \mathcal{Q}, \end{aligned}$$

respectively.

Definition 2.1.8. Hankel Singular Values: [Glo84] Let (2.8)-(2.9) be a stable state-space realization of the transfer matrix $G(z)$, so that $G(z)$ can be written as

$$G(z) = C(zI - A)^{-1}B + D.$$

Then, the Hankel Singular values of $G(z)$ are defined as

$$\sigma_i^{\mathcal{H}}(G(z)) \triangleq \sqrt{\lambda_i(\mathcal{PQ})},$$

where $\mathcal{P}$ and $\mathcal{Q}$ are the controllability and observability Gramians, respectively. It is not difficult to show that the Hankel singular values of $G(z)$ are the singular values of the impulse response block Hankel matrix (2.12) [Glo84]. ♢

Definition 2.1.9. Balance Realizations: [PS82] A stable state-space realization is called *internally balanced*, or simply *balanced* if

$$\mathcal{P} = \mathcal{Q} = \Sigma,$$

where $\Sigma \triangleq \text{diag}\left\{\sigma_1^{\mathcal{H}}, \sigma_2^{\mathcal{H}}, \cdots, \sigma_{n_x}^{\mathcal{H}}\right\}$, $\sigma_1^{\mathcal{H}} \geq \sigma_2^{\mathcal{H}} \geq \cdots \geq \sigma_{n_x}^{\mathcal{H}}$, is a diagonal matrix with the Hankel singular values as diagonal elements. A stable state-space realization is called *input balanced* if

$$\mathcal{P} = I, \qquad \mathcal{Q} = \Sigma^2,$$

and is called *output balanced* if

$$\mathcal{P} = \Sigma^2, \qquad \mathcal{Q} = I.$$

♢

2.1.2 Some Hilbert (Hardy) spaces related to signals and systems

In this subsection, some basic concepts related to general Hilbert spaces are presented, and the particular Hilbert (Hardy) spaces associated with the classes of signals and systems used in this thesis are introduced. An advanced treatment of these topics can be found in the books [You88, Rud74].

Let us first quote the definition of a Hilbert space [You88, Rud74]:

> "A Hilbert space is an inner product space which is complete with respect to the metric induced by (the norm induced by) the inner product."

Here 'inner product space' means a linear vector space H with an inner product operation defined between two of its elements. This inner product is a real-valued function usually denoted[6] $\langle \cdot, \cdot \rangle$, that verifies the following properties:

1. $\langle x, y \rangle = \overline{\langle y, x \rangle}$, for all $x, y \in H$.
2. $\langle \alpha x + \beta y, u \rangle = \alpha \langle x, u \rangle + \beta \langle y, u \rangle$, for all $x, y, u \in H$ and scalar α, β.
3. $\langle x, x \rangle > 0$, and $\langle x, x \rangle = 0$ if and only if $x = \mathbb{O}$.

Where $\overline{(\cdot)}$ indicates complex conjugate, and where $\mathbb{O}$ is the null element in the space H. The inner product can be used to define the norm of an element of the space as follows:

$$\|x\| \triangleq \langle x, x \rangle^{1/2}, \qquad x \in H.$$

Sometimes the notation $\|\cdot\|_H$ will also be used to emphasize that it is the norm induced by the inner product in the space H. This norm can be used now to define a metric in the space measuring the distance between two elements:

$$d(x, y) \triangleq \|x - y\|, \qquad x, y \in H.$$

This concept of distance between two elements of the space can be used in turn to define 'convergence' of sequences of elements as follows:

Definition 2.1.10. A sequence $\{x(k)\}$ of elements of the Hilbert space H converges, with respect to the metric $d(x, y)$, to an element $x \in H$ if

$$\lim_{k \to \infty} d(x(k), x) = 0.$$

The element x is called the 'limit of the sequence', and we write without distinction

$$\lim_{k \to \infty} x(k) = x, \quad \text{or} \quad x(k) \to x.$$

$\diamondsuit$

[6] Sometimes, the notation $\langle \cdot, \cdot \rangle_H$ will also be used to emphasize that it is the inner product on the space H.

Completeness of the space in the metric induced by (the norm induced by) the inner product means that any convergent Cauchy sequence[7] of elements of the space converges to an element of the space.

Fundamental in this thesis will be the concepts of orthogonality and orthonormality of elements and subsets of a Hilbert space leading to the concept of 'orthonormal basis' . These concepts are summarized in the following definition:

Definition 2.1.11.

i. Two elements x, y in a Hilbert space H are said to be orthogonal, written $x \perp y$, if $\langle x, y \rangle = 0$.

ii. The element $x \in H$ is said to be orthogonal to the subset $S \subset H$, written $x \perp S$, if x is orthogonal to every $y \in S$.

iii. Two subsets $S, Q \subset H$ are orthogonal, $S \perp Q$, if $\langle x, y \rangle = 0$ for all $x \in S$ and $y \in Q$.

iv. For a given subset $S \subset H$, the set $S^{\perp}$ defined as

$$S^{\perp} \triangleq \{y \in H : y \perp x, \forall x \in S\}$$

is called the 'orthogonal complement of S'.

v. An element $x \in H$ is said to be 'normal' if $\|x\| = 1$.

vi. A set of elements $S = \{x_i\}$ in H is said to be an 'orthogonal set' if the elements of the set are orthogonal to each other. If in addition, for each $x_i \in S$, $\|x_i\| = 1$, the set is said to be 'orthonormal'. An important property of an orthogonal set is that its elements are linearly independent.

vii. An orthonormal set $S = \{x_i\}_{i=0}^{\infty}$ in an inner product space is said to be complete if its closed linear span[8] is the whole space. For the case of a Hilbert space H, there is a theorem [You88] establishing that an orthonormal set is 'complete' if and only if the only element in H which is orthogonal to each of the x_i is the zero element.

viii. A complete orthonormal set in a Hilbert space is called an **'orthonormal basis'**.

$\diamondsuit$

[7] A sequence $\{x(k)\}$ in a Hilbert space is said to be a Cauchy sequence if $d(x(k), x(\ell)) \to 0$ as $k, \ell \to \infty$. This means that for any $\epsilon > 0$ there exists an integer N_ϵ such that $d(x(k), x(\ell)) \leq \epsilon$ for any $k, \ell \geq N_\epsilon$.

[8] Let A be a set in the normed linear space E, $A \subseteq E$. The *closed linear span* of A, denoted Span $\{A\}$, is the intersection of all closed linear subspaces of E which contain A.

It is possible to show that every Hilbert space has an orthonormal basis. This fact is important because it allows a unique representation of any element of the space as an orthonormal series expansion in terms of the elements of the basis (Generalized Fourier Series). In other words, given an orthonormal basis $\{\mathcal{B}_k\}_{k=0}^{\infty}$ in a (separable [9, 10]) Hilbert space H, every element $x \in H$ can be represented as

$$x = \sum_{k=0}^{\infty} \langle x, \mathcal{B}_k \rangle \, \mathcal{B}_k, \tag{2.14}$$

where the equality has to be interpreted as convergence in the norm induced by the inner product[11], that is as

$$\lim_{n\to\infty} \left\| x - \sum_{k=0}^{n} \langle x, \mathcal{B}_k \rangle \, \mathcal{B}_k \right\| = 0.$$

It is clear that the representation in (2.14) is not useful in practical problems where only a finite number of terms can be handled. The solution is then to approximate x by the element $\widehat{x}$,

$$\widehat{x} = \min_{y \in X_p} \|x - y\|$$

which is the closest element to x belonging to the subspace X_p spanned by the first p elements of the orthonormal basis $\{\mathcal{B}_k\}_{k=0}^{p-1}$. By appealing to the Projection Theorem[12] it can be proved that $\widehat{x}$ is given by

$$\widehat{x} = \sum_{k=0}^{p-1} \langle x, \mathcal{B}_k \rangle \, \mathcal{B}_k.$$

[9] An inner product space is said to be separable if it contains a countable subset which is everywhere dense (see next footnote). All Hilbert spaces considered in this thesis are separable.

[10] A subset S of an inner product space H is everywhere dense if for every $x \in S$ and $\epsilon > 0$ there is a $y \in S$ such that $\|x - y\| < \epsilon$.

[11] When dealing with spaces of functions ($x(t) \in H$) a different interpretation of equation (2.14) is possible. Namely, one can interprete the partial sums

$$\sum_{k=0}^{N} \langle x(t), \mathcal{B}_k \rangle \, \mathcal{B}_k$$

as converging to the element $x(t) \in H$ for all values of t. This is called 'pointwise convergence'. The two concepts are different, and, in general, convergence in the norm does not necessarily imply pointwise convergence.

[12] **Projection Theorem:** Let S be a linear closed subspace in the Hilbert space H, and let $x \in H$ be a vector not in S. Then there exists a unique vector $y_0 \in S$ such that the distance from x to the subspace S is given by

$$d(x, S) = \|x - y_0\|.$$

Furthermore $(x - y_0) \perp S$.

The approximation error for this case lies in the orthogonal complement of X_p.

Some important properties of orthonormal sets in Hilbert spaces are summarized in the following proposition [You88, Rud74].

Proposition 2.1.2.

i. Let $\{\mathcal{B}_k\}_{k=0}^{p}$ be an orthonormal set in the Hilbert space H. Let $\beta_0, \cdots, \beta_p \in \mathbb{C}$, and let x be an element in H. Then

$$\left\| x - \sum_{k=0}^{p} \beta_k \mathcal{B}_k \right\|^2 = \|x\|^2 + \sum_{k=0}^{p} |\beta_k - \langle x, \mathcal{B}_k \rangle|^2 - \sum_{k=0}^{p} |\langle x, \mathcal{B}_k \rangle|^2.$$

ii. Let $\{\mathcal{B}_k\}_{k=0}^{\infty}$ be a countable orthonormal set in the Hilbert space H, and let x be an element in H. Then, the expansion coefficients, $\langle x, \mathcal{B}_k \rangle$, of $x \in H$, and the norm of x satisfy

$$\|x\|^2 \geq \sum_{k=0}^{\infty} |\langle x, \mathcal{B}_k \rangle|^2 , \tag{2.15}$$

which is known as *Bessel's inequality*. If in addition, the orthonormal set $\{\mathcal{B}_k\}_{k=1}^{\infty}$ is complete (i.e., an orthonormal basis), Bessel's inequality becomes an equality, namely

$$\|x\|^2 = \sum_{k=0}^{\infty} |\langle x, \mathcal{B}_k \rangle|^2 , \tag{2.16}$$

which is known as *Parseval's identity* (or *Parseval's Theorem*). The condition (2.16) is a necessary and sufficient condition for the orthonormal set to be an orthonormal basis. An alternative formulation is given by

$$\langle x, y \rangle = \sum_{k=0}^{\infty} \langle x, \mathcal{B}_k \rangle \overline{\langle y, \mathcal{B}_k \rangle}, \qquad \forall x, y \in H.$$

iii. Let $\{\mathcal{B}_k\}_{k=0}^{\infty}$ be an orthonormal basis in the Hilbert space H. Then the closed linear span of $\{\mathcal{B}_k\}_{k=0}^{\infty}$ is the whole space H.

□

In the following, some particular Hilbert and Hardy spaces that will be used in this thesis are introduced.

Let us consider first a Hilbert space related to the impulse response sequence of stable, causal, discrete-time systems. Let $\{g(k)\}$ denote the impulse response sequence of a stable, causal, discrete-time scalar system with input-output representation

$$y_k = G(q)u_k \triangleq \sum_{\ell=0}^{\infty} g(\ell) q^{-\ell} u_k. \tag{2.17}$$

The absolute summability of the impulse response sequence

$$\sum_{k=0}^{\infty} |g(k)| < \infty,$$

ensures the (BIBO[13]) stability of the system[14] [Kai80] and also implies the square summability of the sequence[15]. It is then natural to define the Hilbert space $\ell_2(\mathbb{N}_0)$ of square summable sequences with support in $\mathbb{N}_0$ (the set of non-negative integer numbers), equipped with the inner product

$$\langle f, g \rangle = \sum_{k \in \mathbb{N}_0} f(k)\overline{g(k)},$$

so that the impulse response sequences of all stable, causal, discrete-time systems belong to this space. Of course, not every element of this space can be associated with the impulse response of a stable, causal, discrete-time system, since square summability of the sequence does not necessarily imply its absolute summability. The space $\ell_2(\mathbb{N}_0)$ is a proper subspace of the Hilbert space $\ell_2(\mathbb{Z})$ of (two-sided) square summable sequences with support in $\mathbb{Z}$ (the set of integer numbers), with the same definition for the inner product (substituting the index set $\mathbb{N}_0$ by $\mathbb{Z}$).

Let us now turn our attention to a Hilbert space related to the transfer functions of stable, causal, discrete-time systems. First we recall the definition of the Discrete Fourier Transform (DFT) [OS89, PM92] of a sequence $\{f(k)\} \in \ell_2(\mathbb{Z})$

$$F(e^{\mathrm{j}\omega}) \triangleq \sum_{k=-\infty}^{\infty} f(k)e^{-\mathrm{j}\omega k}. \tag{2.18}$$

[13] Bounded-Input–Bounded Output

[14] **BIBO Stability:** A causal system is said to be externally stable or BIBO stable if a bounded input $u_k < M_u < \infty, 0 \leq k < \infty$ produces a bounded output $y_k < M_y < \infty, 0 \leq k < \infty$. A necessary and sufficient condition for BIBO stability is the absolute summability of the impulse response sequence [Kai80].

[15] **Absolute Summability $\Rightarrow$ Square Summability**

$$\begin{aligned}
\sum_{k=0}^{\infty} |g(k)| < \infty \quad &\Rightarrow \quad \sum_{k=0}^{\infty} |g(k)| \sum_{h=0}^{\infty} |g(h)| < \infty \Rightarrow \\
\sum_{k=0}^{\infty}\sum_{h=0}^{\infty} |g(k)|\,|g(h)| < \infty \quad &\Rightarrow \quad \sum_{k=0}^{\infty} |g(k)|^2 + \sum_{k=0}^{\infty}\sum_{\substack{h=0 \\ h\neq k}}^{\infty} |g(k)|\,|g(h)| < \infty \Rightarrow \\
&\Rightarrow \quad \sum_{k=0}^{\infty} |g(k)|^2 < \infty
\end{aligned}$$

It can be proved that the Discrete Fourier transform of a sequence in $\ell_2(\mathbb{Z})$ belongs to the Hilbert space $L_2(\mathbb{T})$ of Lebesgue square-integrable functions on the unit circle in the complex plane $\mathbb{T} \triangleq \{z : |z| = 1\}$ (or equivalently $\mathbb{T} \triangleq \left\{e^{\mathrm{j}\omega} : -\pi \leq \omega \leq \pi\right\}$), equipped with the inner product

$$\langle F, G \rangle \triangleq \frac{1}{2\pi} \int_{-\pi}^{\pi} F(e^{\mathrm{j}\omega}) \overline{G(e^{\mathrm{j}\omega})} \mathrm{d}\omega. \tag{2.19}$$

Recalling the definition of the 'two-sided' Z-transform of a sequence $\{f(k)\} \in \ell_2(\mathbb{Z})$ [OS89, PM92],

$$F(z) \triangleq \sum_{k=-\infty}^{\infty} f(k) z^{-k}, \tag{2.20}$$

it can be seen that the discrete Fourier transform (equation (2.18)) can be interpreted as the Z-transform computed on the unit circle $\mathbb{T}$. We can then give the following alternative z-domain expression for the inner product in $L_2(\mathbb{T})$[16]

$$\langle F, G \rangle \triangleq \frac{1}{2\pi \mathrm{j}} \oint_{\mathbb{T}} F(z) \overline{G(1/z)} \frac{\mathrm{d}z}{z}. \tag{2.21}$$

A proper subspace of $L_2(\mathbb{T})$ is the Hardy space[17] $H_2(\mathbb{T})$ of Lebesgue square-integrable functions on the unit circle $\mathbb{T}$ that are analytic outside the unit disc $\mathbb{D} \triangleq \{z : |z| < 1\}$. It is clear that the discrete Fourier transforms of the impulse response sequences of all stable, causal, discrete-time systems belong to this space[18]. With some abuse of terminology we will refer to this Hardy space as 'the space of all stable, causal, discrete-time transfer functions', since the discrete Fourier transform (respectively, the Z-transform) of the impulse response sequence is nothing else but the transfer function $G(e^{\mathrm{j}\omega})$ (respectively, $G(z)$) of the system[19].

An important property of Hilbert spaces is that all separable Hilbert spaces with the same cardinality are isometrically isomorphic to each other. This means that between any two separable Hilbert spaces (with the same cardinality) there exists a one to one mapping preserving norms (an isometry). For the spaces $\ell_2(\mathbb{Z})$ and $L_2(\mathbb{T})$, the isometry is the discrete Fourier transform. Parseval's identity then allows us to write

$$\langle f, g \rangle = \langle F, G \rangle .$$

[16] Equation (2.19) follows from equation (2.21) by definition of contour integral, with $z(\omega) = e^{\mathrm{j}\omega}$, $-\pi < \omega \leq \pi$, on the unit circle.

[17] $H_2(\mathbb{T})$ is a Hilbert space with the inner product in $L_2(\mathbb{T})$.

[18] Of course, not every element of this space can be interpreted as the discrete Fourier transform of the impulse response sequence of a stable, causal, discrete-time system.

[19] See also the previous footnote.

When dealing with identification of multivariable systems in chapter 5 we will be interested in a Hilbert space related to the transfer matrices of stable, causal, discrete-time MIMO systems. We will denote with $H_2^{m\times n}(\mathbb{T})$ the Hardy space of $(m \times n)$ matrices whose elements are functions of the complex variable z, belonging to $H_2(\mathbb{T})$. With some abuse of terminology we will refer to this space as 'the space of all stable, causal, discrete-time, $(m \times n)$ transfer matrices'. $H_2^{m\times n}(\mathbb{T})$ is a Hilbert space with the following definition for the inner product

$$\langle \mathcal{B}_\ell, \mathcal{B}_k \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} \mathsf{Tr}\left\{ \mathcal{B}_\ell(e^{\mathrm{j}\omega}) \mathcal{B}_k(e^{\mathrm{j}\omega})^\star \right\} \mathrm{d}\omega,$$

or equivalently

$$\langle \mathcal{B}_\ell, \mathcal{B}_k \rangle = \frac{1}{2\pi\mathrm{j}} \oint_{\mathbb{T}} \mathsf{Tr}\left\{ \mathcal{B}_\ell(z) \mathcal{B}_k(1/z)^\star \right\} \frac{\mathrm{d}z}{z}.$$

2.2 Prediction Error Methods

In this section, system identification methods based on minimization of prediction errors are reviewed. Usually these techniques are called Prediction Error Methods (PEM) (see for instance [Lju87, SS89] for a more detailed discussion).

2.2.1 Problem Formulation

To formalize the problem, let us consider a discrete-time linear time-invariant MIMO system with a general model structure given by

$$\mathcal{M}(\theta): \qquad y_k = G(q,\theta)\, u_k + H(q,\theta)\, e_k, \tag{2.22}$$

where y_k is the m-dimensional output vector, u_k is the n-dimensional input vector, and e_k is a sequence of m-dimensional, independent and identically distributed (iid) random variables with zero mean (i.e. white noise), and covariance matrix

$$\mathsf{E}\left\{ e_k e_s^T \right\} = R_e \delta(k-s),$$

where $\delta(k-s)$ stands for the Kronecker delta[20]. Furthermore, $G(q,\theta)$ and $H(q,\theta)$ are matrices of real, rational, stable, strictly proper[21] transfer functions parameterized by the p-dimensional parameter vector θ lying in the parameter

[20] $\delta(k-s) \triangleq \begin{cases} 1 & \text{for} \quad k = s \\ 0 & \text{for} \quad k \neq s \end{cases}$

[21] A rational transfer function is said to be strictly proper if the relative order (that is the difference between the degrees of the numerator and denominator polynomials) is strictly negative [Kai80].

space $D_{\mathcal{M}} \subset \mathbb{R}^p$. How the set $D_{\mathcal{M}}$ has to be defined will become clear later in the section.

We assume also that an N-point data set

$$Z^N = \{y_k, u_k : \ k = 1, \cdots, N\},$$

consisting of an input sequence $\{u_k\}$ and an output sequence $\{y_k\}$ is available for the purposes of estimating $G(q,\theta)$ and $H(q,\theta)$. We will denote by $\widehat{y}_k(\theta)$ a prediction of y_k given the data up to time $k-1$, and based on the parameter vector θ. In general the predictor can be a linear or a nonlinear filter applied to the data, and it can be constructed in various ways for any given model. It can be determined from the underlying system description, or from other considerations. For instance, a frequently used predictor determined from the model structure (2.22) is the so-called *mean square optimal one-step-ahead predictor*

$$\widehat{y}_k(\theta) = \left[1 - H^{-1}(q,\theta)\right] \ y_k + H^{-1}(q,\theta)G(q,\theta) \ u_k.$$

A more general linear predictor is given as

$$\widehat{y}_k(\theta) = F_1(q,\theta) \ y_k + F_2(q,\theta) \ u_k, \tag{2.23}$$

where the predictor filters $F_1(q,\theta)$ and $F_2(q,\theta)$ are such that $\widehat{y}_k(\theta)$ is a function of past data only.

Given a model structure and a predictor, the *prediction errors* can then be defined as

$$\epsilon_k(\theta) = y_k - \widehat{y}_k(\theta)\,. \tag{2.24}$$

The objective is then to find an estimate $\widehat{\theta}$ of θ that minimizes a given function $V_N(\theta)$ of the prediction errors. This function is called a criterion (loss function or cost function), and it is a scalar-valued function (typically positive) of all the prediction errors $\epsilon_1(\theta), \epsilon_2(\theta), \cdots, \epsilon_N(\theta)$, which will assess the performance of the predictor used. The criterion is minimized with respect to the parameter vector θ to choose the 'best' predictor among the class considered. A criterion which is often adopted is the quadratic one

$$V_N(\theta) = \frac{1}{N}\sum_{k=1}^{N} \mathsf{Tr}\left\{\epsilon_k(\theta)\epsilon_k^T(\theta)\right\}, \tag{2.25}$$

but the range of possibilities is wide. A more general expression for the criterion would be

$$V_N(\theta) = \frac{1}{N}\sum_{k=1}^{N} \upsilon\left(\epsilon_k^F(\theta)\right), \tag{2.26}$$

where $\upsilon(\cdot)$ is a scalar-valued function (typically positive) of some filtered version $\epsilon_k^F(\theta)$ of the prediction errors $\epsilon_k(\theta)$.

The estimate $\widehat{\theta}_N$ is then computed as the minimizing argument of the criterion (2.26), i.e.

$$\widehat{\theta}_N = \arg\min_{\theta \in D_{\mathcal{M}}} \{V_N(\theta)\} . \tag{2.27}$$

It is now clear that the set $D_{\mathcal{M}}$ (the parameter space) has to be defined as those values of θ for which the predictor is asymptotically stable.

Summarizing, a Prediction Error Method (PEM) can be described as follows:

- Choice of the model structure: This concerns the parameterization of the transfer matrices $G(q,\theta)$ and $H(q,\theta)$ as a function of θ.
- Choice of the predictor: This concerns the definition of the predictor. For example, if a linear predictor is specified as in equation (2.23), the user has to choose the prediction filters $F_1(q,\theta)$ and $F_2(q,\theta)$.
- Choice of the criterion: This concerns the choice of the scalar-valued function $\upsilon(\cdot)$ in the general criterion (2.26) which will assess the performance of the predictor.
- Computation of the parameter estimate $\widehat{\theta}_N$ that minimizes the criterion (2.26). Implicit in this step is the choice of the minimization technique.

Particular choices of the model structure, the predictor, the criterion, and the minimization technique, result in particular methods with specific names. For example, the well known Least-Squares, Maximum Likelihood, and Instrumental-Variables Methods can all be considered as Prediction Error Methods [Lju87, SS89].

2.2.2 Asymptotic Analysis

In this subsection, some results regarding the limiting properties of the estimated parameters as the number of data points tends to infinity are reviewed. These results are concerned with two main aspects; namely, consistency of the estimate, and estimation accuracy. Again, the interested reader is referred to [Lju87, SS89], for a thorough treatment of these topics.

We will denote by $\widehat{\theta}_N$ the parameter estimate based on N data points, that is, $\widehat{\theta}_N$ is the minimizing argument of the criterion $V_N(\theta)$.

For the analysis, the following basic assumptions will be made:

A1. The data $\{u_k, y_k\}$ are stationary processes.

A2. The input $\{u_k\}$ is persistently exciting.

A3. The Hessian $V_N''(\theta)$ is nonsingular locally around the minimum points of $V_N(\theta)$.

A4. The transfer matrices $G(q,\theta)$ and $H(q,\theta)$ are smooth (differentiable) functions of the parameter vector θ.

Part of the analysis will require the following additional assumption about the true system:

A5. The set[22]

$$D_T \triangleq \{\theta :\ G(q,\theta) = G(q); H(q,\theta) = H(q)\}$$

consisting of those parameter vectors for which the model structure gives an exact description of the true system consists of precisely one point. This point will be denoted by θ_0, which will be called the true parameter vector.

2.2.2.1 Asymptotic Estimate

We are interested here in the limit value to which the estimate $\widehat{\theta}_N$ converges as the number of data N tends to infinity. This analysis is related to the consistency of the estimation method, that is, with the issue of whether the estimates converge to the 'true parameter' (provided that this value exists) when increasing number of data are considered. The main result is summarized in the following Theorem [Lju87, SS89].

Theorem 2.2.1. Let the estimate $\widehat{\theta}_N$ be defined as in equation (2.27), with the general criterion as in equation (2.26), and suppose that assumptions A1 to A4 hold. Then, the criterion function $V_N(\theta)$ converges uniformly in $\theta \in D_{\mathcal{M}}$ to the limit function $V_\infty(\theta)$, i.e.

$$\sup_{\theta \in D_{\mathcal{M}}} |V_N(\theta) - V_\infty(\theta)| \xrightarrow{\text{a.s.}} 0 \qquad \text{as} \quad N \to \infty$$

where $V_\infty(\theta) = \lim_{N\to\infty} V_N(\theta)$. Moreover, the minimizing argument $\widehat{\theta}_N$ of $V_N(\theta)$ converges to a value $\theta_\star$ in the set D_C of minimizing arguments of $V_\infty(\theta)$. That is

$$\widehat{\theta}_N \xrightarrow{\text{a.s.}} \theta_\star \qquad \text{as} \quad N \to \infty, \quad \theta_\star \in D_C,$$

where D_C is defined as

$$D_C = \arg\min_{\theta \in D_{\mathcal{M}}} \{V_\infty(\theta)\} = \left\{\theta :\ \theta \in D_{\mathcal{M}}, V_\infty(\theta) = \min_{\theta' \in D_{\mathcal{M}}} V_\infty(\theta')\right\}.$$

□

[22] When the set D_T is not empty, the system is said to be 'system identifiable'. If in addition, D_T consists of only one element, the system is said to be 'parameter identifiable'.

The result implies that when the set D_T is empty, the asymptotic estimate will be 'biased' but it will give the best possible approximation of the system which is available in the model set. If the system is parameter identifiable (condition A5 holds), then the set D_T is not empty and consists of only one element. Under some weak assumptions on the data set, it is possible to show that in this case $D_C = D_T = \{\theta_0\}$, so that the estimate is strongly consistent.

2.2.2.2 Asymptotic distribution of the parameter estimate

We analyze here the limiting distribution of the parameter estimates. The following theorem shows that the distribution of the random variable

$$\sqrt{N}(\widehat{\theta}_N - \theta_\star),$$

where $\widehat{\theta}_N$ is the parameter estimate and $\theta_\star$ is the asymptotic estimate (as defined in Theorem 2.2.1), converges to a Gaussian distribution under weak assumptions [Lju87, SS89].

Theorem 2.2.2. Under the conditions of Theorem 2.2.1

$$\sqrt{N}(\widehat{\theta}_N - \theta_\star) \xrightarrow{\text{dist}} \mathcal{N}(0, P),$$

where

$$P = [V''_\infty(\theta_\star)]^{-1} \left[\lim_{N\to\infty} N\mathbf{E}\left\{ V'_N(\theta_\star)^T V'_N(\theta_\star) \right\} \right] [V''_\infty(\theta_\star)]^{-1}, \tag{2.28}$$

with $V'_N(\theta_\star)$ denoting the gradient of $V_N(\theta)$ computed at $\theta = \theta_\star$, and $V''_\infty(\theta_\star)$ denoting the Hessian of $V_\infty(\theta)$ computed at $\theta = \theta_\star$.

□

This result is important because it gives an expression for the asymptotic covariance matrix P (equation (2.28)), that can be used to quantify the estimation accuracy. The asymptotic covariance expression can also be used to derive confidence intervals for each particular estimate $\widehat{\theta}_N$ obtained from the data set [Lju87]. Unfortunately, the expression for the asymptotic covariance (2.28) requires the knowledge of the asymptotic estimate $\theta_\star$ which is unknown to the user. This problem can be solved by instead using an estimate of P. A simple estimate of P in equation (2.28) can be obtained by replacing $\theta_\star$ by $\widehat{\theta}_N$, and the expectation operator by the sample average.

2.2.3 Computational aspects

In general, the minimization of the criterion $V_N(\theta)$ cannot be done analytically. Only for the special case where the prediction error depends linearly on θ (which corresponds to the case of linear regressor) an analytic expression

can be found for the solution of the minimization problem. In most cases the minimization problem is nonlinear and nonconvex, and the solution must be obtained by using some numerical iterative search method. Some commonly used algorithms are the Newton-Raphson Algorithm and the Gauss-Newton Algorithm [SS89, Lju87]. The main problem with these iterative search algorithms is that convergence only to a local minimum of the criterion can be guaranteed. Usually, the way to find the global minimum is to run the algorithm from different initial conditions and then to compare the estimates [Lju87]. In general, this constitutes a computationally intensive procedure.

In this thesis, systems are parameterized using orthonormal bases that lead to linear regressor forms for which the problems of local minima are avoided.

3

Orthonormal Bases on the Unit Circle

In this chapter, the construction of several families of (scalar) rational orthonormal bases on the unit circle will be reviewed. The concept of reproducing kernel associated with the bases will be introduced, and its importance for the analysis of the approximating properties of the bases will be pointed out. The emphasis of the chapter will be on a particular family of orthonormal bases, namely the Orthonormal Bases with Fixed Poles, for which a minimal state space realization will be derived and a closed form expression for the reproducing kernel associated with them will be given. Furthermore, it is shown how families of (matrix) orthonormal bases for the space $H_2^{m\times n}(\mathbb{T})$ can be constructed from orthonormal bases on $H_2(\mathbb{T})$.

3.1 Introduction

In this chapter we study various families of rational orthonormal bases for the Hardy space $H_2(\mathbb{T})$ of functions analytic outside, and square integrable on the unit circle $\mathbb{T}$. The motivation for this is the practical utility of rational orthonormal bases for the approximation of elements of $H_2(\mathbb{T})$ by rational transfer functions [Wal35], even when the element is non rational. These approximating properties of rational orthonormal bases are exploited in this thesis in the context of identification of discrete-time, linear systems. As already mentioned, a main advantage of using orthonormal bases in an identification context is that prior information about the dominant dynamics of the system can be easily incorporated in the process of basis construction, and that the resulting model structures become linear in the parameters which simplifies the estimation problem.

It is not the intention of this chapter to give a complete survey on the area, which would be an overwhelming task. Instead, the purpose is to focus on what this author believes is relevant for the material in the subsequent chapters. In Section 3.2, the concept of *reproducing kernel* associated with the

space spanned by an orthonormal set is introduced and some of its properties are studied. Closed form expressions for the reproducing kernels will be fundamental for the quantification of the accuracy of the estimation we perform in Chapter 4.

In Section 3.3, a brief description of some 'standard' orthonormal bases on the unit circle such as the trigonometric bases (corresponding to the so-called FIR (Finite Impulse Response) model structures), Laguerre [Wah91b] bases and Kautz [Wah94b] bases is given. A more detailed description of more general orthonormal bases that allow prior information about several dominant dynamics of the system to be included in the identification process, and that have the more common FIR, Laguerre and Kautz bases as special cases, is given in Sections 3.4 and 3.5 where the Orthonormal Bases Generated from Inner Functions (OBGIF) [HbVB95, VHB95], and the Orthonormal Bases with Fixed Poles (OBFP) [NG94a, NG97] are respectively considered.

Finally, in Section 3.6 it is shown how families of orthonormal bases for the space of stable $(m \times n)$ transfer matrices $H_2^{m\times n}(\mathbb{T})$ can be constructed from orthonormal bases on $H_2(\mathbb{T})$.

3.2 Reproducing Kernels

As mentioned in the previous chapter, $H_2(\mathbb{T})$ is a Hilbert space when it is endowed with the inner product in $L_2(\mathbb{T})$, as defined in equations (2.19) or (2.21). An orthonormal basis $\{\mathcal{B}_k(z)\}_{k=0}^{\infty}$ in $H_2(\mathbb{T})$ is a complete orthonormal set. The orthonormality is reflected by the property

$$\langle \mathcal{B}_k, \mathcal{B}_\ell \rangle = \begin{cases} 1 & \text{for} \quad (k = \ell) \\ 0 & \text{for} \quad (k \neq \ell) \end{cases} ,$$

while the completeness is characterized by the fact that the closed linear span of the set is the whole space.

In this thesis, rational orthonormal bases on the unit circle are used to represent discrete-time linear systems for the purposes of identification from input-output measurements. The transfer function of the unknown system is modeled as a linear combination of the rational basis functions, and the identification is carried out by estimating a finite number of coefficients in this orthonormal expansion, using least squares techniques. In this context, the accuracy of the estimation is affected by two causes: the noise corruption of the measured data that generates the so-called *variance error*, and the parsimony of the model structure which is too simple to describe the real system, that results in the so-called *bias error*.

In [LY85], Ljung and Yuan show that, for the case of the standard trigonometric bases (or FIR model structure) $\{z^{-k}\}$, the variance error in the frequency response estimate can be approximated (for large model order and length of the

available data) by the product of the noise-to-signal ratio and the model order-to-data length ratio. Essential in the derivation of this result has been the observation that, due to the algebraic structure

$$\mathcal{B}_n(z)\mathcal{B}_m(z) = \mathcal{B}_{m+n}(z), \tag{3.1}$$

enjoyed by the trigonometric bases, the covariance matrix of the transfer function estimate has a Toeplitz structure, and then some classical results on asymptotic properties of Toeplitz matrices [GS58] can be exploited to carry out the analysis. Unfortunately, the Orthonormal Bases with Fixed Poles we consider in this thesis do not have the algebraic structure (3.1) and consequently the covariance matrices do not have a Toeplitz structure. A key rôle in the analysis of the estimation accuracy is then played by the reproducing kernel associated with the bases [Aro50, Dav75], since it allows to generalize the convergence results of Toeplitz matrices for the case in which the orthonormal structure is not the trigonometric one [NHG97b, NHG97a].

In the following, we introduce the concept of reproducing kernel, and describe some of its properties [Aro50, Dav75].

Definition 3.2.1 (Reproducing Kernel). The *reproducing kernel* associated with a Hilbert space X of functions on a set $\mathbb{S}$, is defined as the unique function $K_p(z,\mu)$ of the two variables $z, \mu \in \mathbb{S}$, that satisfies the following two conditions [Aro50]

i. For every $\mu \in \mathbb{S}$, $K_p(z,\mu)$, as a function of z, belongs to X.

ii. $K_p(z,\mu)$ has the *reproducing property*: for every function $G(z) \in X$ and every $\mu \in \mathbb{S}$

$$G(\mu) = \langle G(z), K_p(z,\mu) \rangle .$$

$\diamondsuit$

When the space X is spanned by a finite number of orthonormal basis functions $\{\mathcal{B}_k(z)\}_{k=0}^{p-1}$ with $z \in \mathbb{S}$, it is not difficult to prove that the reproducing kernel can be computed as

$$K_p(z,\mu) = \sum_{k=0}^{p-1} \mathcal{B}_k(z)\overline{\mathcal{B}_k(\mu)}. \tag{3.2}$$

To see this we have to check that conditions **i.** and **ii.** in Definition 3.2.1 are satisfied. That $K_p(z,\mu)$ in equation (3.2) belongs to X is obvious since X is spanned by the basis functions $\{\mathcal{B}_0(z), \mathcal{B}_1(z), \cdots, \mathcal{B}_k(z)\}$. It remains to check

the reproducing property ii. Let $G(z) \in X$, and $\mu \in \mathbb{S}$, then

$$\begin{aligned}
\langle G(z), K_p(z,\mu)\rangle &= \left\langle G(z), \sum_{k=0}^{p-1} \mathcal{B}_k(z)\overline{\mathcal{B}_k(\mu)} \right\rangle, \\
&= \sum_{k=0}^{p-1} \left\langle G(z), \mathcal{B}_k(z)\overline{\mathcal{B}_k(\mu)} \right\rangle, \\
&= \sum_{k=0}^{p-1} \langle G(z), \mathcal{B}_k(z)\rangle\, \mathcal{B}_k(\mu), \\
&= G(\mu). \qquad (3.3)
\end{aligned}$$

In passing to the last line we have used the fact that, since X is spanned by the functions $\{\mathcal{B}_0(z), \mathcal{B}_1(z), \cdots, \mathcal{B}_{p-1}(z)\}$, then any function $G(z) \in X$ has a unique representation of the form

$$G(z) = \sum_{k=0}^{p-1} \langle G(z), \mathcal{B}_k(z)\rangle\, \mathcal{B}_k(z).$$

The existence of the reproducing kernel (3.2) makes the associated Hilbert space a *reproducing kernel Hilbert space* (r.k.H.s.).

In this thesis, the space X is most commonly $H_2(\mathbb{T})$, and the set $\mathbb{S}$ is the open region outside the unit disk in the complex plane $\mathbb{E} \triangleq \{z \in \mathbb{C} : |z| > 1\}$. At times we will need to compute the reproducing kernel[1] on the unit circle $\mathbb{T}$, that is for $z = e^{\mathrm{j}\omega}$, and $\mu = e^{\mathrm{j}\sigma}$. In those cases we will use the shorthand $K_p(\omega, \sigma)$ to denote $K_p(e^{\mathrm{j}\omega}, e^{\mathrm{j}\sigma})$.

We will use equation (3.2) to derive closed form expressions for $K_p(z,\mu)$ for the particular bases. In the context of orthogonal polynomials [Sze59], these closed form expressions for $K_p(z,\mu)$ are known as 'Christoffel-Darboux' type formulas.

Let now $\{\mathcal{B}_k(z)\}_{k=0}^{\infty}$ be a complete orthonormal set in $H_2(\mathbb{T})$. By analogy with (3.2) we can define the function

$$K(z,\mu) \triangleq \sum_{k=0}^{\infty} \mathcal{B}_k(z)\overline{\mathcal{B}_k(\mu)}, \qquad (3.4)$$

for $z, \mu \in \mathbb{E}$. It is not difficult to prove [Reg95] that $K(z,\mu)$ has the reproducing property

$$G(\mu) = \langle G(z), K(z,\mu)\rangle, \qquad (3.5)$$

[1] Properly speaking, $H_2(\mathbb{T})$ is not a function space, but a space of equivalence classes of functions, and then a reproducing kernel is not defined in $H_2(\mathbb{T})$. With some abuse of terminology we will still call a function in $H_2(\mathbb{T})$ satisfying conditions **i.** and **ii.** in Definition 3.2.1, a *reproducing kernel*.

and that as a function of $\mu \in \mathbb{E}$, it belongs to $H_2(\mathbb{T})$. A closed form expression for $K(z,\mu)$ will be very useful in some of the developments of the following chapters. The following Lemma shows that for any complete orthonormal basis in $H_2(\mathbb{T})$, and $\mu \in \mathbb{E}$, $K(z,\mu)$ can be computed as

$$K(z,\mu) = \frac{z\overline{\mu}}{z\overline{\mu}-1}.$$

Lemma 3.2.1. *Let $\{\mathcal{B}_k(z)\}_{k=0}^{\infty}$ be a complete orthonormal set in $H_2(\mathbb{T})$, and let $K(z,\mu)$ be defined as in equation (3.4). Then, independently of the particular basis, $K(z,\mu)$ is given by*

$$K(z,\mu) = \frac{z\overline{\mu}}{z\overline{\mu}-1},$$

for $|z|>1, |\mu|>1$.

Proof: See Appendix 3.A. ■

In the following sections various families of rational orthonormal bases on the unit circle are introduced. For some of them, closed form expressions of the associated reproducing kernels $K_p(z,\mu)$ are derived.

3.3 FIR, Laguerre and Kautz Basis

3.3.1 FIR Basis

The most common orthonormal basis on $L_2(\mathbb{T})$ are the well known trigonometric or FIR basis, that corresponds to the choice

$$\mathcal{B}_k(z) = z^{-k}; \qquad k \geq 0. \tag{3.6}$$

The completeness of the basis in $L_2(\mathbb{T})$ is a standard result of classical Fourier series (the proof can be found for instance in [You88]). A direct calculation gives the following closed form expression for the reproducing kernel for this basis:

$$K_p(z,\mu) = \frac{(z\overline{\mu})^p - 1}{(z\overline{\mu})^p - (z\overline{\mu})^{p-1}}. \tag{3.7}$$

with $|z|>1$ and $|\mu|>1$.

As pointed out by several authors in different contexts (for instance in [GW90, Wah91b, WH93, LW93, NG97] in the context of system approximation and identification, or in [Wil93a, Pdd93, Oli94a, Oli95a, WZ96] in the

context of signal processing) the use of FIR model structures to represent systems with long (possibly infinite) impulse responses has the disadvantage that the number of terms in the series expansion necessary to provide an acceptable approximation of the system is high, and this may lead to poor accuracy in the estimated model. As a counterpart, and as it has already been mentioned, the analysis of the accuracy of the least squares estimation using FIR structures is very tractable by exploiting the algebraic structure (3.1) of the bases, leading to Toeplitz structures of the covariance matrices of the estimates, and then using known results on asymptotics of Toeplitz matrices [GS58, HN77, HW89].

3.3.2 Laguerre Basis

The use of Laguerre series in engineering applications has a long history, that can be traced back to the thirties with the work of Wiener and Lee (see for instance [Lee60]) on synthesis of electrical networks. Since that work, Laguerre bases have been used in many different areas, such as system approximation [Nur87, Mäk90a, Mäk90b, Par91, WAH], system identification [KP79, GW90, GW91, Mäk91, Wah91b, Wah94b], filter design [KP77, den93b, den93a, FD93, den94, Oli94b, Oli95c, Oli95a, Oli95b], and control applications [ZD88, ZBD88, ZDP90].

In the z-domain, the Laguerre bases are given by

$$\mathcal{B}_k(z) = \left(\frac{\sqrt{1-\xi^2}}{z-\xi}\right)\left(\frac{1-\xi z}{z-\xi}\right)^k \qquad ; k \geq 0 \tag{3.8}$$

where $\xi \in \mathbb{R}, |\xi| < 1$ is a free (real) parameter called the Laguerre coefficient, or Laguerre pole position. The orthonormality and completeness of the Laguerre basis in $H_2(\mathbb{T})$ follow from the fact that the bases are the Z-transform of the Laguerre sequences [Sze59] which are a complete orthonormal set in $\ell_2(\mathbb{N}_0)$ [Sze59], and the fact that $H_2(\mathbb{T})$ and $\ell_2(\mathbb{N}_0)$ are isometrically isomorphic (the isomorphism being the Z-transform).

As the reader can easily verify, the FIR model is a special case of the Laguerre structure corresponding to $\xi = 0$.

The following Lemma gives the closed form expression for the reproducing kernel of the Laguerre basis.

Lemma 3.3.1. *Let $\{\mathcal{B}_k(z)\}_{k=0}^{\infty}$ be the Laguerre basis as defined in (3.8). Then the closed form expression for the reproducing kernel associated with the basis is given by*

$$K_p(z,\mu) = \frac{1-\varphi_p(z)\overline{\varphi_p(\mu)}}{z\overline{\mu}-1}, \tag{3.9}$$

with $|z| > 1$ *and* $|\mu| > 1$, *and where the definition*

$$\varphi_p(z) \triangleq \left(\frac{1-\xi z}{z-\xi}\right)^p$$

has been used.

Proof: See Appendix 3.A. ■

If prior information about the dominant dynamics of the system to be approximated is available, then choosing the Laguerre coefficient close to the dominant pole will increase the rate of convergence of the Laguerre series expansion [Wah91b]. In this way, the number of terms needed to obtain an acceptable approximation will also be reduced. This property represents an advantage of the Laguerre bases when compared to the FIR structure where the possibility of incorporating 'a priori' information to accelerate the rate of convergence does not exist.

As pointed out in [WAH], highly resonant systems are very difficult to approximate with the Laguerre basis that only allows the incorporation of prior knowledge about non-resonant dominant dynamics. A more flexible structure that generalizes the Laguerre basis and is better suited for the approximation of systems with highly oscillatory impulse responses is the so-called 'two-parameter' Kautz basis [Kau52].

3.3.3 Kautz Basis

Since the work of Kautz [Kau52] on orthogonalization of a set of continuous exponentials, considerable research effort has been devoted to the study of applications of Kautz basis in system approximation [WAH], identification [Wah91a, LW93, Wah94b, Wah94a], and filter design [den93b, Oli94a, Oli95d, den96].

In the z-domain, the 'two-parameter' Kautz bases are given by [YH62, Bro65]

$$\mathcal{B}_k(z) = \begin{cases} \dfrac{\sqrt{(1-a^2)(1-c^2)}}{z^2-a(c+1)z+c}\left(\dfrac{cz^2-a(c+1)z+1}{z^2-a(c+1)z+c}\right)^{\frac{k-1}{2}} & ;k \quad \text{odd} \\ \dfrac{\sqrt{(1-c^2)}(z-a)}{z^2-a(c+1)z+c}\left(\dfrac{cz^2-a(c+1)z+1}{z^2-a(c+1)z+c}\right)^{\frac{k}{2}} & ;k \quad \text{even} \end{cases} \tag{3.10}$$

with $-1 < a < 1$, $-1 < c < 1$, and $k \geq 0$. The Laguerre structure is a special case of the Kautz one when the poles are real and equal (i.e. for $a^2(c+1)^2 = 4c$).

The condition on the poles for the completeness of the Kautz bases in $H_2(\mathbb{T})$ has been derived in [DD81] (see also [Oli94a, Wah94b]) .

For systems with several resonant dynamics, more general orthonormal bases allowing the incorporation of prior information about several modes would be

more desiderable. Examples of such more general basis are the Orthonormal Basis Generated by Inner Functions (OBGIF) introduced by Heuberger, Van den Hof and co-workers [HbVB95, VHB95], or the Orthonormal Basis with Fixed Poles (OBFP) studied by Ninness and co-workers in [NG94a, NG97]. The Kautz, Laguerre and FIR model structures are all special cases of these methods.

3.4 Orthonormal Basis Generated by Inner Functions

In a series of papers [HB90, HVB92, HVB93, VHB94a, VHB94b, HbVB95, VHB95], Van den Hof, Heuberger and co-workers show how an infinite set of orthonormal functions can be generated from a balanced realization of a square and inner transfer function. The bases are suited for the representation of systems with a wide range of dominant dynamics, since they allow the incorporation of prior information about a set of poles rather than one single pole. By choosing the poles of the bases closed to the actual poles, the speed of convergence of the orthonormal expansion can be increased.

Previous to the introduction of the orthonormal basis, we give the definition of an *inner function*.

Definition 3.4.1. Inner Function. A rational transfer function $G(z)$ is called *inner* if it is stable and satisfies

$$G(z^{-1})G(z) = 1.$$

That is, if it is stable and *all-pass*. ♢

The main result concerning the generation of orthonormal functions for the space $H_2(\mathbb{T})$ from an inner transfer function is summarized in the following theorem due to Van den Hof, Heuberger and co-workers [HbVB95].

Theorem 3.4.1. [HbVB95] *Let $G_x(z)$ be a scalar inner function with McMillan degree $n_x > 0$, having a minimal balanced realization (A, B, C, D). Denote*

$$V_k(z) = z(zI - A)^{-1}BG_x^k(z) \tag{3.11}$$

Then the sequence of scalar rational functions

$$\{\mathcal{B}_{i,k}(z)\} \triangleq \{e_i^T V_k(z)\}, \qquad (i = 1, \cdots, n_x); (k = 0, \cdots, \infty), \tag{3.12}$$

where e_i stands for the i-th Euclidean basis vector in $\mathbb{R}^{n_x}$, forms an orthonormal basis for the Hilbert space $H_2(\mathbb{T})$. Moreover, these orthonormal bases induce associated bases for the signal space $\ell_2(\mathbb{N}_0)$ of squared

summable sequences, through inverse z-transformation to the signal domain. Denoting

$$V_k(z) = \sum_{\ell=0}^{\infty} v_\ell^k z^{-\ell}$$

it follows that $\{e_i^T v_\ell^k\}$ with $(i = 1, \cdots, n_x)$; $(k = 0, \cdots, \infty)$ is an orthonormal basis for the signal space $\ell_2(\mathbb{N}_0)$.

Proof: See [HbVB95]. ■

Since the sequence $\{\mathcal{B}_{i,k}(z)\}$ is an orthonormal basis in $H_2(\mathbb{T})$, then any transfer function $G(z) \in H_2(\mathbb{T})$ has a unique series representation

$$G(z) = \sum_{k=0}^{\infty} \sum_{i=1}^{n_x} \theta_k^i \mathcal{B}_{i,k}(z) = \sum_{k=0}^{\infty} \sum_{i=1}^{n_x} \theta_k^i e_i^T V_k(z) = \sum_{k=0}^{\infty} L_k V_k(z) \tag{3.13}$$

where $L_k \triangleq [\theta_k^1, \theta_k^2, \cdots, \theta_k^{n_x}] \in \ell_2^{1\times n_x}(\mathbb{N}_0)$.

This series representation is schematically depicted in the diagram of Figure 3.1, where we have defined

$$V_0(q) \triangleq q(qI - A)^{-1}B.$$

It can be seen that this orthonormal family for the space of stable systems $H_2(\mathbb{T})$

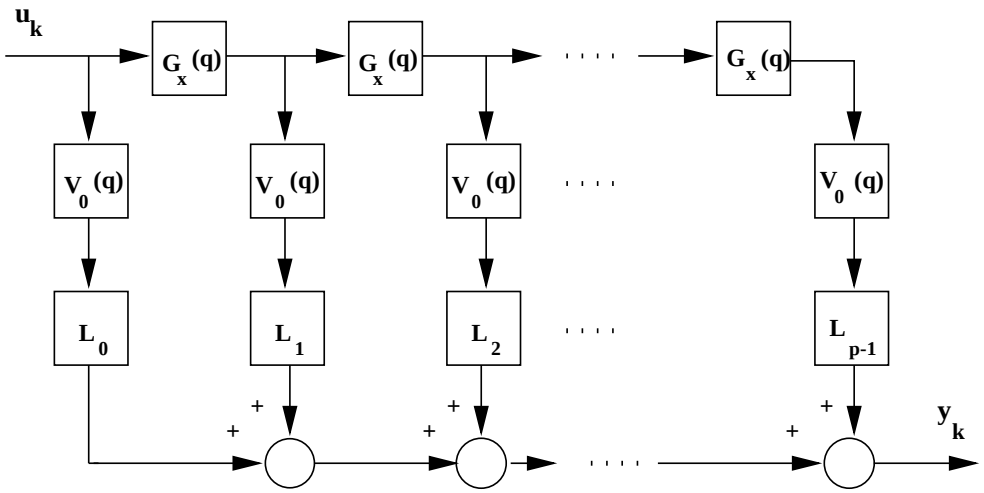

Figure 3.1: *Schematic Representation of the Series Expansion in terms of Orthonormal Bases Generated by Inner Functions*

can be generated by the cascade connection of identical balanced realizations of a stable all-pass filter $G_x(q)$, followed by low pass filters $L_k\, V_0(q)$.

It should be noted that these basis functions can incorporate system dynamics of any complexity by the appropiate choice of the poles of the inner function $G_x(z)$. For specific choices of $G_x(z)$ the classical FIR, Laguerre and 'two-parameter' Kautz orthonormal basis can be generated (see [VHB95, HbVB95] for details).

3.5 Orthonormal Basis with Fixed Poles

The limitation with the orthonormal bases generated from inner functions is that they only allow the incorporation of prior information about one set of poles which is cyclically repeated. In [NG97] it was shown that the set

$$\mathcal{B}_k(z) = \left(\frac{\sqrt{1-|\xi_k|^2}}{z-\xi_k} \right) \prod_{i=0}^{k-1} \left(\frac{1-\overline{\xi}_i z}{z-\xi_i} \right) \tag{3.14}$$

is a complete orthonormal set in $H_2(\mathbb{T})$, but allowing prior knowledge about an arbitrary number of modes $\{\xi_0, \xi_1, \cdots, \xi_{p-1}\} \in \mathbb{D}$ to be incorporated, without the restriction of periodic repetition. The reader can easily check that when all the poles are chosen at the origin ($\xi_k = 0$, for all k), then the construction (3.14) reduces to an FIR model structure, while for the choice $\xi_k = \xi \in \mathbb{R}, |\xi| < 1$, (3.14) reduces to the Laguerre basis.

As pointed out in [NG97], the construction (3.14) has to be modified to accomodate the case of resonant poles, since in this case the bases would have complex valued impulse responses, which would be inappropriate for their use in the representation of physical systems.

The idea in [NG97] is to still use the construction (3.14) for a complex pole, but including also the complex conjugate, and then replace these two basis functions by linear combinations of them in such a way that the resulting new basis functions are orthonormal to one another and to all the preceeding basis functions, and also have real-valued impulse responses.

To be more specific, suppose the n-th pole ξ_n is chosen as complex, and suppose that $\mathcal{B}_n$ is the corresponding basis function (with complex-valued impulse response) computed as in (3.14). Then, the $(n+1)$-th pole has to be chosen as the complex conjugate $\xi_{n+1} = \overline{\xi_n}$, leading to the basis function $\mathcal{B}_{n+1}$ (also with complex-valued impulse response). Now the basis functions $\mathcal{B}_n$ and $\mathcal{B}_{n+1}$ are replaced by the linear combinations

$$\begin{aligned} \mathcal{B}'_n &= \alpha \mathcal{B}_n + \beta \mathcal{B}_{n+1}, \\ \mathcal{B}''_n &= \alpha' \mathcal{B}_n + \beta' \mathcal{B}_{n+1}, \end{aligned}$$

where in order to preserve orthonormality, and to have real-valued impulse

response, the coefficients α, β, α' and β' must satisfy[2]

$$\begin{bmatrix}\alpha' \\ \beta'\end{bmatrix} = \frac{1}{(\xi_n - \overline{\xi_n})\sqrt{1-\mu^2}} \begin{bmatrix}\xi_n & -1 \\ -\overline{\xi_n} & 1\end{bmatrix} \begin{bmatrix}\mu & 1 \\ -1 & -\mu\end{bmatrix} \begin{bmatrix}1 & 1 \\ \overline{\xi_n} & \xi_n\end{bmatrix} \begin{bmatrix}\alpha \\ \beta\end{bmatrix} \tag{3.15}$$

where we have defined

$$\mu \triangleq \frac{\xi_n + \overline{\xi_n}}{1 + |\xi_n|^2}.$$

When only one fixed complex mode $\xi_k = \xi$ is considered and the following choice for α and β, satisfying (3.15), is made

$$\alpha = -\beta = \frac{\sqrt{(1-\mu^2)(1+|\xi|^2)}}{\overline{\xi} - \xi},$$

then the Kautz basis are obtained.

The derivation of the basis (3.14) can be done in different ways. For instance, in [Wal32, NG94b] it is shown how the basis can be derived using the Gram-Schmidt orthonormalization procedure [You88] on the set of functions $\{\mathcal{A}_k(z)\}_{k=0}^{p-1}$ defined as

$$\mathcal{A}_k(z) = \frac{1}{z - \xi_k},$$

with the same fixed poles $\{\xi_0, \xi_1, \cdots, \xi_{p-1}\}$.

In [NG97] it is shown that, under certain conditions on the poles $\{\xi_k\}$, the set $\{z\mathcal{B}_k(z)\}_{p=0}^{\infty}$ is a complete orthonormal set in the space $H_2(\mathbb{T})$, and that under the same conditions the set $\{\mathcal{B}_k(z)\}_{p=0}^{\infty}$ spans the subset of $H_2(\mathbb{T})$ of all rational, causal, stable transfer functions so that we can have the following unique series representation

$$G(z) = \sum_{k=0}^{\infty} \theta_k \mathcal{B}_k(z).$$

The necessary and sufficient condition for the completeness of the basis functions in the space $H_2(\mathbb{T})$ is given in the following theorem due to Ninness and co-workers [NG97].

Theorem 3.5.1. [NG97] *Consider the basis functions defined in (3.14). Then* Span $\{\mathcal{B}_k(e^{\mathrm{j}\omega})\}$ *is dense in* $H_2(\mathbb{T})$ *if and only if*

$$\sum_{k=0}^{\infty}(1 - |\xi_k|) = \infty. \tag{3.16}$$

[2] For the derivation of this result see [NG97].

Proof: See [NG97]. The proof is based on the fact that a set is dense in a Hilbert space if and only if there does not exist a non-zero element of the space that is orthogonal to all the elements in the set. A completely different proof, based on some results by Szász ([Cle63]), is given in [Oli95e]. ■

The series expansion of the system in terms of the orthonormal bases is schematically represented in Figure 3.2, where we have defined

$$A_k(q) \triangleq \frac{1-\overline{\xi_k}q}{q-\xi_k} \qquad \text{(all-pass section)},$$

$$F_k(q) \triangleq \frac{\sqrt{1-|\xi_k|^2}}{q-\xi_k} \qquad \text{(low-pass section)}.$$

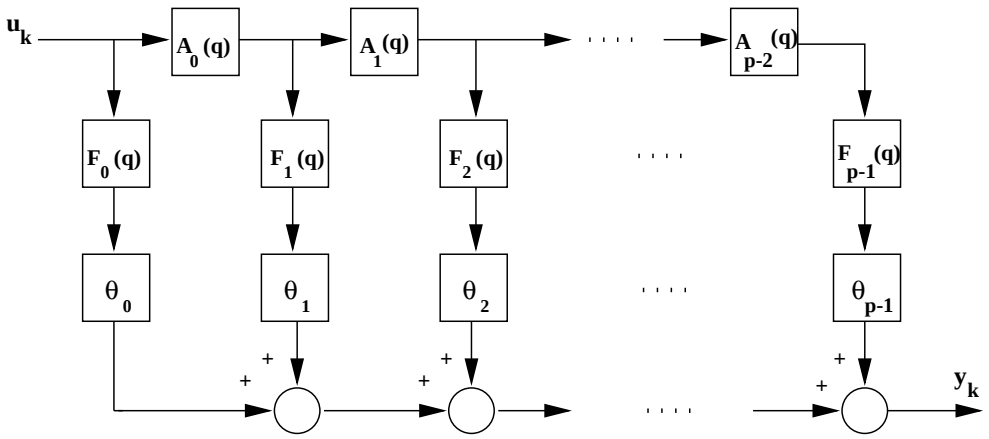

Figure 3.2: *Schematic Representation of the Series Expansion in terms of Orthonormal Bases with Fixed Poles*

It follows that this family of orthonormal bases can be generated by the cascade connection of (different) first order all-pass sections ($A_k(q)$), followed by a first order low pass section ($F_k(q)$).

Remark 3.5.1. Notice the similarity between Figures 3.2 and 3.1. In [Bod95] it is shown how a generalized family of orthonormal bases in $H_2(\mathbb{T})$ can be generated by the cascade connection of stable all-pass filters with input balance realizations. ♦

The Christoffel-Darboux formula for the reproducing kernel for these bases has been derived by Ninness and co-workers in [NHG97a], and is given in the following theorem.

Theorem 3.5.2. [NHG97a] *Define the Blaschke product-like quantity*

$$\varphi_p(z) \triangleq \prod_{k=0}^{p-1} \frac{1-\overline{\xi_k}z}{z-\xi_k}.$$

Then the Reproducing Kernel associated to the orthonormal bases with fixed poles $\{\mathcal{B}_k(z)\}$ *as defined in (3.14) can be expressed as*

$$K_p(z,\mu) = \frac{1 - \overline{\varphi_p(\mu)}\varphi_p(z)}{z\overline{\mu} - 1}, \tag{3.17}$$

with $|z| > 1$ *and* $|\mu| > 1$.

Proof: See [NHG97a]. An alternative proof, by induction, is given in Appendix 3.A. ■

3.5.1 A minimal state-space realization for the OBFP

In this subsection we derive a minimal state-space realization for the orthonormal expansion of the system in terms of the OBFP. The availability of a closed form expression for a minimal state-space realization will be important in the context of system identification since it will allow the implementation of reliable algorithms that can provide closed form estimates directly from input-output data. In addition, a state-space description of the identified system will be important for the purposes of simulation, control design, model order reduction [Glo84], or in the case of digital filters, for an actual hardware implementation of the system [RM87].

A state-space realization of the orthonormal expansion in terms of the OBFP can be obtained from the filter structure of Figure 3.2 by giving each (first order) all-pass section $A_k(q)$ and each (first order) low-pass section $F_k(q)$ a minimal state-space realization (which obviously will be one-dimensional) and then connecting these elemental blocks to obtain the state space realization of the compound system. It is clear that a realization obtained in this way will be in general non-minimal. Indeed, for the case of having p different poles, this procedure will yield a $2p$-dimensional state space realization, while a minimal realization should be of dimension p, since only p different modes are present on the system. The reason for this is that in the filter structure of Figure 3.2 each pole ξ_k appears in both $A_k(q)$ and $F_k(q)$ sections.

An (input-output) equivalent filter structure where each pole appears in only one first order section is represented in Figure 3.3. In the following we will show that the procedure described in the previous paragraph, but applied to the filter structure of Figure 3.3, will yield a minimal state space realization for the OBFP.

Let us consider first the elemental first order section represented in Figure 3.4. An equivalent representation is given in Figure 3.5, where the associated state variable x_k^ℓ has been defined. From this diagram it is straightforward to write the following (minimal) state space realization

$$x_{k+1}^\ell = \xi_\ell x_k^\ell + (1 - \xi_\ell\xi_{\ell-1})u_k^\ell, \tag{3.18}$$

$$\widetilde{y}_k^\ell = x_k^\ell - \xi_{\ell-1}u_k^\ell. \tag{3.19}$$

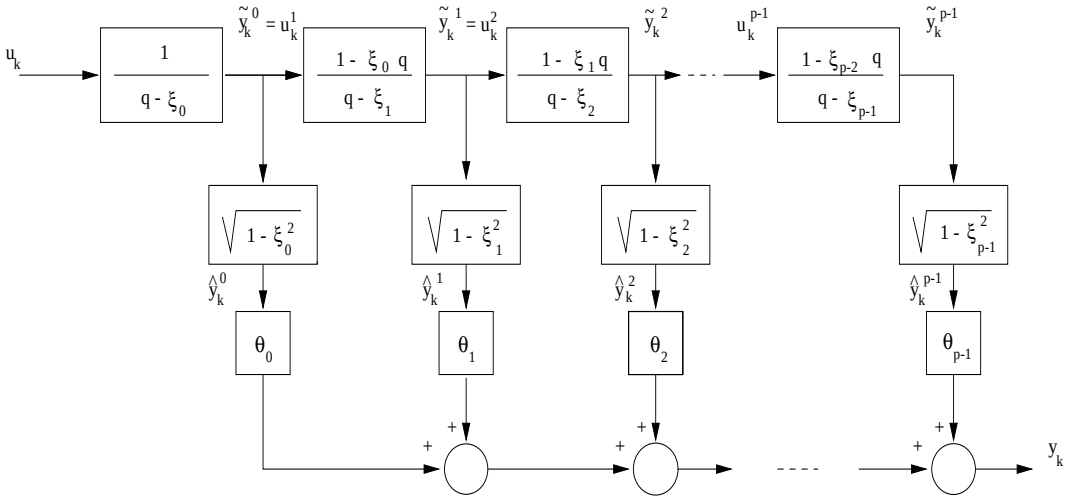

Figure 3.3: *An alternative filter structure for the Orthonormal Bases with Fixed Poles.*

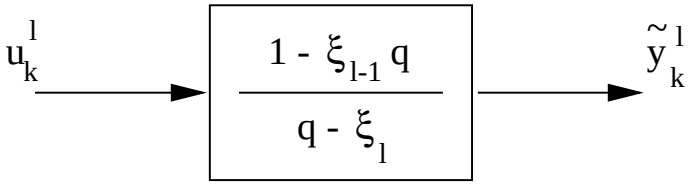

Figure 3.4: *Elemental first order section in the filter structure of Figure 3.3.*

The associated matrices of the state space description are $(A^\ell, B^\ell, C^\ell, D^\ell) = (\xi_\ell, (1-\xi_\ell\xi_{\ell-1}), 1, -\xi_{\ell-1})$.

Let us consider now the cascade of, say, three of these elemental sections. This cascade connection is represented in Figure 3.6.

Defining

$$x_k \triangleq \left(x_k^{\ell-1}, x_k^\ell, x_k^{\ell+1}\right)^T, \tag{3.20}$$

$$\widetilde{y}_k \triangleq \left(\widetilde{y}_k^{\ell-1}, \widetilde{y}_k^\ell, \widetilde{y}_k^{\ell+1}\right)^T, \tag{3.21}$$

the state space realization of the compound system can be written as

$$x_{k+1} = \begin{bmatrix} A^{\ell-1} & 0 & 0 \\ B^\ell C^{\ell-1} & A^\ell & 0 \\ B^{\ell+1}D^\ell C^{\ell-1} & B^{\ell+1}C^\ell & A^{\ell+1} \end{bmatrix} x_k + \begin{bmatrix} B^{\ell-1} \\ B^\ell D^{\ell-1} \\ B^{\ell+1}D^\ell D^{\ell-1} \end{bmatrix} u_k^{\ell-1},$$

$$\widetilde{y}_k = \begin{bmatrix} C^{\ell-1} & 0 & 0 \\ D^\ell C^{\ell-1} & C^\ell & 0 \\ D^{\ell+1}D^\ell C^{\ell-1} & D^{\ell+1}C^\ell & C^{\ell+1} \end{bmatrix} x_k + \begin{bmatrix} D^{\ell-1} \\ D^\ell D^{\ell-1} \\ D^{\ell+1}D^\ell D^{\ell-1} \end{bmatrix} u_k^{\ell-1}.$$

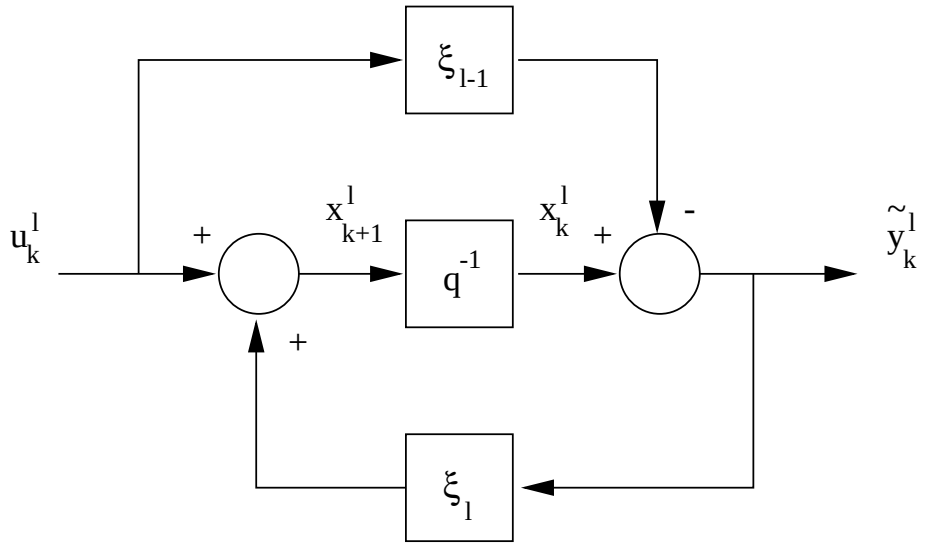

Figure 3.5: *An internal representation of the elemental first order section of Figure 3.4.*

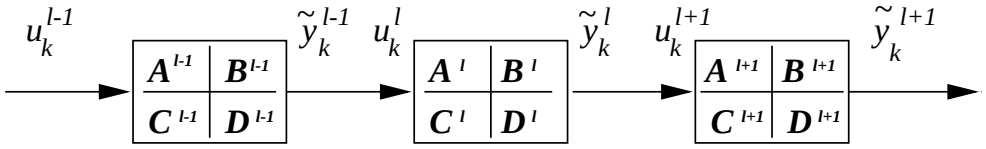

Figure 3.6: *Cascade connection of elemental first order sections.*

We are now able to write a state space realization for an arbitrary number of sections. For the case of p sections the minimal realization is given by

$$\begin{aligned} x_{k+1} &= Ax_k + Bu_k, \\ \widetilde{y}_k &= \widetilde{C}x_k + \widetilde{D}u_k, \end{aligned}$$

where now

$$\begin{aligned} x_k &\triangleq \left(x_k^0, x_k^1, \cdots, x_k^{p-1}\right)^T, \\ \widetilde{y}_k &\triangleq \left(\widetilde{y}_k^0, \widetilde{y}_k^1, \cdots, \widetilde{y}_k^{p-1}\right)^T, \end{aligned}$$

and the matrices $A, B, \widetilde{C}$ and $\widetilde{D}$ are given by

$$A = \begin{bmatrix} A^0 & 0 & \cdots & 0 \\ B^1C^0 & A^1 & \cdots & 0 \\ B^2D^1C^0 & B^2C^1 & \cdots & 0 \\ B^3D^2D^1C^0 & B^3D^2C^1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ B^{p-1}D^{p-2}\cdots D^1C^0 & B^{p-1}D^{p-2}\cdots D^2C^1 & \cdots & A^{p-1} \end{bmatrix}, \qquad (3.22)$$

$$B = \begin{bmatrix} B^0 \\ B^1 D^0 \\ B^2 D^1 D^0 \\ \vdots \\ B^{p-1} D^{p-2} \cdots D^1 D^0 \end{bmatrix} \quad ; \quad \widetilde{D} = \begin{bmatrix} D^0 \\ D^1 D^0 \\ D^2 D^1 D^0 \\ \vdots \\ D^{p-1} D^{p-2} \cdots D^1 D^0 \end{bmatrix},$$

$$\widetilde{C} = \begin{bmatrix} C^0 & 0 & \cdots & 0 \\ D^1 C^0 & C^1 & \cdots & 0 \\ D^2 D^1 C^0 & D^2 C^1 & \cdots & 0 \\ D^3 D^2 D^1 C^0 & D^3 D^2 C^1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ D^{p-1} D^{p-2} \cdots D^1 C^0 & D^{p-1} D^{p-2} \cdots D^2 C^1 & \cdots & C^{p-1} \end{bmatrix}, \tag{3.23}$$

and where for $\ell = 1, \cdots, p-1$

$$\begin{aligned} A^\ell &= \xi_\ell, \\ B^\ell &= 1 - \xi_\ell \xi_{\ell-1}, \\ C^\ell &= 1, \\ D^\ell &= -\xi_{\ell-1}. \end{aligned}$$

The case $\ell = 0$ (that corresponds to the first section in Figure 3.3) is different. It is not difficult to show that a minimal realization for this section is $(A^0, B^0, C^0, D^0) = (\xi_0, 1, 1, 0)$. This simplifies the expressions of matrices B and $\widetilde{D}$ to

$$B = (1, 0, \cdots, 0)^T, \tag{3.24}$$

and $\widetilde{D} = 0$, respectively.

Considering that for $\ell = 0, \cdots, p-1$ we have

$$\widehat{y}_k^\ell = \sqrt{1 - \xi_\ell^2}\ \widetilde{y}_k^\ell,$$

we can write

$$\widehat{y}_k = \Lambda\ \widetilde{y}_k = \Lambda \widetilde{C} x_k,$$

where the matrix Λ is defined as

$$\Lambda \triangleq \begin{bmatrix} \sqrt{1-\xi_0^2} & 0 & \cdots & 0 \\ 0 & \sqrt{1-\xi_1^2} & \cdots & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & \cdots & \sqrt{1-\xi_{p-1}^2} \end{bmatrix}. \tag{3.25}$$

Finally, the output y_k can be computed as

$$y_k = (\theta_0, \theta_1, \cdots, \theta_{p-1})\, \widehat{y}_k = \theta^T \Lambda \widetilde{C}\, x_k. \tag{3.26}$$

Summarizing, a **minimal state-space realization for the OBFP** is given by the quadruplet

$$(A, B, \theta^T \Lambda \widetilde{C}, 0),$$

where matrices $A, B, \widetilde{C}$, and Λ are defined in equations (3.22), (3.24), (3.23) and (3.25) respectively.

3.6 Orthonormal Bases on $H_2^{m\times n}(\mathbb{T})$

In this section we show how orthonormal bases for the space $H_2^{m\times n}(\mathbb{T})$ of stable and causal $(m \times n)$ transfer matrices, can be generated from orthonormal bases on $H_2(\mathbb{T})$. The result is summarized in the following theorem.

Theorem 3.6.1. *Let $\{\mathcal{B}_\ell(z)\}_{\ell=0}^{\infty}$ be a complete orthonormal set (i.e., an orthonormal basis) on the Hilbert space $H_2(\mathbb{T})$ with the usual definition for the inner product:*

$$\langle \mathcal{B}_\ell, \mathcal{B}_k \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} \mathcal{B}_\ell(e^{\mathsf{j}\omega}) \overline{\mathcal{B}_k(e^{\mathsf{j}\omega})} \mathrm{d}\omega = \frac{1}{2\pi \mathsf{j}} \oint_T \mathcal{B}_\ell(z) \overline{\mathcal{B}_k(1/z)} \frac{\mathrm{d}z}{z} \tag{3.27}$$

and let $\left\{\mathcal{B}_\ell^{ij}(z)\right\}_{\ell=0}^{\infty} (i = 1, \ldots, m; j = 1, \ldots, n)$, be a set of transfer matrices whose elements belong to $H_2(\mathbb{T})$, and which are defined as:

$$\mathcal{B}_\ell^{ij}(z) = \begin{array}{c} \begin{bmatrix} 0 & \ldots & 0 & \ldots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \ldots & \mathcal{B}_\ell(z) & \ldots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \ldots & 0 & \ldots & 0 \end{bmatrix} \leftarrow i \\ \qquad\qquad\quad \uparrow \\ \qquad\qquad\quad j \end{array} \tag{3.28}$$

Then $\left\{\mathcal{B}_\ell^{ij}(z)\right\}_{\ell=0}^{\infty}, (i = 1, \ldots, m; j = 1, \ldots, n)$ is a complete orthonormal set in $H_2^{m\times n}(\mathbb{T})$, with the usual definition for an inner product on a space of matrix valued functions

$$\left\langle \mathcal{B}_\ell^{ij}, \mathcal{B}_k^{st} \right\rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} \mathsf{Tr}\left\{ \mathcal{B}_\ell^{ij}(e^{\mathsf{j}\omega}) \mathcal{B}_k^{st}(e^{\mathsf{j}\omega})^\star \right\} \mathrm{d}\omega. \tag{3.29}$$

Proof: See Appendix 3.A. ■

Assuming that $\left\{\mathcal{B}_\ell^{ij}(z)\right\}_{\ell=0}^{\infty}, (i=1,\ldots,m; j=1,\ldots,n)$ is an orthonormal basis for the space $H_2^{m\times n}(\mathbb{T})$ then any stable causal tranfer matrix $G(z)$ in $H_2^{m\times n}(\mathbb{T})$ can be approximated by a linear combination of a finite number of elements of the orthonormal set, that is

$$G(z,\Theta) = \sum_{\ell=0}^{p-1}\sum_{i=1}^{m}\sum_{j=1}^{n}\theta_\ell^{ij}\mathcal{B}_\ell^{ij}(z). \tag{3.30}$$

By choosing $\{\mathcal{B}_\ell(z)\}_{\ell=0}^{\infty}$ as the orthonormal bases with fixed poles of Section 3.5, the previous Theorem allows us to construct a MIMO version of these OBFP. In the following subsection we derive a minimal state-space realization for these MIMO bases.

3.6.1 A Minimal State-Space Realization for the MIMO-OBFP

The corresponding MIMO version of the filter structure in Figure 3.3 is represented in Figure 3.7 for the case $m=n=2$.

A minimal state-space realization for these MIMO-OBFP can be derived from the filter structure in Figure 3.7 by applying a procedure similar to the one employed in the derivation of the minimal realization for the scalar case in Subsection 3.5.1.

We consider first the elemental first order section represented in Figure 3.8, where the second superscript in the input and output variables indicates the corresponding input channel.

An equivalent representation is given in Figure 3.9, where the associated state variable $x_k^{\ell,i}$ has been defined. The following (minimal) state-space realization can be derived from the diagram in Figure 3.9

$$\begin{aligned} x_{k+1}^{\ell,i} &= \xi_\ell x_k^{\ell,i} + (1-\xi_\ell\xi_{\ell-1})u_k^{\ell,i}, &(3.31)\\ \widetilde{y}_k^{\ell,i} &= x_k^{\ell,i} - \xi_{\ell-1}u_k^{\ell,i}. &(3.32)\end{aligned}$$

The associated matrices of the state space description are $(A^{\ell,i}, B^{\ell,i}, C^{\ell,i}, D^{\ell,i}) = (\xi_\ell, (1-\xi_\ell\xi_{\ell-1}), 1, -\xi_{\ell-1})$.

Defining

$$\begin{aligned} x_k^\ell &\triangleq \left[x_k^{\ell,1}, x_k^{\ell,2}, \cdots, x_k^{\ell,n}\right]^T,\\ \widetilde{y}_k^\ell &\triangleq \left[\widetilde{y}_k^{\ell,1}, \widetilde{y}_k^{\ell,2}, \cdots, \widetilde{y}_k^{\ell,n}\right]^T,\\ \widetilde{u}_k^\ell &\triangleq \left[\widetilde{u}_k^{\ell,1}, \widetilde{u}_k^{\ell,2}, \cdots, \widetilde{u}_k^{\ell,n}\right]^T,\end{aligned}$$

we can write in matrix form

$$\begin{aligned} x_{k+1}^\ell &= A^\ell x_k^\ell + B^\ell \widetilde{u}_k^\ell,\\ \widetilde{y}_k^\ell &= C^\ell x_k^\ell + D^\ell \widetilde{u}_k^\ell,\end{aligned}$$

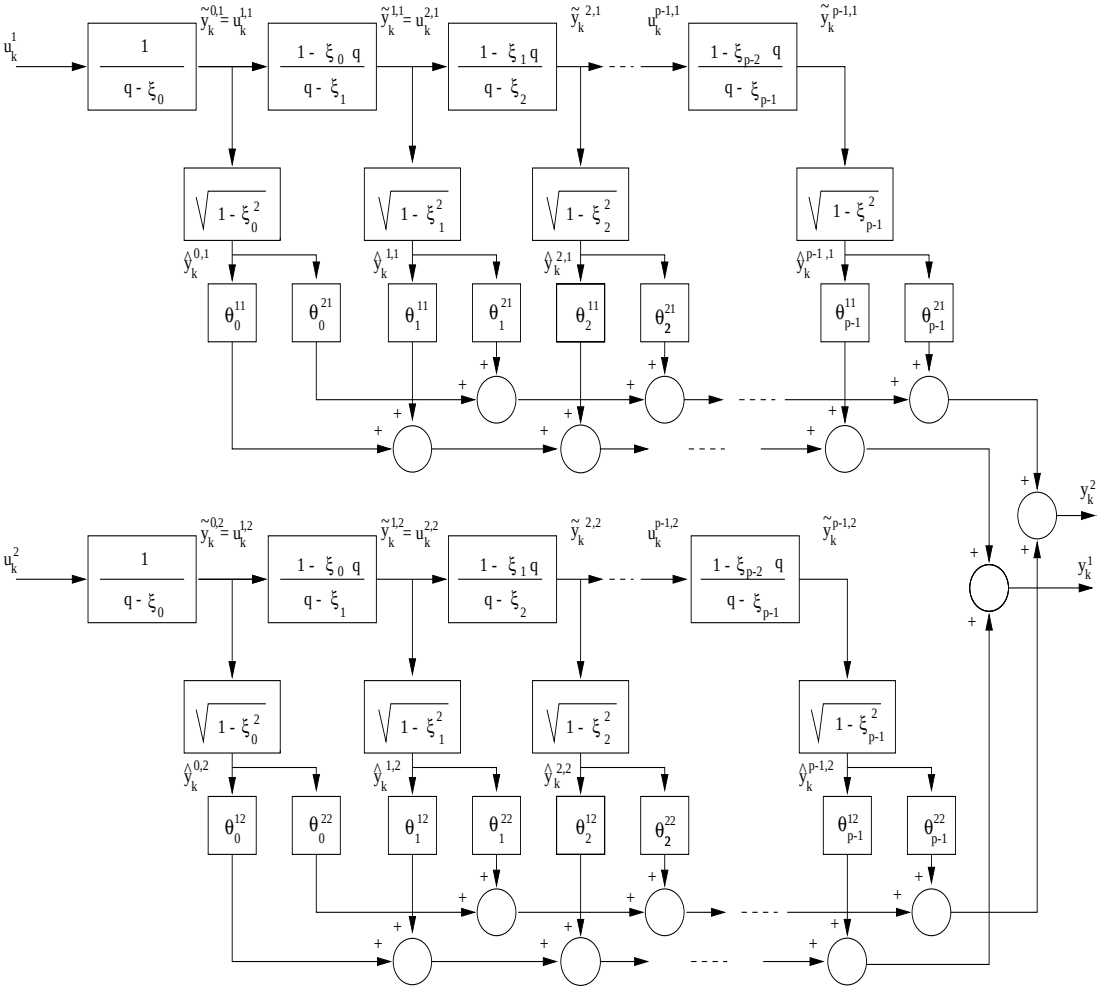

Figure 3.7: *Filter structure for the Multivariable Orthonormal Bases with Fixed Poles (Case $m = n = 2$).*

where

$$A^\ell \triangleq \begin{bmatrix} A^{\ell,1} & 0 & \cdots & 0 \\ 0 & A^{\ell,2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A^{\ell,n} \end{bmatrix} \quad , \quad B^\ell \triangleq \begin{bmatrix} B^{\ell,1} & 0 & \cdots & 0 \\ 0 & B^{\ell,2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & B^{\ell,n} \end{bmatrix},$$

$$C^\ell \triangleq \begin{bmatrix} C^{\ell,1} & 0 & \cdots & 0 \\ 0 & C^{\ell,2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & C^{\ell,n} \end{bmatrix} \quad , \quad D^\ell \triangleq \begin{bmatrix} D^{\ell,1} & 0 & \cdots & 0 \\ 0 & D^{\ell,2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & D^{\ell,n} \end{bmatrix}.$$

Considering the definitions of $A^{\ell,i}, B^{\ell,i}, C^{\ell,i}$ and $D^{\ell,i}$, the expressions for A^ℓ, B^ℓ, C^ℓ

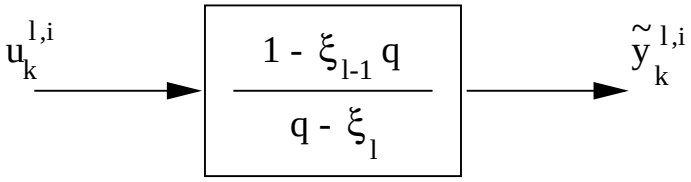

Figure 3.8: *Elemental first order section in the filter structure of Figure 3.7.*

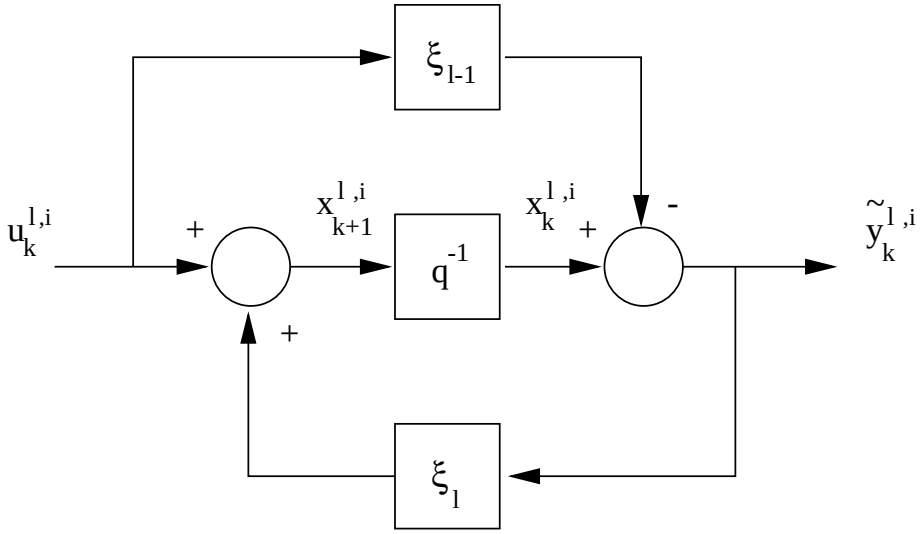

Figure 3.9: *An internal representation of the elemental first order section of Figure 3.8.*

and D^ℓ reduce to

$$\begin{aligned} A^\ell &= \xi_\ell I_n, \\ B^\ell &= (1 - \xi_\ell \xi_{\ell-1}) I_n, \\ C^\ell &= I_n, \\ D^\ell &= -\xi_{\ell-1} I_n. \end{aligned}$$

The filter structure of Figure 3.7 can now be considered as the cascade connection of p sections with minimal realizations given by the quadruplets $(A^\ell, B^\ell, C^\ell, D^\ell)$ with $\ell = 0, \cdots, p-1$. Defining

$$\begin{aligned} x_k &\triangleq \left[x_k^0, x_k^{\ell,1}, \cdots, x_k^{p-1}\right]^T, \\ \widetilde{y}_k &\triangleq \left[\widetilde{y}_k^0, \widetilde{y}_k^1, \cdots, \widetilde{y}_k^{p-1}\right]^T, \end{aligned}$$

we can then write

$$\begin{aligned} x_{k+1} &= Ax_k + Bu_k, \\ \widetilde{y}_k &= \widetilde{C}x_k + \widetilde{D}u_k, \end{aligned}$$

where the matrices $A, B, \widetilde{C}$ and $\widetilde{D}$ are given by

$$A = \begin{bmatrix} A^0 & 0 & \cdots & 0 \\ B^1C^0 & A^1 & \cdots & 0 \\ B^2D^1C^0 & B^2C^1 & \cdots & 0 \\ B^3D^2D^1C^0 & B^3D^2C^1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ B^{p-1}D^{p-2}\cdots D^1C^0 & B^{p-1}D^{p-2}\cdots D^2C^1 & \cdots & A^{p-1} \end{bmatrix}, \tag{3.33}$$

$$B = \begin{bmatrix} B^0 \\ B^1D^0 \\ B^2D^1D^0 \\ \vdots \\ B^{p-1}D^{p-2}\cdots D^1D^0 \end{bmatrix} \quad ; \quad \widetilde{D} = \begin{bmatrix} D^0 \\ D^1D^0 \\ D^2D^1D^0 \\ \vdots \\ D^{p-1}D^{p-2}\cdots D^1D^0 \end{bmatrix},$$

$$\widetilde{C} = \begin{bmatrix} C^0 & 0 & \cdots & 0 \\ D^1C^0 & C^1 & \cdots & 0 \\ D^2D^1C^0 & D^2C^1 & \cdots & 0 \\ D^3D^2D^1C^0 & D^3D^2C^1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ D^{p-1}D^{p-2}\cdots D^1C^0 & D^{p-1}D^{p-2}\cdots D^2C^1 & \cdots & C^{p-1} \end{bmatrix}. \tag{3.34}$$

The expressions for B and D can be further simplified by noting that $B^0 = I_n$ and $D^0 = 0$, so that

$$B = [I_n, \mathbb{O}_n, \cdots, \mathbb{O}_n]^T, \tag{3.35}$$

and $D = 0$, where $\mathbb{O}_n$ is an $(n \times n)$ matrix of zeros.

Now, considering that

$$\widehat{y}_k^{\ell,i} = \sqrt{1-\xi_\ell^2}\ \widetilde{y}_k^{\ell,i},$$

then

$$\widehat{y}_k^{\ell} = \sqrt{1-\xi_\ell^2}\ \widetilde{y}_k^{\ell},$$

so that

$$\widehat{y}_k \triangleq \begin{bmatrix} \widehat{y}_k^0 \\ \widehat{y}_k^1 \\ \vdots \\ \widehat{y}_k^{p-1} \end{bmatrix} = (\Lambda \otimes I_n)\ \widetilde{y}_k,$$

where matrix Λ is given by equation (3.25), and $\otimes$ stands for the Kronecker product (see Appendix A).

Finally, the j-th output is given by

$$\begin{aligned} y_k^j &= \sum_{\ell=0}^{p-1} \left[\theta_\ell^{j1}, \theta_\ell^{j2}, \cdots, \theta_\ell^{jn}\right] \widehat{y}_k^\ell, \\ &= \left[\theta_0^j, \theta_1^j, \cdots, \theta_{p-1}^j\right] \widehat{y}_k, \end{aligned}$$

where we have defined

$$\theta_\ell^j \triangleq \left[\theta_\ell^{j1}, \theta_\ell^{j2}, \cdots, \theta_\ell^{jn}\right].$$

Vectorizing we then have that the output vector is given by

$$y_k \triangleq \begin{bmatrix} y_k^1 \\ y_k^2 \\ \vdots \\ y_k^m \end{bmatrix} = \begin{bmatrix} \theta_0^1 & \theta_1^1 & \cdots & \theta_{p-1}^1 \\ \theta_0^2 & \theta_1^2 & \cdots & \theta_{p-1}^2 \\ \vdots & \vdots & \ddots & \vdots \\ \theta_0^m & \theta_1^m & \cdots & \theta_{p-1}^m \end{bmatrix} \widehat{y}_k = \Theta^T \widehat{y}_k = \Theta^T (\Lambda \otimes I_n) \widetilde{C} x_k.$$

Summarizing, a **minimal state-space realization for the MIMO-OBFP** is given by the quadruplet

$$(A, B, \Theta^T (\Lambda \otimes I_n) \widetilde{C}, 0),$$

where matrices $A, B, \widetilde{C}$, and Λ are defined in equations (3.33), (3.35), (3.34) and (3.25) respectively. The realization is of order np.

3.7 Conclusions

In this chapter, a review of rational orthonormal bases on the unit circle was presented. The concept of reproducing kernel associated with the bases was introduced, and the approximating properties of the more common orthonormal families in $H_2(\mathbb{T})$ were described. Emphasis was put on the Orthonormal Bases with Fixed Poles, since this family has more flexibility in the choice of the pole locations and encompasses the more common FIR, Laguerre and Kautz model structures in a unified formulation. For this family, a closed form expression for the reproducing kernel was given, and a minimal state-space realization was derived. This last issue is important in the context of system identification since it will allow the implementation of simple and reliable algorithms that will provide estimates in closed form (in state-space form) directly from input-output data.

Appendices

3.A Proofs for Chapter 3

Proof of Lemma 3.2.1 Let us compute first the inner product $\left\langle G(z), \frac{z\overline{\mu}}{z\overline{\mu}-1} \right\rangle$. We have

$$\begin{aligned}
\left\langle G(z), \frac{z\overline{\mu}}{z\overline{\mu}-1} \right\rangle &= \frac{1}{2\pi \mathrm{j}} \oint_{\mathbb{T}} G(z) \, \frac{\mu/z}{(\mu/z)-1} \, \frac{\mathrm{d}z}{z}, \\
&= \frac{1}{2\pi \mathrm{j}} \oint_{\mathbb{T}} G(z) \, \frac{1/z}{(1/z)-(1/\mu)} \, \frac{\mathrm{d}z}{z}, \\
&= \frac{1}{2\pi \mathrm{j}} \oint_{\mathbb{T}} \frac{G(1/z)}{z-(1/\mu)} \, \mathrm{d}z, \\
&= G(\mu),
\end{aligned}$$

where in passing to the second last line the variable substitution $z \mapsto 1/z$ was used, and where in passing to the last line use was made of Cauchy Residue Theorem. Now, from equation (3.5) we also have

$$\langle G(z), K(z,\mu) \rangle = G(\mu),$$

so that for all $G(z) \in H_2(\mathbb{T})$, and for all $|\mu| > 1$, we can write

$$\left\langle G(z), \left(\frac{z\overline{\mu}}{z\overline{\mu}-1} - K(z,\mu) \right) \right\rangle = 0,$$

what concludes the proof, since $G(z)$ is an arbitrary element of $H_2(\mathbb{T})$ and the only element of the space which is orthogonal to every other element of the space is the zero element. ■

Proof of Lemma 3.3.1 Considering the expression for the reproducing kernel in (3.2) we can write

$$\begin{aligned}
K_p(z,\mu) &= \sum_{k=0}^{p-1} \mathcal{B}_k(z)\overline{\mathcal{B}_k(\mu)}, \\
&= \sum_{k=0}^{p-1} \frac{(1-\xi^2)}{(z-\xi)(\overline{\mu}-\xi)} \left(\frac{1-\xi z}{z-\xi}\right)^k \left(\frac{1-\xi\overline{\mu}}{\overline{\mu}-\xi}\right)^k, \\
&= \frac{(1-\xi^2)}{(z-\xi)(\overline{\mu}-\xi)} \sum_{k=0}^{p-1} \left(\frac{(1-\xi z)(1-\xi\overline{\mu})}{(z-\xi)(\overline{\mu}-\xi)}\right)^k, \\
&= \frac{(1-\xi^2)}{(z-\xi)(\overline{\mu}-\xi)} \frac{1-\left(\frac{(1-\xi z)(1-\xi\overline{\mu})}{(z-\xi)(\overline{\mu}-\xi)}\right)^p}{1-\frac{(1-\xi z)(1-\xi\overline{\mu})}{(z-\xi)(\overline{\mu}-\xi)}}, \\
&= \frac{1-\left(\frac{(1-\xi z)(1-\xi\overline{\mu})}{(z-\xi)(\overline{\mu}-\xi)}\right)^p}{z\overline{\mu}-1},
\end{aligned}$$

where in passing to the fourth line use was made of the identity

$$\sum_{k=0}^{p-1} x^k = \frac{1-x^p}{1-x}. \tag{3.A.1}$$

Now the result follows by defining

$$\varphi_p(z) \triangleq \left(\frac{1-\xi z}{z-\xi}\right)^p.$$

■

Proof of Theorem 3.5.2 The proof proceeds by induction. We first prove that equation (3.17) holds for $p=1$. Then, assuming that (3.17) holds for $p=q-1$, we prove that it holds also for $p=q$, which concludes the proof.

Let $p=1$, then

$$\begin{aligned}
K_p(z,\mu) = \sum_{k=0}^{p-1} \overline{\mathcal{B}_k(\mu)}\mathcal{B}_k(z) &= \overline{\mathcal{B}_0(\mu)}\mathcal{B}_0(z), \\
&= \frac{1-|\xi_0|^2}{(z-\xi_0)(\overline{\mu}-\overline{\xi_0})}.
\end{aligned} \tag{3.A.2}$$

On the other hand, for $p = 1$

$$\begin{aligned}\frac{1-\overline{\varphi_p(\mu)}\varphi_p(z)}{z\overline{\mu}-1} &= \frac{1-\overline{\varphi_1(\mu)}\varphi_1(z)}{z\overline{\mu}-1},\\ &= \frac{1-\dfrac{(1-\overline{\xi_0}z)(1-\xi_0\overline{\mu})}{(z-\xi_0)(\overline{\mu}-\overline{\xi_0})}}{z\overline{\mu}-1},\\ &= \frac{1-|\xi_0|^2}{(z-\xi_0)(\overline{\mu}-\overline{\xi_0})}. \qquad (3.A.3)\end{aligned}$$

Therefor, the result holds for $p = 1$.

Let us now assume that equation (3.17) holds for $p = q-1$, that is

$$\sum_{k=0}^{q-2}\overline{\mathcal{B}_k(\mu)}\mathcal{B}_k(z) = \frac{1-\overline{\varphi_{q-1}(\mu)}\varphi_{q-1}(z)}{z\overline{\mu}-1}.$$

Then

$$\begin{aligned}\sum_{k=0}^{q-1}\overline{\mathcal{B}_k(\mu)}\mathcal{B}_k(z) &= \overline{\mathcal{B}_{q-1}(\mu)}\mathcal{B}_{q-1}(z) + \sum_{k=0}^{q-2}\overline{\mathcal{B}_k(\mu)}\mathcal{B}_k(z),\\ &= \overline{\mathcal{B}_{q-1}(\mu)}\mathcal{B}_{q-1}(z) + \frac{1-\overline{\varphi_{q-1}(\mu)}\varphi_{q-1}(z)}{z\overline{\mu}-1},\\ &= \frac{(1-|\xi_{q-1}|^2)\overline{\varphi_{q-1}(\mu)}\varphi_{q-1}(z)}{(\overline{\mu}-\overline{\xi_{q-1}})(z-\xi_{q-1})} + \frac{1-\overline{\varphi_{q-1}(\mu)}\varphi_{q-1}(z)}{z\overline{\mu}-1},\\ &= \frac{1}{z\overline{\mu}-1}\left(1-\frac{1-\overline{\mu}\xi_{q-1}}{\overline{\mu}-\overline{\xi_{q-1}}}\overline{\varphi_{q-1}(\mu)}\frac{1-z\overline{\xi_{q-1}}}{z-\xi_{q-1}}\varphi_{q-1}(z)\right),\\ &= \frac{1-\overline{\varphi_q(\mu)}\varphi_q(z)}{z\overline{\mu}-1}, \qquad (3.A.4)\end{aligned}$$

where in passing to the third line use was made of the identity

$$\mathcal{B}_q(z) \triangleq \frac{\sqrt{1-|\xi_q|^2}}{z-\xi_q}\prod_{k=0}^{q-1}\frac{1-\overline{\xi_k}z}{z-\xi_k} = \frac{\sqrt{1-|\xi_q|^2}}{z-\xi_q}\varphi_q(z).$$

Hence, equation (3.17) holds also for $p = q$. This completes the proof. ■

Proof of Theorem 3.6.1 The proof proceeds in two steps. First, it is shown that the set $\left\{\mathcal{B}_\ell^{ij}(z)\right\}$ is orthonormal with respect to the inner product in $H_2^{m\times n}(\mathbb{T})$, and then, that it is complete in that space.

i. **Orthonormality:**
It is clear that the matrices in the set $\left\{\mathcal{B}_\ell^{ij}(z)\right\}$ are linearly independent.

We will use the definition of the inner product in equation (3.29) to prove that $\{\mathcal{B}_\ell^{ij}(z)\}$ is an orthonormal set. Substituting $z = e^{\mathrm{j}\omega}$, equation (3.29) can be written as:

$$\langle \mathcal{B}_\ell^{ij}, \mathcal{B}_k^{st} \rangle = \frac{1}{2\pi \mathrm{j}} \oint_T \mathsf{Tr}\left\{ \mathcal{B}_\ell^{ij}(z) \mathcal{B}_k^{st}(1/z)^\star \right\} \frac{\mathrm{d}z}{z}. \tag{3.A.5}$$

In order to prove that the matrices are orthonormal we have to check that the following conditions hold

$$\langle \mathcal{B}_\ell^{ij}, \mathcal{B}_k^{st} \rangle = \begin{cases} 0 & \text{if} \quad \mathcal{B}_\ell^{ij} \neq \mathcal{B}_k^{st} \\ 1 & \text{if} \quad \mathcal{B}_\ell^{ij} = \mathcal{B}_k^{st} \end{cases} \tag{3.A.6}$$

Let us consider first the inner product $\langle \mathcal{B}_\ell^{ij}, \mathcal{B}_\ell^{ij} \rangle$. We have

$$\begin{aligned} \langle \mathcal{B}_\ell^{ij}, \mathcal{B}_\ell^{ij} \rangle &= \frac{1}{2\pi \mathrm{j}} \oint_T \mathsf{Tr}\left\{ \mathcal{B}_\ell^{ij}(z) \mathcal{B}_\ell^{ij}(1/z)^\star \right\} \frac{\mathrm{d}z}{z}, \\ (\text{by definition of } \mathcal{B}_\ell^{ij}) &= \frac{1}{2\pi \mathrm{j}} \oint_T \mathcal{B}_\ell(z) \overline{\mathcal{B}_\ell(1/z)} \frac{\mathrm{d}z}{z}, \\ (\text{by the orthonormality of } \{\mathcal{B}_\ell\}) &= 1. \end{aligned} \tag{3.A.7}$$

Now, let us consider the inner product $\langle \mathcal{B}_\ell^{ij}, \mathcal{B}_k^{st} \rangle$ for the case $i \neq s$ for any j, t, ℓ, k, or $j \neq t$ for any i, s, ℓ, k. We have

$$\begin{aligned} \langle \mathcal{B}_\ell^{ij}, \mathcal{B}_k^{st} \rangle &= \frac{1}{2\pi \mathrm{j}} \oint_T \mathsf{Tr}\left\{ \mathcal{B}_\ell^{ij}(z) \mathcal{B}_k^{st}(1/z)^\star \right\} \frac{\mathrm{d}z}{z}, \\ (\text{by definition of } \mathcal{B}_\ell^{ij}) &= 0. \end{aligned} \tag{3.A.8}$$

Finally, let us consider the case $i = s, j = t$ and $\ell \neq k$. We have

$$\langle \mathcal{B}_\ell^{ij}, \mathcal{B}_k^{st} \rangle = \frac{1}{2\pi \mathrm{j}} \oint_T \mathsf{Tr}\left\{ \mathcal{B}_\ell^{ij}(z) \mathcal{B}_k^{st}(1/z)^\star \right\} \frac{\mathrm{d}z}{z},$$

$$(\text{by definition of } \mathcal{B}_\ell^{ij}) = \frac{1}{2\pi \mathrm{j}} \oint_T \mathcal{B}_\ell(z) \overline{\mathcal{B}_k(1/z)} \frac{\mathrm{d}z}{z}, \tag{3.A.9}$$

$$(\text{by the orthonormality of } \{\mathcal{B}_\ell\}) = 0. \tag{3.A.10}$$

Hence, we can conclude that $\{\mathcal{B}_\ell^{ij}(z)\}$ is an orthonormal set in the space $H_2^{m\times n}(\mathbb{T})$.

ii. Completeness:
We use the idea that an orthonormal set in a Hilbert space H is complete if and only if the only element of H which is orthogonal to every element of the set is the null element. Suppose, in order to obtain a contradiction, that there exist a nonzero matrix $F(z) = (f_{ij}(z)) \in H_2^{m\times n}(\mathbb{T})$ which is orthonormal to every element of the set $\{\mathcal{B}_\ell^{ij}(z)\}$. Suppose also, without

loss of generality, that the entry $f_{st}(z)$ of $F(z)$ is a nonzero function. Then, the inner product of F with any one of the matrices in $\left\{\mathcal{B}_\ell^{ij}(z)\right\}$ is given by

$$\begin{aligned}
\left\langle F, \mathcal{B}_k^{st} \right\rangle &= \frac{1}{2\pi \mathsf{j}} \oint_T \mathsf{Tr}\left\{F(z)\mathcal{B}_k^{st}(1/z)^\star\right\} \frac{\mathrm{d}z}{z}, \\
(\text{by definition of } \mathcal{B}_\ell^{ij}) &= \frac{1}{2\pi \mathsf{j}} \oint_T f_{st}(z)\overline{\mathcal{B}_k(1/z)}\frac{\mathrm{d}z}{z}, \qquad (3.A.11) \\
&= 0, \qquad (3.A.12)
\end{aligned}$$

where the last equality should hold for each k, which means that there exist a nonzero function, namely $f_{st}(z)$, which is orthogonal to every element in $\{\mathcal{B}_\ell(z)\}$, which represents a contradiction, since the set $\{\mathcal{B}_\ell(z)\}$ is complete by hypothesis. Hence, we can conclude that the orthonormal set $\left\{\mathcal{B}_\ell^{ij}(z)\right\}$ is complete in $H_2^{m\times n}(\mathbb{T})$.

■

4

SISO Identification using Orthonormal Bases

In this chapter, the problem of least squares identification (from input-output data in the time domain) of Discrete-Time (DT), Linear Time-Invariant (LTI), Single-Input Single-Output (SISO) systems represented using orthonormal model structures will be analyzed. We concentrate on identification using the orthonormal bases with fixed poles of Section 3.5, that have the most common FIR, Laguerre and Kautz bases as special cases. The estimation accuracy will be quantified by providing bounds on the undermodelling error and by deriving an asymptotic (in model order and data-length) expression for the noise induced error. Fundamental for the analysis of the noise induced error will be the extension, to the OBFP, of some known results on convergence of Toeplitz matrices available for FIR model structures.

4.1 Introduction

In the last years there has been significant interest in the use of orthonormal basis functions for approximation of dynamical systems [GKB89, WAH, Mäk90a, Mäk90b, Par91, Oli95a, Oli94a], system identification [KP79, Lju85, LY85, Nur87, Wah91b, Mäk91, Wah94b, NG97, VHB95, HbVB95, PT91], signal processing [KP77, Wil95, Pdd93, WZ96, MJM89, den93b], and control applications [ZDP90, ZBD88, ZD88]. Particularly in the area of system identification, several schemes have been proposed for the identification of linear systems from input-output data using orthonormal model structures and least squares techniques [Lju85, LY85, Wah91b, Wah94b, NG94a, NG97, VHB95, HbVB95]. In these methods, the transfer function of the system, say $G(z)$ for the discrete time case, is represented as a series expansion in terms of orthonormal basis functions $\{\mathcal{B}_k(z)\}$, which are stable-causal transfer functions, and then the identification is performed by estimating a finite number of expansion coefficients using least squares techniques.

One of the main motivations for using orthonormal bases to represent the system is that the resulting model structure becomes linear in the parameters (a linear regressor form), with the regressors depending only on the observed input signal. It is well known that in this case the least squares estimate has a closed form solution which corresponds to a global minimum of the quadratic criterion [Lju87, SS89]. In this way, the need for costly iterative optimization procedures for the parameter estimation, and the associated problems of local minima are avoided.

A second factor that has been pointed out in the literature [Wah91b, Wah91a, Wah94b, Oli95a, Bod95] is the numerical robustness of these methods when compared to estimation using non-orthonormal structures. This is so since a worst case numerical conditioning of the least squares estimation problem can be guaranteed for the case of using orthonormal structures while in general this result can not be established for the non-orthonormal case. We defer the study of this issue until the next chapter, since it is in the context of multivariable systems, in which a large number of parameters need to be estimated, where the problem of numerical robustness is more relevant [Vd94a, Vib94].

Besides the above mentioned advantages, a third aspect that we want to emphasize in this thesis, following the lead of Ninness and co-workers [NHG97a], is the use of orthonormal bases as an analysis tool which is particularly suited for the study of estimation methods that employ fixed denominator model structures. This is so because fixed denominator model structures can be linearly re-parameterized using orthonormal bases with the same fixed poles. The analysis of estimation accuracy can then be carried out by extending convergence results of the well known trigonometric bases $\{e^{j\omega n}\}$ (i.e., results of classical Fourier analysis) to the more general orthonormal bases employed in this thesis.

In this chapter we concentrate on the study of identification of DT-LTI-SISO systems using orthonormal model structures and least squares techniques. The material in this chapter will be used as a paradigm to extend these results to the multivariable case in Chapter 5.

The remainder of the chapter is organized as follows. In Section 4.2 we consider the identification of DT-LTI-SISO systems from input-output data in the time domain using rational orthonormal bases and least squares techniques. We then particularized the study of the estimation accuracy for the case of using the OBFP introduced in Section 3.5. This analysis is carried out in Sections 4.3 and 4.4, where the bias error and the variance error are (respectively) considered. Fundamental in this analysis is the use of new results regarding the convergence of generalized Toeplitz-like matrices. These convergence results are also presented in this section. A new phenomenom of accuracy limitation that arises in the estimation using orthonormal bases with fixed poles is illustrated in Section 4.5. Specifically, it is shown that at a given frequency there is a

trade-off between bias and variance errors regarding the choice of the poles of the bases.

4.2 Problem Formulation

It is assumed that the LTI-SISO system is described by the standard model

$$y_k = G(q)\ u_k + \nu_k, \tag{4.1}$$

and that an N point data record of input and output sequences $\{y_k, u_k\}_{k=0}^{N-1}$ is available for the identification of the assumed stable (unknown) transfer function $G(q)$ describing the system dynamics.

In the model (4.1), $\{\nu_k\}$ is a zero-mean stationary sequence representing some measurement noise which is assumed to have finite variance $\mathbf{E}\{\nu_k^2\} = \sigma_\nu^2$, and to be uncorrelated from the input sequence $\{u_k\}$. It is also assumed that the input sequence $\{u_k\}$ is a quasi-stationary process [Lju87], with spectral density $\Phi_u(\omega)$.

Let $\{\mathcal{B}_k(z)\}_{k=0}^{\infty}$ be an orthonormal basis in $H_2(\mathbb{T})$. Now, since the system was assumed to be asymptotically stable, then its transfer function $G(z)$ belongs to $H_2(\mathbb{T})$, and can be uniquely represented by the series expansion

$$G(z) = \sum_{k=0}^{\infty} \theta_k \mathcal{B}_k(z), \tag{4.2}$$

where θ_k are the 'Generalized Fourier Coefficients' defined as

$$\theta_k \ \triangleq\ \langle G, \mathcal{B}_k \rangle \qquad (k = 0, 1, \cdots). \tag{4.3}$$

Of course, since the transfer function $G(z)$ is unknown, the coefficients can not be computed as in (4.3). Instead, our objective will be to estimate the parameters of a finite dimensional model

$$G(z, \theta) \triangleq \sum_{k=0}^{p-1} \theta_k \mathcal{B}_k(z), \tag{4.4}$$

so that the transfer function can be (approximately) identified by using the estimate $\widehat{\theta} \triangleq \left[\widehat{\theta}_0, \widehat{\theta}_1, \cdots, \widehat{\theta}_{p-1}\right]^T$ of the parameter vector $\theta \triangleq [\theta_0, \theta_1, \cdots, \theta_{p-1}]^T$ as follows

$$G(z, \widehat{\theta}) \triangleq \sum_{k=0}^{p-1} \widehat{\theta}_k \mathcal{B}_k(z). \tag{4.5}$$

It remains now to choose the parameter estimation method. The model structure (4.4) leads to the linear regressor form

$$y_k = \phi_k^T \ \theta + \nu_k,$$

where the regressor vector ϕ_k is defined as

$$\phi_k^T \triangleq [\mathcal{B}_0(q)u_k, \mathcal{B}_1(q)u_k, \cdots, \mathcal{B}_{p-1}(q)u_k].$$

The obvious choice for the parameter estimation method is the 'least squares' one, since it provides a closed form solution, and leads asymptotically to an efficient estimate.

The least squares estimate $\widehat{\theta}$ of θ is the minimizing argument of the quadratic criterion

$$V_N(\theta) = \frac{1}{N}\sum_{k=0}^{N-1}(y_k - \phi_k^T\theta)^2. \tag{4.6}$$

That is

$$\widehat{\theta} \triangleq \underset{\theta \ \in \ \mathbb{R}^p}{\arg\min} \left\{ \frac{1}{N}\sum_{k=0}^{N-1}(y_k - \phi_k^T\theta)^2 \right\}.$$

It is well known that the solution of this minimization problem can be written in closed form as [Lju87, SS89, GP77]

$$\widehat{\theta} = R_p(N)^{-1} \left\{ \frac{1}{N}\sum_{k=0}^{N-1} \phi_k y_k \right\}, \tag{4.7}$$

where

$$R_p(N) \triangleq \frac{1}{N}\sum_{k=0}^{N-1} \phi_k \phi_k^T, \tag{4.8}$$

with the subscript p indicating the model order (number of terms in the parameterized model (4.4)).

Given the parameter estimate $\widehat{\theta}$, the transfer function estimate $G(z, \widehat{\theta})$ can then be computed as in equation (4.5).

Remark 4.2.1. For the case of $\{\mathcal{B}_k(z)\}_{k=0}^{\infty}$ being the Orthonormal Bases with fixed poles of Section 3.5, once an estimate $\widehat{\theta}$ has been computed, a minimal state-space realization of the estimated model is immediately available by appealing to the results of Subsection 3.5.1. The described identification technique can be easily implemented in software, for instance in a MATLAB environment [Mat94]. ♦

We are interested now in analyzing the performance of the proposed system identification scheme. This performance will be evaluated by quantifying the estimation error. In the frequency domain, the estimation error can be written as

$$G(e^{j\omega},\widehat{\theta}) - G(e^{j\omega}) = \underbrace{G(e^{j\omega},\widehat{\theta}) - G(e^{j\omega},\theta_\star)}_{\text{noise induced}} + \underbrace{G(e^{j\omega},\theta_\star) - G(e^{j\omega})}_{\text{undermodelling induced}} \tag{4.9}$$

where $\theta_\star$ is the convergence value of the estimate $\widehat{\theta}$ when the number of available data points N tends to infinity. We can then recognize two components of the estimation error:

- A component corresponding to the term

  $$G(e^{j\omega},\widehat{\theta}) - G(e^{j\omega},\theta_\star),$$

  which is due to the noise corruption of the data. Typically, the size of this term is measured as ensemble average as

  $$\mathsf{E}\left\{|G(e^{j\omega},\widehat{\theta}) - G(e^{j\omega},\theta_\star)|^2\right\},$$

  and it is also called *variance error*.

- A component corresponding to the term

  $$G(e^{j\omega},\theta_\star) - G(e^{j\omega}),$$

  which is due to the fact that the model (4.5) is too simple to represent the real system. We call this term *undermodelling error* or *bias error*.

In this thesis, the study of estimation accuracy will focus on using the Orthonormal Bases with Fixed Poles introduced in Section 3.5 as an effective analysis tool for quantifying bias and variance error. One of the contributions of this thesis will be to show how the bias and variance errors depend on the choice of the poles of the basis functions $\{\mathcal{B}_k(z)\}$.

The undermodelling error is analyzed in Section 4.3, and the noise induced error in Section 4.4 .

4.3 Undermodelling Error

The undermodelling induced error arises from the parsimony of the model structure (4.4) (a finite-length series expansion) which cannot completely describe the true dynamics $G(z)$. This error can be quantified in terms of the deviation between the real system and the model

$$|G(e^{j\omega},\theta_\star) - G(e^{j\omega})|.$$

Unfortunately, the convergence value $\theta_\star$ of the estimate $\widehat{\theta}$ is unknown. To quantify the undermodelling error we then compute the error involved in the approximation of $G(z)$ with $\widehat{G}_p(z)$ which is the best $H_2(\mathbb{T})$ approximation of $G(z)$ belonging to the subspace spanned by the first p basis functions $\{\mathcal{B}_k(z)\}_{k=0}^{p-1}$.

As already mentioned, when compared to the FIR, Laguerre, Kautz or the more general OBGIF, the OBFP enjoy greater flexibility in the possible choice of the pole location without the need of a cyclic repetition. Unfortunately, there is a price to be paid for this increased flexibility in that these bases do not have the algebraic structure (3.1) so that the analysis of the undermodelling error can not be reduced to that of the FIR case by a simple transformation of the system or change of variables. This is not the case for the Laguerre, Kautz and OBGIF, all of them having the algebraic structure (3.1). As a consequence, the derivation of results quantifying the undermodelling error for the case of the OBFP is considerably more complicated. In [NHG97a], Ninness and co-workers derive an upper bound on the undermodelling error based on the Christoffel-Darboux formula for the reproducing kernel associated with the bases (equation (3.17)). The result is summarized in the following theorem.

Theorem 4.3.1. [NHG97a] *Let the transfer function of the system $G(z)$ have partial fraction expansion*

$$G(z) = \sum_{\ell=0}^{r-1} \frac{\alpha_\ell}{z - \gamma_\ell}, \tag{4.10}$$

where all the poles satisfy $|\gamma_\ell| < 1$. Let $\widehat{G}_p(z)$ denote the best H_2 approximation to $G(z)$ with respect to the p basis functions $\{\mathcal{B}_k(z)\}_{k=0}^{p-1}$ as defined in (3.14), with poles $\{\xi_k\}_{k=0}^{p-1}$, i.e.

$$\widehat{G}_p(z) = \sum_{k=0}^{p-1} \langle G, \mathcal{B}_k \rangle \, \mathcal{B}_k(z).$$

Then

$$\left| G(e^{\mathrm{j}\omega}) - \widehat{G}_p(e^{\mathrm{j}\omega}) \right| < \sum_{\ell=0}^{r-1} \left| \frac{\alpha_\ell}{e^{\mathrm{j}\omega} - \gamma_\ell} \right| \prod_{k=0}^{p-1} \left| \frac{\gamma_\ell - \xi_k}{1 - \overline{\xi_k}\gamma_\ell} \right| \tag{4.11}$$

Proof: The proof is given in Appendix 4.A. ■

It is obvious that the theorem also provides an upper bound in the undermodelling error for the FIR and Laguerre bases, since these bases are special cases of the OBFP corresponding to poles $\xi_k = 0, \forall k$, and $\xi_k = \xi \in \mathbb{R}, \forall k$, respectively.

It can be seen from equation (4.11) that if the poles of the system γ_ℓ are exactly known, then choosing $\xi_\ell = \gamma_\ell, \forall\ \ell$ gives a zero upper bound on the

undermodelling error. The result also implies that the convergence rate of the series expansion can be faster than that of the special cases of FIR, Laguerre, or Kautz basis, if the guesses for the poles ξ_k approach the true poles γ_k. To illustrate this, let us consider the following examples.

Example 4.3.1. Let the transfer function of the true system be

$$G(z) = \frac{0.8}{z + 0.8},$$

and let us consider a 10-th order expansion (i.e. $p = 10$), with guesses for the poles satisfying $|\xi_k| = 0.4, \forall\ k$. Then, from equation (4.11) the upper bound on the undermodelling error is

$$UB(\omega) = \left| \frac{0.8}{e^{j\omega} + 0.8} \right| \ 0.5882^{10}.$$

The corresponding upper bound using a 10-th order FIR model is

$$UB_{FIR}(\omega) = \left| \frac{0.8}{e^{j\omega} + 0.8} \right| \ 0.8^{10}.$$

We can see that the upper bound using OBFP is $(0.8/0.5882)^{10} = 21.6$ times smaller than the corresponding one using an FIR expansion, even with a 50 % discrepancy between the guesses for the poles and the true poles. ▼

Example 4.3.2. Let us consider now the second order system

$$G(z) = \frac{1}{z + 0.8} + \frac{1}{z + 0.4},$$

with only two terms in the orthonormal model structure ($p = 2$). In Figure 4.1, the upper bound on the undermodelling error for different choices of the poles of the basis is plotted as a function of the frequency ω. In that figure, Curve A corresponds to an FIR model structure (that is the guesses for the poles are $\{0, 0\}$), Curve B corresponds to a Laguerre structure with poles at $\{-0.2, -0.2\}$, Curve C corresponds to the guesses $\{-0.6, -0.2\}$, and Curve D corresponds to poles at $\{-0.7, -0.3\}$. It can be seen that as the guesses for the poles approach the true poles, the upper bound on the undermodelling error decreases. This also illustrates our claim that when (approximate) 'a priori' information about the dominating dynamics of the system is available, the use of OBFP is preferable over FIR structures since a smaller undermodelling error will be obtained for the same model order. ▼

A result of the same type has been obtained by Van den Hof, Heuberger and co-workers [HbVB95] for the Orthonormal Bases Generated from Inner Functions introduced in Section 3.4. In [HbVB95], the authors show that if the

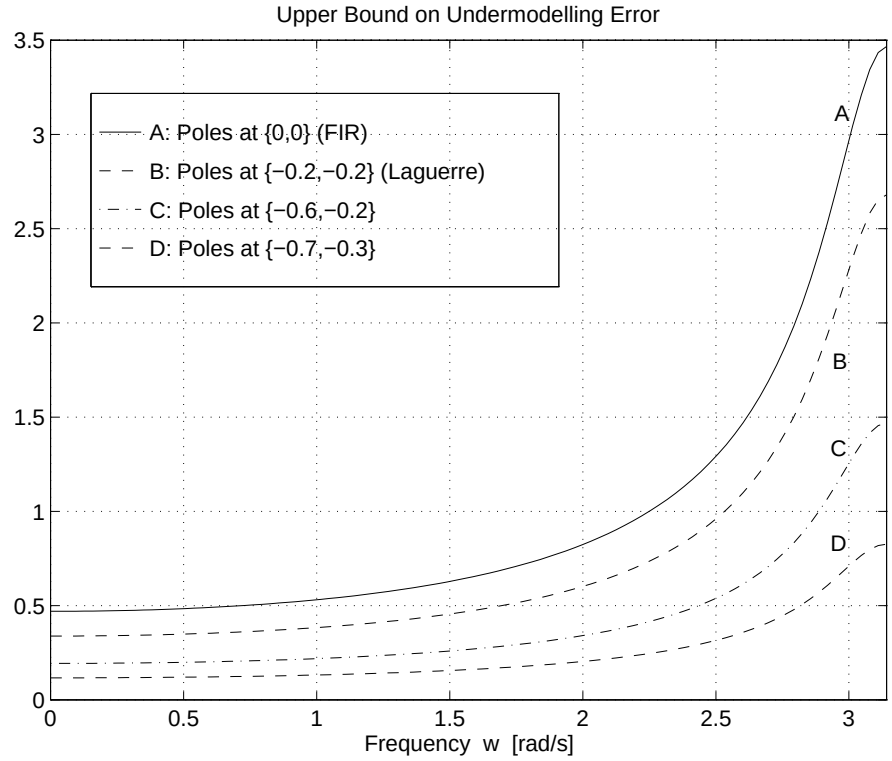

Figure 4.1: *Upper Bound on the Undermodelling Error for different choices of the poles*

dynamics of the inner function generating the orthonormal system $G_x(z)$ and the dynamics of the system to be identified $G(z)$ approach each other, then the convergence rate of the series expansion representation of the system becomes very fast. This implies that the number of coefficients to be estimated in order to accurately model the system becomes smaller. An upper bound of this convergence rate is given in Theorem 4.3.2, adapted from [VHB95, HbVB95].

Before presenting this result, we need to introduce some notation. Let $\{\mathcal{B}_{i,k}(z)\}$ be a set of orthonormal bases generated from an inner transfer function $G_x(z)$ as in Theorem 3.4.1. Then any causal, stable system $G(z) \in H_2(\mathbb{T})$ has a unique series representation as in (3.13), i.e.

$$\begin{aligned} G(z) &= \sum_{k=0}^{\infty}\sum_{i=1}^{n_x}\theta_k^i\mathcal{B}_{i,k}(z) = \sum_{k=0}^{\infty}\sum_{i=1}^{n_x}\theta_k^i e_i^T V_k(z) \\ &= \sum_{k=0}^{\infty} L_k V_k(z) \end{aligned}$$

$$= \sum_{k=0}^{p-1} L_k V_k(z) + \sum_{k=p}^{\infty} L_k V_k(z)$$

$$= \widehat{G}_p(z) + \sum_{k=p}^{\infty} L_k V_k(z), \tag{4.12}$$

where $L_k = [\theta_k^1, \theta_k^2, \cdots, \theta_k^{n_x}] \in \ell_2^{1\times n_x}(\mathbb{N}_0)$, and $\widehat{G}_p(z)$ is the best L_2 approximation to $G(z)$ with respect to the basis functions $\{\mathcal{B}_{i,k}(z)\}$.

Theorem 4.3.2. [VHB95, HbVB95] *Let the transfer function $G(z)$ of the system have poles $\mu_i, (i = 1, \cdots, n_s)$, and let the inner function $G_x(z)$ generating the orthonormal system $\{\mathcal{B}_{i,k}(z)\}$ have poles $\rho_j, (j = 1, \cdots, n_x)$. Denote*

$$\lambda \triangleq \max_i \prod_{j=1}^{n_x} \left| \frac{\mu_i - \rho_j}{1 - \mu_i \rho_j} \right|.$$

Then there exists a constant $c \in \mathbb{R}$ such that, for all $\eta > \lambda$,

$$\sqrt{\sum_{k=p}^{\infty} L_k L_k^T} \leq c \, \frac{\eta^{p+1}}{\sqrt{1-\eta^2}}.$$

Proof: See [VHB95, HbVB95]. In the proof, the authors exploit the algebraic structure of the bases

$$\mathcal{B}_{i,m}(z)\mathcal{B}_{i,n}(z) = \mathcal{B}_{i,0}(z)\mathcal{B}_{i,(m+n)}(z) \tag{4.13}$$

to establish a transformation of signal and systems (the so-called 'Hambo' transform [VHB94a, VHB94b, VHB95, HV96]) so that the system in the transform domain can be obtained by a simple variable transformation from the original system represented with the standard trigonometric bases $\{z^{-n}\}$. The result then follows by using known properties of Fourier series approximation. ■

The previous theorem implies that when the two sets of poles converge to each other, λ will tend to zero, and the upper bound on the 2-norm of the tail of the series expansion will decrease drastically, reducing in this way the undermodelling error.

4.4 Noise Induced Error

Our interest is now in the quantification of the component of the frequency response estimation error that is induced by the measurement noise (the 'variance error'). Since the analysis for finite data is too complicated, we follow the work

of other authors [Lju85, LY85, Wah91b, Wah94b] and consider an asymptotic analysis when the data-length N tends to infinity. Since we are interested here only in the noise induced error, we will also allow the model order p to tend to infinity to avoid the presence of undermodelling error. The quantification of the frequency response noise induced error can be achieved by using known results [Lju87] on the asymptotic statistics of the parameter estimation error and by noting that the frequency response estimate is linearly related to the parameter estimate.

Using the results in [Lju87] (as summarized in Theorems 2.2.1 and 2.2.2) we can draw the following conclusions on the asymptotic (as the number of data N tends to infinity) statistics of the parameter estimate (4.7):

- **Asymptotic Estimate:** For fixed model order p

$$\widehat{\theta} \xrightarrow{\text{a.s.}} \theta_\star \quad \text{as} \quad N \to \infty$$

where

$$\theta_\star = \arg\min_{\theta \in \mathbb{R}^p} \left\{ \frac{1}{2\pi} \int_{-\pi}^{\pi} \left| G(e^{\mathrm{j}\omega}) - G_p(e^{\mathrm{j}\omega}, \theta) \right|^2 \Phi_u(\omega) \, \mathrm{d}\omega \right\} \tag{4.14}$$

- **Asymptotic Distribution of the parameter estimate:** For fixed model order p

$$\sqrt{N}(\widehat{\theta} - \theta_\star) \xrightarrow{\text{dist}} \mathcal{N}(0, P_p) \qquad \text{as} \qquad N \longrightarrow \infty$$

where

$$\begin{aligned} P_p &\triangleq R_p^{-1} Q_p R_p^{-1}, \\ R_p &\triangleq \lim_{N\to\infty} R_p(N) = \lim_{N\to\infty} \mathbf{E}\left\{ V_N''(\theta_\star) \right\}, \\ Q_p &\triangleq \lim_{N\to\infty} N \mathbf{E}\left\{ V_N'(\theta_\star)^T V_N'(\theta_\star) \right\}, \end{aligned}$$

Notice now that the transfer function estimate (4.5) can be written as

$$G(z, \widehat{\theta}) \triangleq \sum_{k=0}^{p-1} \widehat{\theta}_k \mathcal{B}_k(z) = \Gamma_p^T(z) \, \widehat{\theta}, \tag{4.15}$$

where we have defined

$$\Gamma_p(z) \triangleq [\mathcal{B}_0(z), \mathcal{B}_1(z), \cdots, \mathcal{B}_{p-1}(z)]^T. \tag{4.16}$$

This linear relationship between the transfer function estimate and the parameter estimate, together with the above results on the statistics of the parameter estimate, allows us to give the following frequency domain characterization of the asymptotic distribution of the (transfer function) estimate:

- **Asymptotic Distribution of the Frequency Response Estimate:** For fixed model order p

$$\sqrt{N}\begin{bmatrix} G(e^{\mathrm{j}\omega_1},\widehat{\theta}) - G(e^{\mathrm{j}\omega_1},\theta_\star) \\ G(e^{\mathrm{j}\omega_2},\widehat{\theta}) - G(e^{\mathrm{j}\omega_2},\theta_\star) \end{bmatrix} \xrightarrow{\text{dist}} \mathcal{N}(0,\Lambda'_p(\omega_1,\omega_2))$$

as $N \longrightarrow \infty$, with

$$\Lambda'_p(\omega_1,\omega_2) \triangleq \begin{bmatrix} \Gamma_p^T(e^{\mathrm{j}\omega_1}) \\ \Gamma_p^T(e^{\mathrm{j}\omega_2}) \end{bmatrix} P_p \begin{bmatrix} \Gamma_p^T(e^{\mathrm{j}\omega_1}) \\ \Gamma_p^T(e^{\mathrm{j}\omega_2}) \end{bmatrix}^\star$$

if $\omega_1 \neq \omega_2$, and where $\cdot^\star$ denotes the conjugate transpose.

A measure of the transfer function estimation error induced by the noise is then given by the covariance matrix $\Lambda'_p(\omega_1,\omega_2)$. The exact expression for the covariance matrices P_p and $\Lambda'_p(\omega_1,\omega_2)$ will depend on the particular choice for the orthonormal basis. Unfortunately, these exact expressions for a fixed (finite) model order p are so complicated that they have no practical utility. As mentioned before, the standard approach in the literature (see for instance: [LY85, Lju85] for FIR basis, [Wah91b] for Laguerre basis, [Wah94b] for Kautz basis, [VHB95] for OBGIF, and [NHG97a] for OBFP) has been to provide an approximate quantification of the noise induced error by considering an asymptotic analysis when both the model order p and the number of data N are allowed to tend to infinity.

4.4.1 The FIR case

Historically, the first results regarding the quantification of the noise induced error in this identification setup were obtained by Ljung and Yuan [LY85, Lju85] for the case of FIR model structures. The FIR variance results are summarized in the following theorem.

Theorem 4.4.1. [LY85] *Let $G(z,\theta)$ be represented as in (4.4) and let $\{\mathcal{B}_k(z)\}$ be the standard FIR basis. Let $\widetilde{G}(e^{\mathrm{j}\omega})$ be defined as*

$$\widetilde{G}(e^{\mathrm{j}\omega}) \triangleq G(e^{\mathrm{j}\omega},\widehat{\theta}) - G(e^{\mathrm{j}\omega},\theta_\star).$$

Then provided that $p \to \infty$ as $N \to \infty$,

$$\sqrt{\frac{N}{p}}\,\widetilde{G}(e^{\mathrm{j}\omega}) \xrightarrow{dist} \mathcal{N}\left(0,\frac{\Phi_\nu(\omega)}{\Phi_u(\omega)}\right)$$

as $N \to \infty$. In addition

$$\frac{N}{p}\,\mathsf{Cov}\left\{\widetilde{G}(e^{\mathrm{j}\omega_1}),\widetilde{G}(e^{\mathrm{j}\omega_2})\right\} \to \begin{cases} 0 & \text{if } \omega_1 \neq \omega_2 \\ \dfrac{\Phi_\nu(\omega_1)}{\Phi_u(\omega_1)} & \text{if } \omega_1 = \omega_2 \end{cases}$$

as $N \to \infty$.

Proof: See [LY85]. ■

The theorem implies that the variance of the tranfer function estimate at a particular frequency ω, and for large model order and data-length can be approximated by

$$\mathsf{Var}\left\{G(e^{\mathrm{j}\omega},\widehat{\theta})\right\} \approx \frac{p}{N}\,\frac{\Phi_\nu(\omega)}{\Phi_u(\omega)}, \tag{4.17}$$

which is the noise-to-signal ratio with a weighting factor which is the ratio between model order and data-length.

4.4.2 The Fixed Denominator case

Our interest now is to determine if the FIR result in Theorem 4.4.1 can also be applied for the case of a fixed denominator model structure. We can write

$$y_k = G(q,\widehat{\theta})u_k + \nu_k = \frac{C(q,\widehat{\theta})}{D(q)}u_k + \nu_k, \tag{4.18}$$

where the poles in $D(q)$ are fixed (and known), and the unknown parameters are the coefficients of the numerator polynomial $C(q,\widehat{\theta})$. We can then write

$$y_k = C(q,\widehat{\theta})\left(\frac{1}{D(q)}u_k\right) + \nu_k = C(q,\widehat{\theta})\tilde{u}_k + \nu_k, \tag{4.19}$$

where

$$\tilde{u}_k \triangleq \frac{1}{D(q)}u_k$$

is a filtered version of u_k. The fixed denominator model structure estimation problem (4.18), being then by (4.19) really an FIR estimation problem with prefiltered input $\tilde{u}_k$, should be amenable to FIR variance analysis by Theorem 4.4.1 to lead to the conclusion that for a given frequency ω, and for large model order and data-length we can approximate the numerator variance as

$$\mathsf{Var}\left\{C(e^{\mathrm{j}\omega},\widehat{\theta})\right\} \approx \frac{p}{N}\,\frac{\Phi_\nu(\omega)}{\Phi_{\tilde{u}}(\omega)},$$

and then considering that

$$\Phi_{\tilde{u}}(\omega) = \frac{\Phi_u(\omega)}{|D(e^{\mathrm{j}\omega})|^2},$$

we have

$$\mathsf{Var}\left\{C(e^{\mathrm{j}\omega},\widehat{\theta})\right\} \approx \frac{p|D(e^{\mathrm{j}\omega})|^2}{N}\frac{\Phi_\nu(\omega)}{\Phi_u(\omega)}.$$

Therefore, the variability of the full frequency response estimate could be expected by Theorem 4.4.1 to be approximated as

$$\mathsf{Var}\left\{G(e^{\mathrm{j}\omega},\widehat{\theta})\right\} \approx \frac{1}{|D(e^{\mathrm{j}\omega})|^2}\mathsf{Var}\left\{C(e^{\mathrm{j}\omega},\widehat{\theta})\right\}$$

which is (4.17), and therefore that in the general fixed denominator case, the estimation **variance does not appear to depend on the location of the poles** in the model structure.

This line of reasoning is, however, flawed. The problem is that the filter $D(e^{\mathrm{j}\omega})$ is changing as the model order p increases, and therefore the spectrum $\Phi_{\tilde{u}}(\omega)$ is also changing with p, so that we are applying a result (Theorem 4.4.1) which was derived assuming a fixed input spectrum to a case where this spectrum is not fixed.

The remedy for this problem turns out to be to reparameterize the problem in a special orthonormal form which is specifically adapted to the fixed denominator being used. Developing these methods consume the remainder of this chapter.

4.4.3 Variance error using OBFP

Let $\{\mathcal{B}_k(z)\}$ be a set of orthonormal bases with fixed poles $\{\xi_k\}$ as defined in (3.14), and let us assume that a system $G(z) \in H_2(\mathbb{T})$ is identified using these bases and least squares techniques as described in Section 4.2. The asymptotic (in model order and data-length) distribution of the transfer function estimate for this case is as follows.

Theorem 4.4.2. *Let the input spectral density $\Phi_u(\omega)$ have a finite dimensional spectral factorization, and let the poles $\{\xi_k\}$ be chosen to satisfy the completeness condition*

$$\sum_{k=0}^{\infty}(1-|\xi_k|) = \infty.$$

Then for $N \to \infty$ and $p \to \infty$

$$\sqrt{N}\begin{bmatrix} \gamma_p(\omega_1) & 0 \\ 0 & \gamma_p(\omega_2) \end{bmatrix}^{-1/2}\begin{bmatrix} G(e^{\mathrm{j}\omega_1},\widehat{\theta}) - G(e^{\mathrm{j}\omega_1},\theta_\star) \\ G(e^{\mathrm{j}\omega_2},\widehat{\theta}) - G(e^{\mathrm{j}\omega_2},\theta_\star) \end{bmatrix} \xrightarrow{dist} \mathcal{N}\left(0,\Lambda(\omega_1,\omega_2)\right)$$

where

$$\Lambda(\omega_1,\omega_2) = \begin{bmatrix} \dfrac{\Phi_\nu(\omega_1)}{\Phi_u(\omega_1)} & 0 \\ 0 & \dfrac{\Phi_\nu(\omega_2)}{\Phi_u(\omega_2)} \end{bmatrix}$$

if $\omega_1 \neq \omega_2$, *and*

$$\gamma_p(\omega) \triangleq K_p(\omega,\omega) = \sum_{k=0}^{p-1} |\mathcal{B}_k(e^{\mathrm{j}\omega})|^2.$$

Proof: The main difficulty encountered in the proof of these asymptotic results has been that the bases (3.14) employed here do not have the algebraic structure (3.1), so that the problem can not be reduced to the FIR one by a change of variables. Fundamental in the analysis will be the derivation of some results regarding the convergence properties of generalized Toeplitz-like matrices. These results are summarized in Appendix 4.B, and their derivation proceeds based only on the orthonormality of the bases and the Christoffel-Darboux formula for the reproducing kernel associated with them. The proof is given in Appendix 4.A. ■

As a corollary of the previous theorem we have the following quantification of the noise induced error in the transfer function estimate.

Corollary 4.4.3. *Under the same conditions of the previous theorem, but with the strengthened requirement that* $\mathsf{E}\{e_k^8\} < \infty$ *then*

$$\lim_{p\to\infty}\lim_{N\to\infty} \frac{N}{\gamma_p(\omega)} \mathsf{E}\left\{ |G(e^{\mathrm{j}\omega},\widehat{\theta}) - G(e^{\mathrm{j}\omega},\theta_\star)|^2 \right\} = \frac{\Phi_\nu(\omega)}{\Phi_u(\omega)}.$$

Proof: Follows along the same lines in the developments in Appendix 9B of [Lju87]. ■

These results imply that the variance of the transfer function estimate at a given frequency ω, and for large N and model order p, can be well approximated by

$$\mathsf{Var}\left\{ G(e^{\mathrm{j}\omega},\widehat{\theta}) \right\} \approx \frac{\gamma_p(\omega)}{N} \frac{\Phi_\nu(\omega)}{\Phi_u(\omega)}, \tag{4.20}$$

which is the noise-to-signal ratio weighted with a frequency dependent factor that is determined by the basis functions. This variance expression explicitly shows (through the factor $\gamma_p(\omega) \triangleq K_p(\omega,\omega)$) how the choice of the poles of the basis functions (the poles in the fixed denominator model) affects the noise induced error. In Figure 4.2, the factor $\gamma_p(\omega)$ is plotted for model order $p = 4$ and for various pole choices.

In addition, this result generalizes similar results available for FIR model structures [Lju85, LY85], Laguerre basis [Wah91b], and Kautz basis [Wah94b]. For example, the FIR model structure corresponds to the choice $\xi_k = 0, \forall\, k$ for the poles of the bases, and in this case the factor $\gamma_p(\omega) = p$, so that the variance expression (4.20) becomes

$$\mathsf{Var}\left\{ G(e^{\mathrm{j}\omega},\widehat{\theta}) \right\} \approx \frac{p}{N} \frac{\Phi_\nu(\omega)}{\Phi_u(\omega)},$$

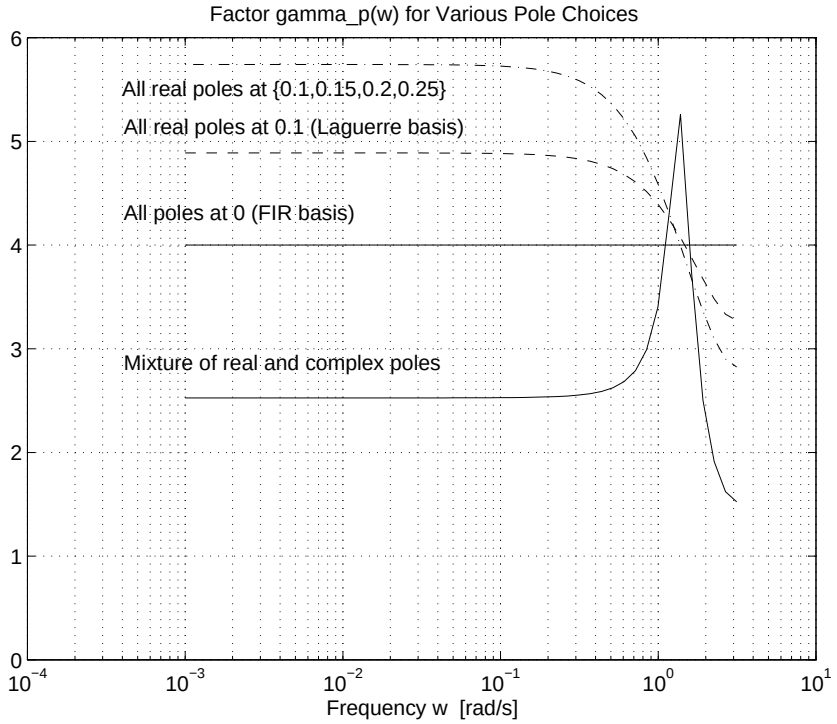

Figure 4.2: *Reproducing kernel* $\gamma_p(\omega) \triangleq K_p(\omega, \omega)$ *as a function of frequency* ω*, and for model order* $p = 4$ *and various pole choices.*

which is the same result of Ljung and Yuan [Lju85, LY85] as summarized in Theorem 4.4.1. The Laguerre basis instead corresponds to the choice $\xi_k = \xi \in \mathbb{R}, \forall\, k$ for the poles, so that the variance expression (4.20) becomes

$$\text{Var}\left\{G(e^{\mathrm{j}\omega}, \widehat{\theta})\right\} \approx \frac{p}{N} \frac{(1-\xi^2)}{|e^{\mathrm{j}\omega} - \xi|^2} \frac{\Phi_\nu(\omega)}{\Phi_u(\omega)},$$

which is exactly the result obtained originally by Wahlberg [Wah91b].

A result of the same nature as Theorem 4.4.2 has been derived by Van den Hof and co-workers [VHB95] for the orthonormal bases generated from inner functions of Section 3.4. The authors exploit the algebraic structure (4.13) of the bases in order to establish a transformation of systems (the 'Hambo Transform') so that the system in the transformed domain can be obtained by a simple change of variables from the original system represented with the FIR bases $\{z^{-k}\}$. See [VHB95] for the details.

The following theorem summarizes these variance results.

Theorem 4.4.4. [VHB95] *Assume that the input spectral density* $\Phi_u(\omega)$ *is bounded away from zero and sufficiently smooth. Then, for* $N, p \longrightarrow \infty$*,*

$p^2/N \longrightarrow 0$

$$\frac{N}{pn_x}\mathsf{Cov}\left\{G(e^{\mathrm{j}\omega_1},\widehat{\theta}_N),G(e^{\mathrm{j}\omega_2},\widehat{\theta}_N)\right\} \longrightarrow \begin{cases} 0 \quad \text{for} \quad G_x(e^{\mathrm{j}\omega_1}) \neq G_x(e^{\mathrm{j}\omega_2}) \\ V_0^T(e^{\mathrm{j}\omega_1})V_0(e^{-\mathrm{j}\omega_1})\dfrac{\Phi_\nu(\omega_1)}{\Phi_u(\omega_1)} \\ \qquad\qquad \text{for} \quad \omega_1=\omega_2 \end{cases} \tag{4.21}$$

where $\mathsf{Cov}\left\{G(e^{\mathrm{j}\omega_1},\widehat{\theta}_N),G(e^{\mathrm{j}\omega_2},\widehat{\theta}_N)\right\}$ *is the cross-covariance matrix in the joint distribution of*

$$\left[G(e^{\mathrm{j}\omega_1},\widehat{\theta}_N)-G(e^{\mathrm{j}\omega_1},\theta_\star),G(e^{\mathrm{j}\omega_2},\widehat{\theta}_N)-G(e^{\mathrm{j}\omega_2},\theta_\star)\right],$$

and where $\Phi_\nu(\omega)$ *is the measurement noise spectral density.*

Proof: See [VHB95]. The key idea of the proof is to exploit the algebraic structure (4.13) of the bases in order to establish a transformation of systems (the 'Hambo Transform') so that the system in the transformed domain can be obtained by a simple change of variables from the original system represented with the standard trigonometric basis $\{z^{-k}\}$. The use of convergence results available for FIR model structures [Lju85, LY85] together with asymptotic properties of Toeplitz matrices [GS58] then gives the result. ■

As pointed out in [VHB95], the interpretation of the previous theorem is that the variance of the transfer function estimate at a given frequency ω, and for large N and model order p, can be approximated by

$$\mathsf{Var}\left\{G(e^{\mathrm{j}\omega},\widehat{\theta})\right\} \approx \frac{pn_x}{N}V_0^T(e^{\mathrm{j}\omega})V_0(e^{-\mathrm{j}\omega})\frac{\Phi_\nu(\omega)}{\Phi_u(\omega)} \tag{4.22}$$

which is the noise-to-signal ratio weighted with a frequency dependent factor that is determined by the basis functions. This result also generalizes the available FIR, and Laguerre results in [Lju85, LY85], and [Wah91b], respectively. See [VHB95] for the details.

This thesis, via Theorem 4.4.2 has developed a result in a similar vein, but with one very important difference - it is asymptotic in the number of poles, not asymptotic in the number of *repetitions* of poles as the pre-existing result of Theorem 4.4.4 is. This difference is considered very substantial, since the point of results such as Theorem 4.4.4 or our Theorem 4.4.2 is to allow the progression to approximations like (4.22), and this can only be done by assuming that convergence has occurred in results like Theorems 4.4.4 and 4.4.2. If, as commonly occurs in practice, one is using a model with all the poles different (so as to distribute them as much as possible in the hope of minimizing the undermodelling induced error), then there has been no repetition of poles,

and it cannot be argued that convergence in Theorem 4.4.4 is likely to have occurred, and hence for (4.22) to be a realistic approximation.

In contrast, in this same scenario of all poles being different, but using new and completely different analysis techniques to those used in deriving Theorem 4.4.4, the new result of Theorem 4.4.2 is relevant since, provided there is a reasonable number of fixed poles (5 or more appears 'reasonable' in our experience), then it can be argued that convergence may have approximately occurred in Theorem 4.4.2 and hence that an approximation like (4.20) can be reasonably argued to be appropriate.

4.5 Bias/Variance Trade-off

The variance expression (4.20) together with the upper bound on the undermodelling error (4.11) provides a complete characterization of the accuracy of the estimates. The results show the well known trade-off that exists in the choice of model order p with regard to the relative size of both error components. Specifically an increase in model order to reduce the undermodelling error will be at the cost of an increase of the variance error.

The results also show how the estimation accuracy is influenced by the choice of the poles of the basis functions. Here, an until now unappreciated phenomenom is manifested, namely, a trade-off in the choice of the poles of the bases regarding the magnitude of the bias and variance errors. More specifically, assuming that the noise induced error and the undermodelling error are uncorrelated we can write

$$\begin{aligned}\mathsf{E}\left\{\left|G(e^{\mathrm{j}\omega},\widehat{\theta})-G(e^{\mathrm{j}\omega})\right|^2\right\} &= \mathsf{E}\left\{\left|G(e^{\mathrm{j}\omega},\widehat{\theta})-G(e^{\mathrm{j}\omega},\theta_\star)\right|^2\right\} \\ &+ \mathsf{E}\left\{\left|G(e^{\mathrm{j}\omega},\theta_\star)-G(e^{\mathrm{j}\omega})\right|^2\right\}\end{aligned}$$

Use of Theorems 4.3.1 and 4.4.2 then allows us to upper bound the frequency response estimation error as follows

$$\mathsf{E}\left\{\left|G(e^{\mathrm{j}\omega},\widehat{\theta})-G(e^{\mathrm{j}\omega})\right|^2\right\} \leq r\sum_{\ell=0}^{r-1}\left|\frac{\alpha_\ell}{e^{\mathrm{j}\omega}-\gamma_\ell}\right|^2\prod_{k=0}^{p-1}\left|\frac{\gamma_\ell-\xi_k}{1-\overline{\xi_k}\gamma_\ell}\right|^2+\frac{\gamma_p(\omega)}{N}\frac{\Phi_\nu(\omega)}{\Phi_u(\omega)}$$

We can see that if we want to decrease the undermodelling error for a given model order, the poles of the bases have to be chosen close to the true poles, but then the noise induce error cannot be reduced at the frequencies of the poles (due to the presence of the factor $\gamma_p(\omega)$). On the other hand, if we want to reduce the noise induced error for a fixed model order, the poles of the bases have to be chosen well below the frequency at which the noise is dominating, but then if the true poles are not at these frequencies, the undermodelling error will be incremented. The following example illustrates this phenomenom.

Example 4.5.1. Let the true system be given by

$$G(q) = \frac{0.1548q + 0.0939}{(q - 0.6065)(q - 0.3679)} = \frac{0.7871}{(q - 0.6065)} - \frac{0.6323}{(q - 0.3679)},$$

and let us consider that $N = 1000$ samples are available for the estimation in the identification setup of Section 4.2. It is assumed that the measurement noise is a zero mean Gaussian white noise process of variance $\sigma_\nu^2 = 0.01$, and that the input is a Gaussian white noise process with variance $\sigma_u^2 = 0.1$.

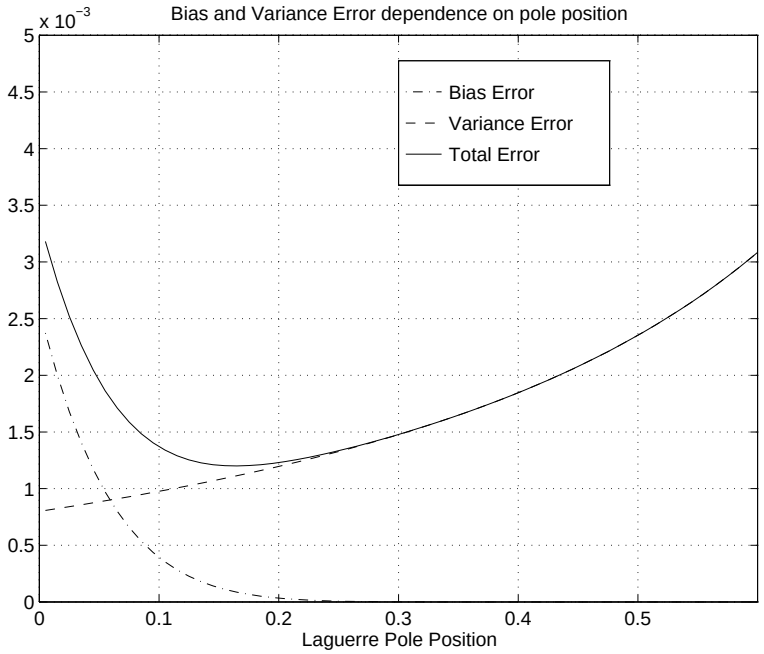

Figure 4.3: *Illustration of Bias/Variance trade-off with respect to the choice of Laguerre pole position.*

Figure 4.3 shows the dependence on the fixed pole position of the variance error, the bias error, and the total error at a frequency $\omega = 0.1$ rad/s, when using an 8th order Laguerre model structure. ▼

4.6 Conclusions

In this chapter we have studied the problem of identification, from input-output data in the time domain, of discrete-time single-input, single-output linear systems using rational orthonormal bases and least squares techniques. The focus of the chapter has been the derivation of results concerning the accuracy of the estimation for the case of using orthonormal bases with fixed poles. The results generalize previous works corresponding to FIR, Laguerre and Kautz

bases. The way the location of the poles in the orthonormal structure affects the two components of the estimation error has been determined. This shows a new phenomenom of bias/variance trade-off regarding the choice of the pole locations.

Appendices

4.A Proofs for Chapter 4

Proof of Theorem 4.3.1 Using the Christoffel-Darboux formula for the reproducing kernel (equation (3.17)), and the partial fraction expansion for $G(z)$ in (4.10) we can write

$$\begin{aligned} G(\mu) - \widehat{G}_p(\mu) &= \langle G(z), K(z,\mu) - K_p(z,\mu) \rangle \\ &= \left\langle G(z),\, 1 + \frac{\varphi_p(z)\overline{\varphi_p(\mu)}}{z\overline{\mu} - 1} \right\rangle \end{aligned}$$

$$\begin{aligned} &= \langle G(z),\, 1 \rangle + \left\langle G(z),\, \frac{\varphi_p(z)\overline{\varphi_p(\mu)}}{z\overline{\mu} - 1} \right\rangle \\ &= \left\langle G(z),\, \frac{\varphi_p(z)\overline{\varphi_p(\mu)}}{z\overline{\mu} - 1} \right\rangle \\ &= -\varphi_p(\mu)\, \frac{1}{2\pi \mathrm{j}} \oint_{\mathbb{T}} \frac{G(z)\overline{\varphi_p(z^{-1})}}{z - \mu}\, \mathrm{d}z \\ &= -\varphi_p(\mu) \sum_{\ell=0}^{r-1} \alpha_\ell\, \frac{1}{2\pi \mathrm{j}} \oint_{\mathbb{T}} \frac{\overline{\varphi_p(z^{-1})}}{(z - \gamma_\ell)(z - \mu)}\, \mathrm{d}z. \end{aligned}$$

In passing to the fourth line, use was made of the fact that due to Cauchy Residue Theorem and the analyticity of $G(z)$ outside the unit circle, then

$$\langle G(z),\, 1 \rangle = \frac{1}{2\pi \mathrm{j}} \oint_{\mathbb{T}} G(z^{-1}) \frac{\mathrm{d}z}{z} = 0.$$

Considering now that

$$\overline{\varphi_p(z^{-1})} = \prod_{k=0}^{p-1} \frac{z - \xi_k}{1 - z\overline{\xi_k}},$$

we can write appealing once again to Cauchy Residue Theorem

$$\begin{aligned} G(\mu) - \widehat{G}_p(\mu) &= -\varphi_p(\mu) \sum_{\ell=0}^{r-1} \alpha_\ell \, \frac{1}{2\pi \mathrm{j}} \oint_{\mathbb{T}} \prod_{k=0}^{p-1} \frac{(z-\xi_k)}{(z-\gamma_\ell)(z-\mu)(1-z\overline{\xi_k})} \, \mathrm{d}z \\ &= -\varphi_p(\mu) \sum_{\ell=0}^{r-1} \frac{\alpha_\ell}{(\gamma_\ell - \mu)} \prod_{k=0}^{p-1} \frac{\gamma_\ell - \xi_k}{1-\gamma_\ell \overline{\xi_k}}. \end{aligned}$$

Now, the result follows by computing the previous equation on the unit circle ($\mu = e^{\mathrm{j}\omega}$), taking the module on both sides of the equation, considering that $\left|\varphi_p(e^{\mathrm{j}\omega})\right| = 1$, and using triangle inequality. ■

Proof of Theorem 4.4.2 As mentioned at the beginning of Section 4.4, by appealing to the results in [Lju87] it is possible to give the following frequency domain characterization of the asymptotic distribution of the transfer function estimate.

- **Asymptotic distribution of the transfer function estimate.** For fixed model order p

$$\sqrt{N} \begin{bmatrix} G(e^{\mathrm{j}\omega_1}, \widehat{\theta}) - G(e^{\mathrm{j}\omega_1}, \theta_\star) \\ G(e^{\mathrm{j}\omega_2}, \widehat{\theta}) - G(e^{\mathrm{j}\omega_2}, \theta_\star) \end{bmatrix} \xrightarrow{\text{dist}} \mathcal{N}(0, \Lambda'_p(\omega_1, \omega_2))$$

as $N \longrightarrow \infty$, with

$$\Lambda'_p(\omega_1, \omega_2) \triangleq \begin{bmatrix} \Gamma_p^T(e^{\mathrm{j}\omega_1}) \\ \Gamma_p^T(e^{\mathrm{j}\omega_2}) \end{bmatrix} P_p \begin{bmatrix} \Gamma_p^T(e^{\mathrm{j}\omega_1}) \\ \Gamma_p^T(e^{\mathrm{j}\omega_2}) \end{bmatrix}^\star$$

if $\omega_1 \neq \omega_2$, and where

$$\begin{aligned} P_p &\triangleq R_p^{-1} Q_p R_p^{-1}, \\ R_p &\triangleq \lim_{N\to\infty} R_p(N) = \lim_{N\to\infty} \mathbf{E}\left\{V_N''(\theta_\star)\right\}, \\ Q_p &\triangleq \lim_{N\to\infty} N\mathbf{E}\left\{V_N'(\theta_\star)^T V_N'(\theta_\star)\right\}. \end{aligned}$$

It is then straightforward to conclude that, for fixed model order p

$$\sqrt{N} \begin{bmatrix} \gamma_p(\omega_1) & 0 \\ 0 & \gamma_p(\omega_2) \end{bmatrix}^{-1/2} \begin{bmatrix} G(e^{\mathrm{j}\omega_1}, \widehat{\theta}) - G(e^{\mathrm{j}\omega_1}, \theta_\star) \\ G(e^{\mathrm{j}\omega_2}, \widehat{\theta}) - G(e^{\mathrm{j}\omega_2}, \theta_\star) \end{bmatrix} \xrightarrow{\text{dist}} \mathcal{N}(0, \Lambda_p(\omega_1, \omega_2)) \tag{4.A.1}$$

as $N \longrightarrow \infty$, where now $\Lambda_p(\omega_1, \omega_2)$ is defined as

$$\Lambda_p(\omega_1, \omega_2) \triangleq \begin{bmatrix} \gamma_p(\omega_1) & 0 \\ 0 & \gamma_p(\omega_2) \end{bmatrix}^{-1/2} \Lambda'_p(\omega_1, \omega_2) \begin{bmatrix} \gamma_p(\omega_1) & 0 \\ 0 & \gamma_p(\omega_2) \end{bmatrix}^{-1/2}.$$

After some matrix manipulation, $\Lambda_p(\omega_1, \omega_2)$ can be written as

$$\Lambda_p(\omega_1,\omega_2) = \begin{bmatrix} \dfrac{\Gamma_p^\star(e^{\mathrm{j}\omega_1})P_p\Gamma_p(e^{\mathrm{j}\omega_1})}{\gamma_p(\omega_1)} & \dfrac{\Gamma_p^\star(e^{\mathrm{j}\omega_1})P_p\Gamma_p(e^{\mathrm{j}\omega_2})}{\gamma_p^{1/2}(\omega_1)\gamma_p^{1/2}(\omega_2)} \\ \dfrac{\Gamma_p^\star(e^{\mathrm{j}\omega_2})P_p\Gamma_p(e^{\mathrm{j}\omega_1})}{\gamma_p^{1/2}(\omega_1)\gamma_p^{1/2}(\omega_2)} & \dfrac{\Gamma_p^\star(e^{\mathrm{j}\omega_2})P_p\Gamma_p(e^{\mathrm{j}\omega_2})}{\gamma_p(\omega_2)} \end{bmatrix}. \tag{4.A.2}$$

It is now clear that in order to analyze the asymptotic distribution (4.A.1) when p tends to infinity, we need only to study the asymptotic behaviour of the term

$$\frac{\Gamma_p^\star(e^{\mathrm{j}\omega_1})P_p\Gamma_p(e^{\mathrm{j}\omega_2})}{\gamma_p^{1/2}(\omega_1)\gamma_p^{1/2}(\omega_2)}. \tag{4.A.3}$$

To proceed with this analysis, we will provide a frequency domain expression for the matrix P_p.

We introduce first some notation. For any positive function $f : [-\pi, \pi] \to (0, \infty)$ we define the $p \times p$ matrix $M_p(f)$ as

$$M_p(f) \triangleq \frac{1}{2\pi}\int_{-\pi}^{\pi} \Gamma_p(e^{\mathrm{j}\omega})f(\omega)\Gamma_p^\star(e^{\mathrm{j}\omega})\mathrm{d}\omega. \tag{4.A.4}$$

We will call $M_p(f)$ a *Toeplitz-like* matrix on account of the fact that for the case of the basis functions being the FIR ones, then $M_p(f)$ has effectively a Toeplitz structure.[1] [GS58].

Considering the expression of $R_p(N)$ in equation (4.8), the matrix R_p can be written as

$$R_p = \lim_{N\to\infty}\frac{1}{N}\sum_{k=0}^{N-1}\mathbf{E}\left\{\phi_k\phi_k^T\right\}.$$

Using now Parseval's Theorem (in the form given by equation (2.3)), and the definition of $M_p(f)$ in (4.A.4), R_p can be written as

$$R_p = \frac{1}{2\pi}\int_{-\pi}^{\pi}\Gamma_p(e^{\mathrm{j}\omega})\Phi_u(\omega)\Gamma_p^\star(e^{\mathrm{j}\omega})\mathrm{d}\omega = M_p(\Phi_u).$$

The derivation of a frequency domain expression for the matrix Q_p is more difficult. In [NHG97a] it is shown that Q_p can be written as

$$Q_p = M_p(\Phi_u\Phi_\nu) + \Delta_p,$$

[1] A matrix is said to be Toeplitz if its i,j-th entry depends only on the difference $(i-j)$ of the indices. Some basic properties of Toeplitz matrices are:

- A Toeplitz matrix is constant along its diagonals.
- A lower (upper) triangular Toeplitz matrix is completely specified by the elements of the first column (row).

where Δ_p is given by

$$\Delta_p = \frac{1}{2\pi}\int_{-\pi}^{\pi} \Gamma_p(e^{\mathrm{j}\omega})\Phi_u^2(\omega)|\widetilde{G}_p(e^{\mathrm{j}\omega})|^2\Gamma_p^\star(e^{\mathrm{j}\omega})\mathrm{d}\omega,$$

with

$$\widetilde{G}_p(e^{\mathrm{j}\omega}) \triangleq G(e^{\mathrm{j}\omega}) - G(e^{\mathrm{j}\omega},\theta_\star).$$

It can be also proved [NHG97a] that[2]

$$\lim_{p\to\infty} \|\Delta_p\|_2 = 0.$$

Finally, a frequency domain expression of the matrix P_p is then given by

$$P_p = M_p^{-1}(\Phi_u)M_p(\Phi_u\Phi_\nu)M_p^{-1}(\Phi_u) + M_p^{-1}(\Phi_u)\Delta_p M_p^{-1}(\Phi_u).$$

The term (4.A.3) can then be written as

$$\begin{aligned}\frac{\Gamma_p^\star(e^{\mathrm{j}\omega_1})P_p\Gamma_p(e^{\mathrm{j}\omega_2})}{\gamma_p^{1/2}(\omega_1)\gamma_p^{1/2}(\omega_2)} &= \underbrace{\frac{\Gamma_p^\star(e^{\mathrm{j}\omega_1})M_p^{-1}(\Phi_u)M_p(\Phi_u\Phi_\nu)M_p^{-1}(\Phi_u)\Gamma_p(e^{\mathrm{j}\omega_2})}{\gamma_p^{1/2}(\omega_1)\gamma_p^{1/2}(\omega_2)}}_{\text{Term 1}} \\ &+ \underbrace{\frac{\Gamma_p^\star(e^{\mathrm{j}\omega_1})M_p^{-1}(\Phi_u)\Delta_p M_p^{-1}(\Phi_u)\Gamma_p(e^{\mathrm{j}\omega_2})}{\gamma_p^{1/2}(\omega_1)\gamma_p^{1/2}(\omega_2)}}_{\text{Term 2}}. \qquad (4.A.5)\end{aligned}$$

Now, the asymptotic analysis when $p \to \infty$ can be carried out by resorting to the results on convergence of Toeplitz-like matrices in Appendix 4.B. It can be proved that for any choices of ω_1 and ω_2, Term 2 on the right hand side of equation (4.A.5) tends to zero as $p \to \infty$. On the other hand, as $p \to \infty$, Term 1 tends to $\Phi_\nu(\omega)/\Phi_u(\omega)$ for $\omega_1 = \omega_2 = \omega$, and tends to zero for $\omega_1 \neq \omega_2$. We refer the reader to [NHG97a, NHG97b] for the remainder details of the proof. ■

4.B Convergence of Toeplitz-like Matrices

Although for general orthonormal bases the matrix form $M_p(f)$ defined in (4.A.4) will not have a Toeplitz structure, some results on convergence of Toeplitz matrices can be extended to this form. For the orthonormal bases with fixed poles introduced in Section 3.5, these convergence results are as follows.

[2]Here $\|\cdot\|_2$ stands for the matrix induced 2-norm or spectral norm (i.e. the maximum singular value).

Theorem 4.B.1. *Suppose $f(\omega)$ is a real valued and continuous function on $[-\pi, \pi]$. Then provided*

$$\sum_{k=0}^{\infty}(1 - |\xi_k|) = \infty,$$

the following limit result holds

$$\lim_{p\to\infty} \frac{\Gamma_p^\star(e^{\mathrm{j}\omega})M_p(f)\Gamma_p(e^{\mathrm{j}\omega})}{\gamma_p(\omega)} = f(\omega).$$

Proof: [NHG97b]

$$\begin{aligned}
&\Gamma_p(e^{\mathrm{j}\omega})^\star M_p(f)\Gamma_p(e^{\mathrm{j}\omega}) = \\
&= \sum_{m=0}^{p-1}\sum_{n=0}^{p-1} \overline{\mathcal{B}_m(e^{\mathrm{j}\omega})}\mathcal{B}_n(e^{\mathrm{j}\omega})[M_p(f)]_{m,n}, \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\sigma) \sum_{m=0}^{p-1}\sum_{n=0}^{p-1} \overline{\mathcal{B}_m(e^{\mathrm{j}\omega})}\mathcal{B}_n(e^{\mathrm{j}\omega})\mathcal{B}_m(e^{\mathrm{j}\sigma})\overline{\mathcal{B}_n(e^{\mathrm{j}\sigma})}\,\mathrm{d}\sigma, \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\sigma)\,|K_p(\omega,\sigma)|^2\,\mathrm{d}\sigma.
\end{aligned}$$

Therefore, for any $\delta > 0$

$$\begin{aligned}
&\frac{1}{2\pi}\left|\frac{\Gamma_p^\star(e^{\mathrm{j}\omega})M_p(f)\Gamma_p(e^{\mathrm{j}\omega})}{\gamma_p(\omega)} - f(\omega)\right| = \\
&= \frac{1}{2\pi\gamma_p(\omega)}\left|\Gamma_p^\star(e^{\mathrm{j}\omega})M_p(f)\Gamma_p(e^{\mathrm{j}\omega}) - \gamma_p(\omega)f(\omega)\right|, \\
&= \frac{1}{2\pi\gamma_p(\omega)}\left|\int_{-\pi}^{\pi}(f(\sigma) - f(\omega))\,|K_p(\omega,\sigma)|^2\,\mathrm{d}\sigma\right|, \\
&\le \frac{1}{2\pi\gamma_p(\omega)}\left|\int_{\sigma\in[\omega-\delta,\omega+\delta]}(f(\sigma) - f(\omega))\,|K_p(\omega,\sigma)|^2\,\mathrm{d}\sigma\right| + \\
&\quad + \frac{1}{2\pi\gamma_p(\omega)}\left|\int_{\sigma\notin[\omega-\delta,\omega+\delta]}(f(\sigma) - f(\omega))\,|K_p(\omega,\sigma)|^2\,\mathrm{d}\sigma\right|.
\end{aligned}$$

Now, since $f(\omega)$ is continuous, then for δ sufficiently small

$$|f(\sigma) - f(\omega)| \le \epsilon \ \text{ on } [\omega - \delta, \omega + \delta].$$

and hence

$$\begin{aligned}
&\frac{1}{2\pi\gamma_p(\omega)}\left|\int_{\sigma\in[\omega-\delta,\omega+\delta]}(f(\sigma) - f(\omega))\,|K_p(\omega,\sigma)|^2\,\mathrm{d}\sigma\right| \le \\
&\le \frac{\epsilon}{2\pi\gamma_p(\omega)}\int_{-\pi}^{\pi}|K_p(\omega,\sigma)|^2\,\mathrm{d}\sigma = \epsilon.
\end{aligned}$$

Also, since f is continuous on compact $[-\pi,\pi]$ then f is bounded by some $M/2 < \infty$. Therefore

$$\begin{aligned}\frac{1}{2\pi\gamma_p(\omega)}\left|\int_{\sigma\notin[\omega-\delta,\omega+\delta]}(f(\sigma)-f(\omega))\,|K_p(\omega,\sigma)|^2\,\mathrm{d}\sigma\right| \leq \\ \leq \quad \frac{M}{2\pi\gamma_p(\omega)}\int_{\sigma\notin[\omega-\delta,\omega+\delta]}|K_p(\omega,\sigma)|^2\,\mathrm{d}\sigma.\end{aligned}$$

This gives

$$\frac{1}{2\pi}\left|\frac{\Gamma_p^\star(e^{\mathrm{j}\omega})M_p(f)\Gamma_p(e^{\mathrm{j}\omega})}{\gamma_p(\omega)}-f(\omega)\right| \leq \epsilon + \frac{M}{2\pi\gamma_p(\omega)}\int_{\sigma\notin[\omega-\delta,\omega+\delta]}|K_p(\omega,\sigma)|^2\,\mathrm{d}\sigma$$

Using the result in Lemma 4.C.1 and considering that ϵ is arbitrary then completes the proof. ■

Theorem 4.B.2. *Suppose $f(\omega) \in L_2([-\pi,\pi])$ is positive definite and has finite dimensional spectral factorization. Then provided*

$$\sum_{k=0}^{\infty}(1-|\xi_k|)=\infty$$

the following limit result holds

$$\lim_{p\to\infty}\frac{\Gamma_p(e^{\mathrm{j}\omega})^\star M_p^{-1}(f)\Gamma_p(e^{\mathrm{j}\omega})}{\gamma_p(\omega)} = f^{-1}(\omega)$$

Proof:

$$\begin{aligned}\frac{\Gamma_p(e^{\mathrm{j}\mu})^\star M_p(f)^{-1}\Gamma_p(e^{\mathrm{j}\omega})}{\gamma_p(\omega)} &= \frac{\Gamma_p(e^{\mathrm{j}\mu})^\star M_p(1/f)\Gamma_p(e^{\mathrm{j}\omega})}{\gamma_p(\omega)} + \\ &+ \frac{\Gamma_p(e^{\mathrm{j}\mu})^\star M_p(f)^{-1}[I-M_p(f)M_p(1/f)]\Gamma_p(e^{\mathrm{j}\omega})}{\gamma_p(\omega)}\end{aligned}$$

Now by construction, the elements of the vector $\Gamma(e^{\mathrm{j}\omega})$ are bounded in magnitude by some finite number K_1, as defined in Lemma 5.E.2. Similarly, by Lemma 4.C.3 the elements of the vector $M_p(f)^{-1}\Gamma_p(e^{\mathrm{j}\omega})$ can also be bounded by some finite number K. In this case, using Lemma 4.C.2 gives that for some

$|\eta| < 1$

$$\begin{aligned}
&\left|\Gamma_p(\omega)^\star M_p(f)^{-1}[I - M_p(f)M_p(1/f)]\Gamma_p(\omega)\right| \leq \\
&\leq \sum_{m=0}^{p-1}\sum_{n=0}^{p-1}\left|\left[\Gamma_p(\omega)^\star M_p(f)^{-1}\right]_m\right| \times \\
&\times \left|[M_p(f)M_p(1/f)]_{m,n} - [M_p(1)]_{m,n}\right| \left|[\Gamma_p(\omega)]_n\right| \\
&\leq K_1K^2\sum_{m=0}^{p-1}\sum_{n=0}^{p-1}(\eta^{p-m}+\eta^m)(\eta^{p-n}+\eta^n) \\
&= K_1K^2\left(\frac{1-\eta^p}{1-\eta}\right)^2(\eta^p+\eta)^2 < \infty.
\end{aligned}$$

But by Lemma 5.E.3

$$\frac{1}{2}\sum_{k=0}^{p-1}(1-|\xi_k|) \leq \gamma_p(\omega),$$

so that under the conditions of the lemma

$$\lim_{p\to\infty}\frac{\Gamma_p(\omega)^\star M_p(f)^{-1}[I - M_p(f)M_p(1/f)]\Gamma_p(\omega)}{\gamma_p(\omega)} = 0.$$

Therefore, by using Theorem 4.B.1

$$\lim_{p\to\infty}\frac{\Gamma_p(\omega)^\star M_p(f)^{-1}\Gamma_p(\omega)}{\gamma_p(\omega)} = \lim_{p\to\infty}\frac{\Gamma_p(\omega)^\star M_p(1/f)\Gamma_p(\omega)}{\gamma_p(\omega)} = f^{-1}(\omega).$$

■

4.C Technical Lemmas

Lemma 4.C.1. [NHG97b] *Let $K_p(\omega,\sigma)$ denote the reproducing kernel of the OBFP $\{\mathcal{B}_k\}$ introduced in Section 3.5. Then*

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}|K_p(\omega,\sigma)|^2\,\mathrm{d}\sigma = \sum_{m=0}^{p-1}|\mathcal{B}_m(e^{\mathrm{j}\omega})|^2 \triangleq \gamma_p(\omega).$$

Furthermore, for any $\delta > 0$, provided

$$\sum_{k=0}^{\infty}(1-|\xi_k|) = \infty,$$

then

$$\lim_{p\to\infty}\frac{1}{\gamma_p(\omega)}\int_{\sigma\notin[\omega-\delta,\omega+\delta]}|K_p(\omega,\sigma)|^2\,\mathrm{d}\sigma = 0.$$

Proof: See [NHG97b]. ■

Lemma 4.C.2. *Let $f, g \in L_2([-\pi, \pi])$ have finite dimensional spectral factorizations. Then there exists $|\eta| < 1$ and $K < \infty$ such that*

$$\left| [M_p(f)M_p(g)]_{m,n} - [M_p(fg)]_{m,n} \right| \leq K(\eta^{p-m} + \eta^m)(\eta^{p-n} + \eta^n).$$

Proof: See [NHG96]. ■

Lemma 4.C.3. *Suppose $f \in L_2([-\pi, \pi])$ has a finite dimensional spectral factorization. Then $\exists K < \infty$ which is independent of p such that*

$$\left| [M_p(f)\Gamma_p(\omega)]_\ell \right| < K, \quad \left| \left[M_p^{-1}(f)\Gamma_p(\omega)\right]_\ell \right| < K.$$

Proof: See [NHG96]. ■

5

MIMO Identification using Orthonormal Bases

In this chapter the idea of using orthonormal bases and PEM (least squares techniques) for the identification of discrete-time linear time-invariant systems will be extended from the SISO to the MIMO (Multiple-Input Multiple-Output) setting. It will be shown how the rational orthonormal bases with fixed poles introduced in Section 3.5 (or the corresponding MIMO bases generated as in Section 3.6) can be used to linearly parameterize any multivariable fixed denominator model structure. The use of these bases will allow the incorporation in the identification process of prior knowledge about dominant dynamics of the system, and will facilitate the analysis of the estimation accuracy. As done for the SISO case, the accuracy of the estimation will be quantified by deriving expressions for an upper bound on the undermodelling error and for the asymptotic (in data-length and model order) covariance of the transfer matrix estimate (noise induced error). The asymptotic covariance analysis will be based on the derivation of convergence properties of some block Toeplitz-like matrices. The recently popular Subspace-based State Space System IDentification (4SID) methods for multivariable systems [Vd96] will be briefly reviewed, and through simulation experiments, their performance will be compared with that of the orthonormal basis-based methods proposed here.

5.1 Introduction

To the best of our knowledge, except for the well known special case of FIR multivariable model structure studied in [YL84, Zhu89, Zhu94], the use of more general bases in MIMO system identification has not been studied to date except for the work in [HbVB95] and work by Ninness, the current author and coworkers in [NnGW95, NG96].

Following the SISO paradigm for identification using orthonormal bases and least squares techniques described in the previous chapter, the obvious exten-

sion to the MIMO setting would be to construct general orthonormal bases for the space $H_2^{m\times n}(\mathbb{T})$ of $(m \times n)$ transfer matrices whose elements are in $H_2(\mathbb{T})$, and then to parameterize the transfer matrix of the system as a series expansion in terms of these bases. The identification would then be performed by estimating a finite number of expansion coefficients using least squares techniques. However, in contrast to the SISO case, several parameterizations of the system using orthonormal bases are possible in the MIMO context. For instance, in [YL84], Yuan and Ljung use the standard FIR scalar bases $\{z^{-k}\}$ to parameterize the $(m \times n)$ transfer matrix of the system as a linear combination of the bases, where the coefficients are $(m \times n)$ matrices (the impulse response matrices or *Markov (matrix) parameters*).

This idea of using scalar bases in identification of MIMO systems is applied here for the case of the more general orthonormal bases studied in Section 3.5. In addition, it is shown that the parameterization of the MIMO system using scalar bases (with matrix coefficients) is equivalent to the parameterization using matrix bases (generated from the scalar ones via Theorem 3.6.1) with scalar coefficients.

As mentioned in the introduction of the previous chapter, some emphasis has been placed in the literature [Wah91b, Wah94b, Wah91a, Oli95a, Bod95] on the use of orthonormal model structures as an implementational tool with certain numerical properties ensuring the well-posedness of the least squares estimation problem. In this chapter, an upper bound on the condition number of the least squares estimation will be derived. This guarantees a worst case numerical conditioning of the estimation using orthonormal structures. However, it will be shown that since this upper bound is completely specified by the input spectral density and is not affected by the particular bases chosen (as far as they are orthonormal), the numerical conditioning can still be very bad depending on the nature of the input.

In consideration of this, and as has already been mentioned, the approach in this thesis, following [NHG97a], is to consider the orthonormal structure as an analysis tool rather than an implementational tool. We will show that any multivariable fixed denominator model structure can be linearly re-parameterized using rational orthonormal bases with the same fixed poles. The analysis of the estimation accuracy can then be carried out on the orthonormal structure in a more tractable way by exploiting the orthonormality property of the bases.

In this chapter, we extend the single-input, single-output results concerning the accuracy of the estimation presented in [Wah91b, Wah94b] for Laguerre and Kautz models, and in [VHB95, NG97, NHG97a] (as summarized in Sections 4.3 and 4.4) for more general models, to the multivariable setting; and also the multivariable results in [YL84] from the FIR setting to more general model structures which encompass the FIR structure as a special case.

The main contribution of the chapter is the extension of the asymptotic

FIR results of [YL84] to the case of using general orthonormal bases with fixed poles. More specifically, in [YL84] the variance of the FIR transfer function matrix estimate was shown to be approximately (for large data-length N and large model order p) equal to

$$\frac{p}{N}\ \Phi_u^{-1}(\omega) \otimes \Phi_\nu(\omega)$$

where $\Phi_u(\omega)$ is the input spectral density, $\Phi_\nu(\omega)$ is the output measurement noise spectral density, and $\otimes$ is the Kronecker matrix product (see Appendix A for the definition and properties). In this chapter we show that for the OBFP described in Section 3.5, the above expression should be changed to

$$\frac{\gamma_p(\omega)}{N}\ \Phi_u^{-1}(\omega) \otimes \Phi_\nu(\omega).$$

where $\gamma_p(\omega) \triangleq \sum_{k=0}^{p-1} |\mathcal{B}_k(e^{j\omega})|^2$ (see Figure 4.2). Note that for FIR models $\gamma_p(\omega) = p$ so that the new expression contains the previously known FIR model structure result [YL84] as a special case. The expression is also in formal analogy with the single-input, single-output result of Theorem 4.4.2. The derivation, however, is considerably more complicated. A first difficulty (inherited from the SISO case) is that the OBFP we use in this thesis do not have the algebraic structure that would allow to reduce the problem to the FIR case. A second difficulty is that in the MIMO case several quantities do not conmute. Fundamental for the analysis will be the derivation of some results concerning the convergence of block Toeplitz-like matrices.

The rest of the chapter is organized as follows. The identification problem is stated in Section 5.2. In that section we also show how the orthonormal basis with fixed poles introduced in Section 3.5 can be used to linearly reparameterized any multivariable fixed denominator model structure. In Section 5.3, it is shown that the parameterization of the MIMO system using (matrix) orthonormal bases (with scalar coefficients) is equivalent to the parameterization using scalar bases with matrix coefficients. The MIMO identification problem using orthonormal bases and least squares techniques is solved in Section 5.4. The numerical robustness of the identification algorithms is analyzed in Section 5.5. In Section 5.6 we analyze the undermodelling induced error that results from the parsimony of the model structure (due to a finite number of expansion terms), which is too simple to exactly represent the system. In Section 5.7, the main contribution of the chapter is derived. Namely, an expression for the asymptotic (in number of observed data and in model order) covariance of the transfer matrix estimate is obtained. This expression is used to quantify the estimation error induced by the presence of measurement noise. The derived result is consistent with that presented in [YL84] for the particular case of FIR multivariable model structures, and indicates that the noise induced error is asymptotically proportional to the (generalized) noise-to-signal

ratio. The analysis is based on new results concerning the convergence of block Toeplitz-like matrices. For the purposes of comparison with the orthonomal basis-based identification method proposed in this chapter, a brief review of Subspace-based State Space System IDentification (4SID) methods for multivariable systems is given in Section 5.9. Finally, some simulation examples are presented in Section 5.10, and some conclusions in Section 5.11.

5.2 Problem Formulation

We address the problem of identification of Discrete-Time Linear Time-Invariant MIMO systems from observed input-output data in the time domain. To be more specific, it is assumed that the system has n inputs and m outputs and that N samples of n input sequences $\{u_k^1\}, \{u_k^2\}, \ldots, \{u_k^n\}$, as well as m output sequences $\{y_k^1\}, \{y_k^2\}, \ldots, \{y_k^m\}$ are available for the identification. We assume also that the data are related according to

$$y_k^i = \sum_{j=1}^{n} G_{ij}(q) u_k^j + \sum_{s=1}^{m} H_{is}(q) e_k^s, \tag{5.1}$$

with $i = 1, 2, \cdots, m$.

The scalar transfer functions $\{G_{ij}(q)\}$ and $\{H_{ij}(q)\}$ describe respectively the unknown (assumed stable) system dynamics and the disturbance model that are to be identified. A notational simplification is possible by vectorizing:

$$\begin{aligned} y_k &\triangleq (y_k^1, y_k^2, \ldots, y_k^m)^T, \\ u_k &\triangleq (u_k^1, u_k^2, \ldots, u_k^n)^T, \\ e_k &\triangleq (e_k^1, e_k^2, \ldots, e_k^m)^T, \end{aligned}$$

$$G(q) = \begin{bmatrix} G_{11}(q) & G_{12}(q) & \ldots & G_{1n}(q) \\ G_{21}(q) & G_{22}(q) & \ldots & G_{2n}(q) \\ \vdots & \vdots & \ldots & \vdots \\ G_{m1}(q) & G_{m2}(q) & \ldots & G_{mn}(q) \end{bmatrix},$$

$$H(q) = \begin{bmatrix} H_{11}(q) & H_{12}(q) & \ldots & H_{1m}(q) \\ H_{21}(q) & H_{22}(q) & \ldots & H_{2m}(q) \\ \vdots & \vdots & \ldots & \vdots \\ H_{m1}(q) & H_{m2}(q) & \ldots & H_{mm}(q) \end{bmatrix},$$

so that (5.1) can be rewritten in matrix form as

$$y_k = G(q)\ u_k + H(q)\ e_k = G(q)\ u_k + \nu_k. \tag{5.2}$$

Here $\{e_k\}$ is assumed to be a stationary (zero mean) white noise vector processs with covariance matrix $\mathbf{E}\left\{e_k e_k^T\right\} = \Lambda$. In this case, the disturbance term $\nu_k = H(q)\ e_k$ is also a stationary process with spectral density [Lju87]

$$\Phi_\nu(\omega) = H(e^{\mathrm{j}\omega})\Lambda H^\star(e^{\mathrm{j}\omega}). \tag{5.3}$$

A standard approach is to provide the model (5.2) with a finite dimensional parameterization

$$y_k = G(q,\theta)\ u_k + H(q,\theta)\ e_k, \tag{5.4}$$

so that the system can be identified by estimating the (finite dimensional) parameter vector θ.

There are many options available for the estimation of $G(q,\theta)$ and $H(q,\theta)$ within this problem setting. For example, a general prediction error technique (as described in Chapter 2) using a multivariable Box–Jenkins [BJ76] or a state–space model structure could be employed [Lju87] (see [Lju91] for an example of identification of arbitrarily parameterized state-space models within the framework of the *System Identification Toolbox* [Lju95] for use with MATLAB[1][Mat94]). Unfortunately, as mentioned in Chapter 2, apart from the various difficulties concerning identifiability of these model structures [Lju87, GW74, Gui75, Gui81], this strategy may also result in a numerically intensive iterative (possibly nonlinear and nonconvex) optimization procedure. In addition, the analysis of the (finite data) estimation accuracy becomes very difficult (sometimes intractable) for this case.

Another possibility is to employ one of the recently popular Subspace-based State-Space System IDentification (4SID)[2] methods for multivariable systems [Vd96], such as the N4SID [Vd94a, Vd94b, Vd91a, Vib94, Vib95, OV94, VOWL93, VOWL91], MOESP [Ver91, VD91b, VD92], and CVA [Lar90, Lar94] methods. These schemes provide accurate state-space models for multivariable systems directly from input-output data, and have the advantage that no iterative procedures are involved. Instead, they employ reliable numerical algorithms such as Singular Value Decomposition (SVD), and QR-decomposition [GV89]. Work is still progressing on quantifying the estimation error involved with the use of such methods [VWO97, DPS95, DPS94].

In this thesis, our interest is not on the estimation of the model for the additive output noise (that is, on the estimation of $H(q,\theta)$), whose second order statistics will be assumed to be known, but only on the estimation of the transfer matrix $G(q,\theta)$ describing the system dynamics. We concentrate then on the parameterized model

$$y_k = G(q,\theta)\ u_k + \nu_k, \tag{5.5}$$

[1] MATLAB is a registered trademark of The MathWorks, Inc.

[2] Pronounced 'force it'.

rather than on (5.4).

Furthermore, and for the sake of ease of error quantification, lack of identifiability problems and small computational load, we will focus on the study of prediction error methods with a quadratic criterion and with a particular model structure that has fixed poles that are chosen according to prior knowledge about the system. To facilitate the analysis of estimation accuracy we will further re-parameterize this model structure using orthonormal bases with the same fixed poles.

The particular model structure we will consider is given as

$$G(q,\beta) = \mathcal{D}_p^{-1}(q)\sum_{k=0}^{p-1}\beta_k q^k, \tag{5.6}$$

$$\mathcal{D}_p(q) = \prod_{\ell=0}^{p-1}(q-\xi_\ell), \tag{5.7}$$

where $\beta_k \in \mathbb{R}^{m\times n}, k=0,\cdots,p-1$, are matrices of parameters to be estimated (namely, the coefficients of the numerator polynomials of the individual transfer functions $\{G_{ij}(q)\}$), and $\{\xi_0,\xi_1,\cdots,\xi_{p-1}\}$ are the poles chosen by the user to reflect prior knowledge about the true system $G(q)$. The advantage of this model structure is that it allows the input-output relationship to be easily cast in linear regressor form as

$$y_k = \beta^T\psi_k + \nu_k, \tag{5.8}$$

with

$$\beta^T \triangleq (\beta_0,\beta_1,\cdots,\beta_{p-1}), \tag{5.9}$$

$$\psi_k \triangleq \begin{bmatrix} \mathcal{D}_p^{-1}(q)I_n \\ q\mathcal{D}_p^{-1}(q)I_n \\ q^2\mathcal{D}_p^{-1}(q)I_n \\ \vdots \\ q^{p-1}\mathcal{D}_p^{-1}(q)I_n \end{bmatrix} u_k = \Upsilon_p(q)\ u_k, \tag{5.10}$$

$$\Upsilon_p(q) \triangleq \left[\mathcal{D}_p^{-1}(q)I_n, q\mathcal{D}_p^{-1}(q)I_n,\cdots,q^{p-1}\mathcal{D}_p^{-1}(q)I_n\right]^T, \tag{5.11}$$

so that if a quadratic criterion is used, the resulting least squares estimate can be found in a computationally cheap manner. It is well known that the least squares estimate can be written in closed form as

$$\widehat{\beta} = \left(\sum_{k=0}^{N-1}\psi_k^T\psi_k\right)^{-1}\sum_{k=0}^{N-1}\psi_k y_k^T. \tag{5.12}$$

Our main interest is now to quantify the accuracy of the resulting transfer matrix estimate

$$G(q,\widehat{\beta}) = \widehat{\beta}^T \ \Upsilon_p(q), \tag{5.13}$$

and particularly to study how the estimation error is affected by the choice of the poles $\{\xi_k\}$ in $\mathcal{D}_p(q)$.

To undertake this study let us first notice that since via equations (5.6), (5.7), (5.9), (5.11), and (5.12) the estimate $G(q,\widehat{\beta})$ is linear in the data, then it is invariant under linear re-parameterization of the model structure (5.6) (See Lemma 5.E.1 in Appendix 5.E). This implies that the same transfer matrix estimate $G(q,\widehat{\beta})$ is obtained if one instead chooses the model structure

$$G(q,\Theta) = \sum_{k=0}^{p-1} \theta_k^T \mathcal{B}_k(q) = \Theta^T \ \Gamma_p(q), \tag{5.14}$$

where the transfer functions $\{\mathcal{B}_\ell(q)\}$ are the orthonormal bases with fixed poles (OBFP) introduced in Section 3.5, with the same poles as in $\mathcal{D}_p(q)$, and where now

$$\Theta \triangleq \left[\theta_0^T, \theta_1^T, \cdots, \theta_{p-1}^T\right]^T,$$

with $\theta_k^T \in \mathbb{R}^{m\times n}$, and $\Gamma_p(q) \triangleq [\mathcal{B}_0(q)I_n, \mathcal{B}_1(q)I_n, \cdots, \mathcal{B}_{p-1}(q)I_n]^T$.

Although the two formulations (5.6)-(5.7) and (5.14) are equivalent in the sense that the obtained tranfer matrix estimates are identical (i.e. $G(q,\widehat{\Theta}) = G(q,\widehat{\beta})$), the latter structure (5.14) is much preferable from an analytical point of view, since the basis functions $\{\mathcal{B}_\ell(q)\}$ are orthonormal, and this property can be exploited in the quantification of the estimation accuracy. Due to this equivalence, it is not difficult to see that the bias and variance error properties of the estimates derived from the two model structures are also identical. However, since the orthonormal structure (5.14) is more tractable, this is the model structure we will employ for the analysis in this chapter, keeping in mind that the results of this analysis can also be applied to the model structure (5.6)-(5.7) or any other linearly equivalent one.

As mentioned in the introduction of the chapter, several parameterizations using orthonormal basis are possible in the MIMO framework. Besides the model structure (5.14) that corresponds to the case of scalar bases $\{\mathcal{B}_\ell(z)\}$ (i.e. bases of the space $H_2(\mathbb{T})$) with matrix coefficients $\theta_\ell^T \in \mathbb{R}^{m\times n}$, we will also consider an orthonormal structure corresponding to $(m\times n)$-matrix bases (i.e. bases of the space $H_2^{m\times n}(\mathbb{T})$, as the ones introduced in Section 3.6) with scalar coefficients $\theta_\ell^T \in \mathbb{R}$. In the following section we prove that both parameterizations are equivalent.

5.3 An equivalent MIMO parameterization

Let $\left\{\mathcal{B}_\ell^{ij}(z)\right\}_{\ell=0}^{\infty}, (i=1,\ldots,m; j=1,\ldots,n)$ be an orthonormal basis for the space $H_2^{m\times n}(\mathbb{T})$ generated from the scalar bases $\{\mathcal{B}_\ell(z)\}_{\ell=0}^{\infty}$ as in Theorem 3.6.1. Then any stable causal tranfer matrix $G(z)$ in $H_2^{m\times n}(\mathbb{T})$ can be approximated by a linear combination of a finite number of elements of the orthonormal set, that is

$$G(z,\Theta) = \sum_{\ell=0}^{p-1}\sum_{i=1}^{m}\sum_{j=1}^{n}\theta_\ell^{ij}\mathcal{B}_\ell^{ij}(z). \tag{5.15}$$

The following Lemma shows that, for the appropriate definition of the parameter matrices θ_ℓ, the above expression is equivalent to

$$G(z,\Theta) = \sum_{\ell=0}^{p-1}\theta_\ell^T\mathcal{B}_\ell(z). \tag{5.16}$$

Lemma 5.3.1. *Let $\{\mathcal{B}_\ell^{ij}(z)\}_{\ell=0}^{\infty}, (i=1,\ldots,m; j=1,\ldots,n)$ be an orthonormal basis on $H_2^{m\times n}(\mathbb{T})$ generated from the scalar basis $\{\mathcal{B}_\ell(z)\}_{\ell=0}^{\infty}$ as in Theorem 3.6.1. Then the following identity holds*

$$\sum_{\ell=0}^{p-1}\sum_{i=1}^{m}\sum_{j=1}^{n}\theta_\ell^{ij}\mathcal{B}_\ell^{ij}(z) = \sum_{\ell=0}^{p-1}\theta_\ell^T\mathcal{B}_\ell(z),$$

where the parameter matrices θ_ℓ are defined as

$$\theta_\ell^T \triangleq \begin{bmatrix} \theta_\ell^{11} & \ldots & \theta_\ell^{1n} \\ \vdots & \ddots & \vdots \\ \theta_\ell^{m1} & \ldots & \theta_\ell^{mn} \end{bmatrix}. \tag{5.17}$$

Proof: See Appendix 5.A. ■

Remark 5.3.1. This result means that the two orthonormal structures we have considered thus far (given by equations (5.16) and (5.15)) are completely equivalent, so that we can use any of them indistinctively to represent the transfer matrix of the system. Most of the analysis of the following sections will be carried out for the parameterization with scalar bases and matrix coefficients (5.16). However, when convenient, the parameterization with matrix bases will also be used. ♦

5.4 Parameter Estimation

The representation in (5.14) leads to the convenient linear regressor form

$$\begin{aligned} y_k &= G(q,\Theta)u_k + \nu_k, \\ &= \left(\theta_0^T, \theta_1^T, \cdots, \theta_{p-1}^T\right) \begin{bmatrix} I_n\mathcal{B}_0(q) \\ I_n\mathcal{B}_1(q) \\ \vdots \\ I_n\mathcal{B}_{p-1}(q) \end{bmatrix} u_k + \nu_k, \\ &= \Theta^T \left(\mathsf{B}_p(q) \otimes I_n\right) u_k + \nu_k, \\ &= \Theta^T \Gamma_p(q)\, u_k + \nu_k, \\ &= \Theta^T \phi_k + \nu_k, \end{aligned} \tag{5.18}$$

where we have defined[3]

$$\Theta^T \triangleq \left(\theta_0^T, \theta_1^T, \cdots, \theta_{p-1}^T\right), \tag{5.19}$$

$$\mathsf{B}_p(q) \triangleq \left[\mathcal{B}_0(q), \cdots, \mathcal{B}_{p-1}(q)\right]^T, \tag{5.20}$$

$$\Gamma_p(q) \triangleq \mathsf{B}_p(q) \otimes I_n, \tag{5.21}$$

$$\phi_k \triangleq \Gamma_p(q)\, u_k. \tag{5.22}$$

With the above definitions for Θ and $\Gamma_p(q)$, the transfer matrix is given by

$$G(q,\Theta) = \Theta^T \Gamma_p(q). \tag{5.23}$$

With the system in linear regressor form (5.18), the most obvious scheme for estimating the parameter matrix Θ is the least squares method. The least squares estimate $\widehat{\Theta}$ of Θ is the minimizing argument of the quadratic criterion

$$V_N(\Theta) = \frac{1}{N}\mathsf{Tr}\left\{\sum_{k=0}^{N-1} \varepsilon_k(\Theta)\varepsilon_k^T(\Theta)\right\}, \tag{5.24}$$

where $\varepsilon_k(\Theta) = y_k - G(q,\Theta)u_k$. That is

$$\widehat{\Theta} = \arg\min_{\Theta}\left\{V_N(\Theta)\right\} = \arg\min_{\Theta}\left\{\frac{1}{N}\mathsf{Tr}\left\{\sum_{k=0}^{N-1} \varepsilon_k(\Theta)\varepsilon_k^T(\Theta)\right\}\right\}. \tag{5.25}$$

It is well known [Lju87] that this optimization problem has an explicit solution given by

$$\widehat{\Theta} = \left(\frac{1}{N}\sum_{k=0}^{N-1}\phi_k\phi_k^T\right)^{-1}\left(\frac{1}{N}\sum_{k=0}^{N-1}\phi_k y_k^T\right) = \widetilde{R}_p^{-1}(N)\left(\frac{1}{N}\sum_{k=0}^{N-1}\phi_k y_k^T\right), \tag{5.26}$$

[3]Please don't confuse $\mathsf{B}_p(q)$ (a column vector of tranfer functions), with $\mathcal{B}_p(q)$ (a scalar transfer function).

where we have defined

$$\widetilde{R}_p(N) \triangleq \left(\frac{1}{N} \sum_{k=0}^{N-1} \phi_k \phi_k^T \right). \tag{5.27}$$

Adopting the vectorized notation

$$\begin{aligned} Y^T &= (y_0, y_1, \cdots, y_{N-1}), \\ \Phi^T &= (\phi_0, \phi_1, \cdots, \phi_{N-1}), \\ V^T &= (\nu_0, \nu_1, \cdots, \nu_{N-1}), \end{aligned}$$

the model for the N point observed data record can be written as:

$$Y = \Phi\Theta + V, \tag{5.28}$$

so that the estimate $\widehat{\Theta}$ defined in equation (5.25) can be written in a more compact form as

$$\widehat{\Theta} = \Phi^{\dagger} Y, \tag{5.29}$$

where $\Phi^{\dagger}$ is the *Moore-Penrose* pseudoinverse of Φ. If there is sufficient input excitation for the indicated inverse to exist this will be given by

$$\Phi^{\dagger} = (\Phi^T \Phi)^{-1} \Phi^T. \tag{5.30}$$

The frequency response estimate is then given by

$$G(e^{\mathrm{j}\omega}, \widehat{\Theta}) = \widehat{\Theta}^T \Gamma_p(e^{\mathrm{j}\omega}). \tag{5.31}$$

Remark 5.4.1. Although the proposed identification scheme can be used with any orthonormal basis $\{\mathcal{B}_\ell(z)\} \in H_2(\mathbb{T})$, we will restrict the analysis of estimation accuracy to the generalized orthonormal basis with fixed poles (OBFP) introduced in Section 3.5 (or the equivalent MIMO version generated via Theorem 3.6.1), keeping in mind that these bases allow the representation of a more extensive class of models. In particular, they can be used to represent the multivariable fixed denominator model structure with arbitrary poles in (5.6)-(5.7). This is not the case with other orthonormal bases such as FIR, Laguerre, Kautz, or the more general OBGIF, where the poles are restricted either to be all the same, or to come from the same set which is repeated cyclically as higher model orders are assumed. ♦

In the following sections we analyze the numerical properties of the proposed identification algorithm, as well as the accuracy of the estimation (bias and variance errors).

5.5 Numerical properties

The numerical robustness of identification algorithms is particularly important in the context of multivariable systems where large numbers of parameters need to be estimated [Vd94a, Vib94]. Furthermore, it is well known [GV89] that when least squares procedures are employed, these numerical properties are strongly governed by the condition number of the 'covariance' matrix $\widetilde{R}_p(N)$ (defined in (5.27)) in the normal equation (5.26).

For the single-input, single-output case this condition number has been upper bounded in terms of the input spectral density $\Phi_u(\omega)$ when particular orthonormal basis model structures are employed [Wah91b, Wah94b, VHB95, NHG97b, GS58].

This result can be extended to arbitrary orthonormal bases and to the multivariable setting. This is done in the following Lemma, where (as in [Wah91b, Wah94b, NHG97b]) we compute lower and upper bounds on the singular values of the asymptotic (in data-length N) covariance matrix $\widetilde{R}_p$ defined as

$$\widetilde{R}_p \triangleq \lim_{N\to\infty} \widetilde{R}_p(N) = \lim_{N\to\infty} \left(\frac{1}{N} \sum_{k=0}^{N-1} \phi_k \phi_k^T \right) = \overline{\mathbf{E}} \left\{ \phi_k \phi_k^T \right\}.$$

Lemma 5.5.1. *Let the set of singular values of $\widetilde{R}_p$ be denoted as $\sigma[\widetilde{R}_p]$ and let $\underline{\sigma}[\Phi_u(\omega)]$ and $\overline{\sigma}[\Phi_u(\omega)]$ denote the smallest and the largest singular values of the input spectral density $\Phi_u(\omega)$, respectively. Then*

$$\sigma[\widetilde{R}_p] \subset \left[\inf_\omega \underline{\sigma}[\Phi_u(\omega)], \sup_\omega \overline{\sigma}[\Phi_u(\omega)] \right].$$

Proof: See Appendix 5.A. ■

Using this lemma, the condition number $\kappa[\widetilde{R}_p]$ of $\widetilde{R}_p$ may be straightforwardly upper bounded in terms of the singular values of the input spectral density as

$$1 \leq \kappa[\widetilde{R}_p] \leq \frac{\sup_\omega \overline{\sigma}[\Phi_u(\omega)]}{\inf_\omega \underline{\sigma}[\Phi_u(\omega)]}. \tag{5.32}$$

This upper bound provides a guaranteed worst-case limit for the numerical conditioning. Notice however that since the upper bound depends exclusively on the input excitation, the numerical conditioning can be very bad depending on the nature of this input. For the particular case of white input, the input spectral density $\Phi_u(\omega)$ is constant, so that equation (5.32) implies

$$\kappa[\widetilde{R}_p] = 1,$$

which means that the least squares estimation using the orthonormal structure is perfectly numerically conditioned.

Notice that the way the numerical conditioning is affected by an increasing model order (which is the case where the numerical issue is more important [GV89]) or by the particular choices for the poles of the bases, is not reflected by the upper bound we have derived.

As already mentioned, this guarantee of the worst–case numerical conditioning appears to have been one of the main motivations for using orthonormal bases in a system identification setting. However, to the best of our knowledge, there is no theoretical evidence to suggest that an orthonormal structure will necessarily provide a better numerical conditioning when compared with the one obtained with an equivalent non-orthonormal structure. In the following, lower and upper bounds on the ratio between the condition numbers using orthonormal and non-orthonormal model structures are derived.

Let the transfer matrix of the system be represented as a truncated series expansion in terms of the orthonormal basis $\{\mathcal{B}_k(q)\}_{k=0}^{p-1}$. This representation leads to the linear regressor form of equation (5.18), or its vectorized form (5.28). The asymptotic (in data-length) numerical conditioning of the least squares estimation is then given by the condition number of the matrix $\overline{\mathbf{E}}\left\{\phi_k\phi_k^T\right\}$.

Similarly, the transfer matrix can be represented as a linear combination of the elements of a (non-orthonormal) set $\{\mathcal{A}_k(q)\}_{k=0}^{p-1}$, spanning the same space as $\{\mathcal{B}_k(q)\}_{k=0}^{p-1}$. For example, for the case of $\{\mathcal{B}_k(q)\}_{k=0}^{p-1}$ been the OBFP introduced in Section 3.5, a possible non-orthonormal set spanning the same space is given by the functions $\mathcal{A}_k(q) = 1/(q-\xi_k)$, with the same poles as in the set $\{\mathcal{B}_k(q)\}$. The representation of the system using the non-orthonormal set also leads to a linear regressor form

$$y_k = G(q,\alpha)u_k + \nu_k = \alpha^T\psi_k + \nu_k,$$

where $\alpha \in \mathbb{R}^{np\times m}$ is the parameter matrix to be estimated, and where

$$\begin{aligned}
\psi_k &\triangleq \Upsilon_p(q)u_k,\\
\Upsilon_p(q) &\triangleq \mathsf{A}_p(q)\otimes I_n,\\
\mathsf{A}_p(q) &\triangleq [\mathcal{A}_0(q),\cdots,\mathcal{A}_{p-1}(q)]^T.
\end{aligned}$$

In vectorized form we can write

$$\begin{aligned}
Y &= \Psi\alpha + V,\\
Y^T &\triangleq (y_0, y_1, \cdots, y_{N-1}),\\
\Psi^T &\triangleq (\psi_0, \psi_1, \cdots, \psi_{N-1}),\\
V^T &\triangleq (\nu_0, \nu_1, \cdots, \nu_{N-1}).
\end{aligned}$$

In this case the asymptotic (in data-length) numerical conditioning of the least squares estimation of the parameter matrix α is determined by the condition number of the matrix $\overline{\mathbf{E}}\left\{\psi_k\psi_k^T\right\}$.

Our objective is now to relate the matrices $\overline{\mathbf{E}}\left\{\phi_k\phi_k^T\right\}$ and $\overline{\mathbf{E}}\left\{\psi_k\psi_k^T\right\}$ in order to compare the numerical conditioning of the estimation using orthonormal and non-orthonormal model structures. To proceed with this, notice that since $\mathcal{A}_k(z) \in \mathsf{Span}\left\{\{\mathcal{B}_k(z)\}_{k=0}^{p-1}\right\}$ we can write

$$\begin{aligned}\mathcal{A}_k(z) &= \sum_{j=0}^{p-1}\langle\mathcal{A}_k(z),\mathcal{B}_j(z)\rangle\,\mathcal{B}_j(z),\\ &= \left[\langle\mathcal{A}_k(z),\mathcal{B}_0(z)\rangle,\cdots,\langle\mathcal{A}_k(z),\mathcal{B}_{p-1}(z)\rangle\right]\mathsf{B}_p(z).\end{aligned}$$

Then $\mathsf{A}_p(z)$ and $\mathsf{B}_p(z)$ are related according to

$$\mathsf{A}_p(z) = J^{-1}\mathsf{B}_p(z),$$

where $J \in \mathbb{R}^{p\times p}$ is the nonsingular matrix

$$J = \begin{bmatrix}\langle\mathcal{A}_0(z),\mathcal{B}_0(z)\rangle & \cdots & \langle\mathcal{A}_0(z),\mathcal{B}_{p-1}(z)\rangle\\ \vdots & \vdots & \vdots\\ \langle\mathcal{A}_{p-1}(z),\mathcal{B}_0(z)\rangle & \cdots & \langle\mathcal{A}_{p-1}(z),\mathcal{B}_{p-1}(z)\rangle\end{bmatrix}^{-1}.$$

Based on this it is not difficult to show that the regressors ψ_k and ϕ_k are related according to

$$\psi_k = \left(J^{-1}\otimes I_n\right)\phi_k, \tag{5.33}$$

and then

$$\overline{\mathbf{E}}\left\{\psi_k\psi_k^T\right\} = \left(J^{-1}\otimes I_n\right)\overline{\mathbf{E}}\left\{\phi_k\phi_k^T\right\}\left(J^{-1}\otimes I_n\right)^T, \tag{5.34}$$

which is the relation we were looking for. Considering now that for a given norm and matrices A and B of appropriate dimensions [GV89]

$$\kappa[AB] \le \kappa[A]\kappa[B],$$

from equation (5.34) we can write

$$\frac{1}{\kappa[(J\otimes I_n)]^2} \le \frac{\kappa[\overline{\mathbf{E}}\left\{\psi_k\psi_k^T\right\}]}{\kappa[\overline{\mathbf{E}}\left\{\phi_k\phi_k^T\right\}]} \le \kappa[(J\otimes I_n)]^2. \tag{5.35}$$

The above equation shows that it is not necessarily true that the use of orthonormal model structures represents an improvement on the numerical conditioning of the estimation with respect to the case of using non-orthonormal model structures. The following example illustrates this fact.

Example 5.5.1. Let us consider the scalar case where the orthonormal functions $\{\mathcal{B}_k(z)\}$ are the Laguerre bases

$$\mathcal{B}_k(z) = \left(\frac{\sqrt{1-\xi^2}}{z-\xi}\right)\left(\frac{1-\xi z}{z-\xi}\right)^k \quad ; \quad k \geq 0, \tag{5.36}$$

and where the non-orthonormal functions $\{\mathcal{A}_k(z)\}$ are given by

$$\mathcal{A}_k(z) = \frac{1}{(z-\xi)^{k+1}} \quad ; \quad k \geq 0, \tag{5.37}$$

with the same fixed pole ξ, and spanning the same space as $\{\mathcal{B}_k(z)\}$. For the case of model order $p = 3$, matrix J is given by

$$J = \begin{bmatrix} \sqrt{1-\xi^2} & 0 & 0 \\ -\xi\sqrt{1-\xi^2} & (1-\xi^2)\sqrt{1-\xi^2} & 0 \\ \xi^2\sqrt{1-\xi^2} & 2\xi(\xi^2-1)\sqrt{1-\xi^2} & \sqrt{1-\xi^2}(1-\xi^4+2\xi^2(\xi^2-1)) \end{bmatrix}.$$

Figure 5.1 shows the upper bound $\kappa[(J \otimes I_n)]^2$ in equation (5.35) as a function of the pole location (ξ) for this case.

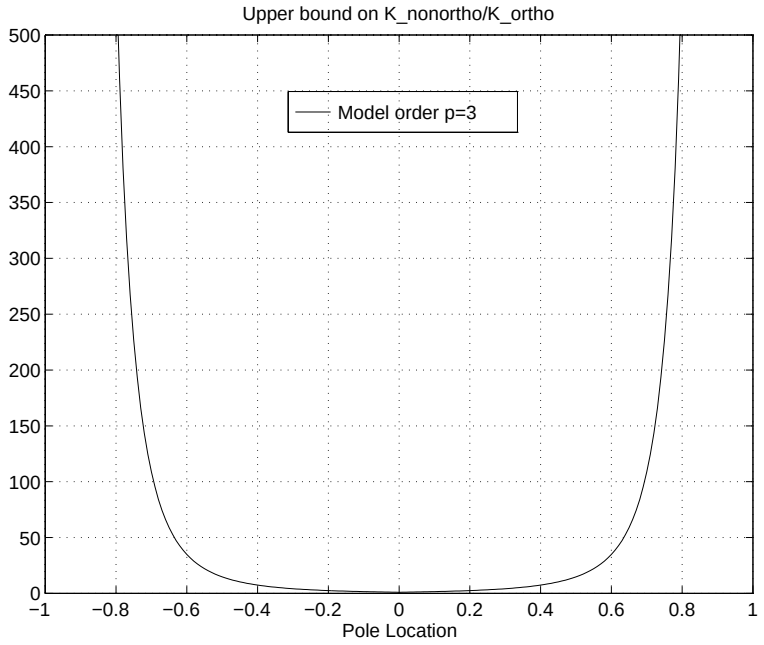

Figure 5.1: *Upper bound on the ratio between the condition numbers for the non-orthonormal and the orthonormal structures* ($\kappa[\overline{\mathbf{E}}\{\psi_k\psi_k^T\}]/\kappa[\overline{\mathbf{E}}\{\phi_k\phi_k^T\}]$) *as a function of the pole location* ξ*, and for model order* $p = 3$.

The left hand plot of Figure 5.2 shows the condition numbers $\kappa[\overline{\mathbf{E}}\left\{\psi_k\psi_k^T\right\}]$ (dashed line) and $\kappa[\overline{\mathbf{E}}\left\{\phi_k\phi_k^T\right\}]$ (solid line) corresponding to the non-orthonormal and the orthonormal structures respectively, as a function of the model order p, for the case $\xi = -0.1$. The right hand plot of the figure shows the ratio $\kappa[\overline{\mathbf{E}}\left\{\psi_k\psi_k^T\right\}]/\kappa[\overline{\mathbf{E}}\left\{\phi_k\phi_k^T\right\}]$ between these conditions numbers. The simulations were performed with a unit amplitude, square wave input signal of fundamental frequency 0.01 Hz.

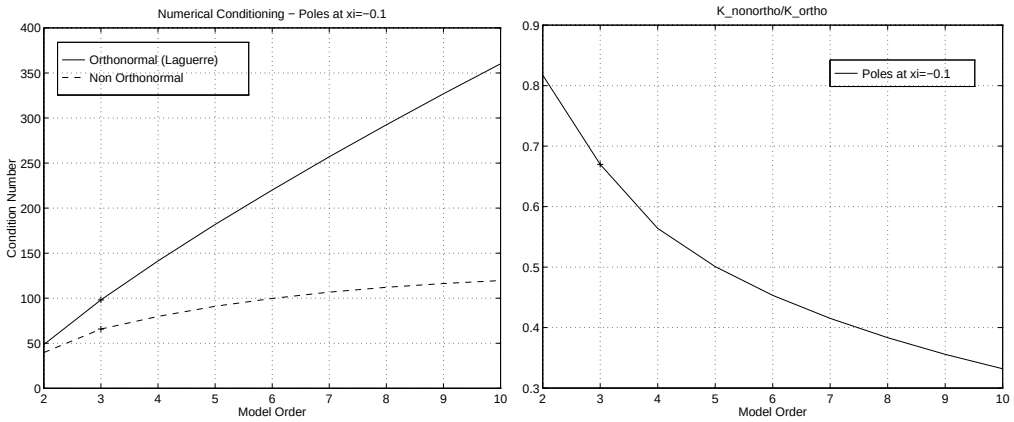

Figure 5.2: *Left hand figure shows the condition numbers* $\kappa[\overline{\mathbf{E}}\left\{\psi_k\psi_k^T\right\}]$ *(dashed line) and* $\kappa[\overline{\mathbf{E}}\left\{\phi_k\phi_k^T\right\}]$ *(solid line) corresponding to the non-orthonormal and the orthonormal structures respectively, as a function of the model order* p *for the case* $\xi = -0.1$. *Right hand figure shows the ratio* $\kappa[\overline{\mathbf{E}}\left\{\psi_k\psi_k^T\right\}]/\kappa[\overline{\mathbf{E}}\left\{\phi_k\phi_k^T\right\}]$ *as a function of* p *for this case.*

The simulations show that for the case $p = 3$ and $\xi = -0.1$ the upper and lower bounds on the ratio between the condition numbers of the non-orthonormal and orthonormal covariance matrices are

$$\kappa[(J \otimes I_n)]^2 = 1.5688 \quad , \quad \frac{1}{\kappa[(J \otimes I_n)]^2} = 0.6374,$$

while the value of this ratio is

$$\frac{\kappa[\overline{\mathbf{E}}\left\{\psi_k\psi_k^T\right\}]}{\kappa[\overline{\mathbf{E}}\left\{\phi_k\phi_k^T\right\}]} = 0.6698.$$

It is clear from the plots in Figure 5.2 that, in this case, the use of the orthonormal structure (5.36) instead of the more natural non-orthonormal form (5.37) results in a deterioration of the numerical conditioning rather than in an improvement. ▼

5.6 Undermodelling Error

In this section we derive results regarding the quantification of the component of the transfer matrix estimation error that is produced by the restricted complexity of the model used to represent the system. For the single-input, single-output case, the result of Theorem 4.3.1, in the form of an upper bound on the undermodelling error, shows that the flexibility in the choice of the poles of the bases can be exploited in order to provide a more accurate estimation. A similar result can be established for the multivariable case '*mutatis mutandis*' from the single-input, single-output result in Theorem 4.3.1.

Let $\widehat{G}(z)$ denote the best $H_2^{m\times n}(\mathbb{T})$ approximation to $G(z)$ with respect to the pmn basis matrices $\left\{\mathcal{B}_0^{11},\cdots,\mathcal{B}_0^{mn},\cdots,\mathcal{B}_{p-1}^{11},\cdots,\mathcal{B}_{p-1}^{mn}\right\}$ defined by equations (3.28)-(3.14) . Namely

$$\widehat{G}(z) \triangleq \sum_{k=0}^{p-1}\sum_{i=1}^{m}\sum_{j=1}^{n}\left\langle G,\mathcal{B}_k^{ij}\right\rangle \mathcal{B}_k^{ij}(z) \tag{5.38}$$

where $\langle\cdot,\cdot\rangle$ stands for the inner product in the space $H_2^{m\times n}(\mathbb{T})$ as defined in (3.29). Considering the definition of $\mathcal{B}_k^{ij}(z)$ in (3.28), equation (5.38) can be written as

$$\widehat{G}(z)=\sum_{k=0}^{p-1}\begin{bmatrix}\langle G,\mathcal{B}_k^{11}\rangle & \cdots & \langle G,\mathcal{B}_k^{1n}\rangle\\ \vdots & \ddots & \vdots\\ \langle G,\mathcal{B}_k^{m1}\rangle & \cdots & \langle G,\mathcal{B}_k^{mn}\rangle\end{bmatrix}\mathcal{B}_k(z)$$

Then, the i,j-th component $\widehat{G}_{ij}(z)$ of $\widehat{G}(z)$ is given by

$$\widehat{G}_{ij}(z)=\sum_{k=0}^{p-1}\left\langle G,\mathcal{B}_k^{ij}\right\rangle \mathcal{B}_k(z). \tag{5.39}$$

Using again the definition of $\mathcal{B}_k^{ij}(z)$, it is straightforward to show that

$$\left\langle G,\mathcal{B}_k^{ij}\right\rangle = \langle G_{ij},\mathcal{B}_k\rangle\,,$$

where in the RHS, the symbol $\langle\cdot,\cdot\rangle$ stands now for the inner product in the space $H_2(\mathbb{T})$ as defined in (3.27)[4]. Substituting back in equation (5.39), the expression for $\widehat{G}_{ij}(z)$ becomes

$$\widehat{G}_{ij}(z)=\sum_{k=0}^{p-1}\langle G_{ij},\mathcal{B}_k\rangle\,\mathcal{B}_k(z), \tag{5.40}$$

[4]The fact that the same symbol is being used to denote both the inner product in $H_2(\mathbb{T})$ and in $H_2^{m\times n}(\mathbb{T})$ should lead to no confusion, since the meaning becomes clear from the context.

which is the best $H_2(\mathbb{T})$ approximation of $G_{ij}(z)$ with respect to the p basis functions $\{\mathcal{B}_0(z), \mathcal{B}_1(z), \cdots, \mathcal{B}_{p-1}(z)\}$. In similar way, it can be proved that the i,j-th component $G_{ij}(z)$ of the true transfer matrix $G(z)$ is given by

$$G_{ij}(z) = \sum_{k=0}^{\infty} \langle G_{ij}, \mathcal{B}_k \rangle \, \mathcal{B}_k(z). \tag{5.41}$$

We are now able to quantify (component-wise) the error between the true transfer function $G_{ij}(z)$ and the restricted complexity approximation $\widehat{G}_{ij}(z)$. The result is summarized in the following theorem.

Theorem 5.6.1. *Let the i,j-th component $G_{ij}(z)$ of the transfer matrix $G(z)$ of the system have partial fraction expansion*

$$G_{ij}(z) = \sum_{\ell=0}^{r^{ij}-1} \frac{\alpha_\ell^{ij}}{z - \gamma_\ell^{ij}},$$

where all the poles satisfy $|\gamma_\ell^{ij}| < 1$. Let $\widehat{G}_{ij}(z)$ denote the i,j-th component of best $H_2^{m\times n}(\mathbb{T})$ approximation $\widehat{G}(z)$ to $G(z)$ with respect to the pmn basis matrices $\left\{\mathcal{B}_0^{11}, \cdots, \mathcal{B}_0^{mn}, \cdots, \mathcal{B}_{p-1}^{11}, \cdots, \mathcal{B}_{p-1}^{mn}\right\}$ defined by equations (3.28)-(3.14), with fixed poles $\{\xi_k\}(k = 0, \cdots, p-1)$. Then

$$\left|G_{ij}(e^{\mathrm{j}\omega}) - \widehat{G}_{ij}(e^{\mathrm{j}\omega})\right| < \sum_{\ell=0}^{r^{ij}-1} \left|\frac{\alpha_\ell^{ij}}{e^{\mathrm{j}\omega} - \gamma_\ell^{ij}}\right| \prod_{k=0}^{p-1} \left|\frac{\gamma_\ell^{ij} - \xi_k}{1 - \overline{\xi_k}\gamma_\ell^{ij}}\right|. \tag{5.42}$$

Proof: Considering the derived expressions for $G_{ij}(z)$ and $\widehat{G}_{ij}(z)$ in equations (5.41) and (5.40) respectively, it becomes clear that the proof proceeds in an identical way as that of Theorem 4.3.1 for the single-input, single-output case, and therefore it is omitted. ■

The same comments made on page 60 in relation to the result in Theorem 4.3.1 also apply component-wise to this case. The theorem indicates that the approximation error that results from the use of the restricted complexity model structure (5.38) can be reduced by choosing the poles $\{\xi_k\}$ of the bases close to the true poles $\{\gamma_\ell^{ij}\}$ of the system, since this minimizes the product term in the upper bound in (5.42).

Theorem 5.6.1 can also be used to provide a quantification of the undermodelling induced error in the least squares estimate $G(q, \widehat{\Theta})$ by noting that under the given assumptions, by the results in [Lju87] (as summarized in Theorems 2.2.1 and 2.2.2)

$$\widehat{\Theta} \xrightarrow{\text{a.s.}} \Theta_\star \quad \text{as } N \to \infty,$$

where

$$\Theta_\star = \underset{\Theta\in\mathbb{R}^{m\times np}}{\arg\min}\left\{\lim_{N\to\infty}\frac{1}{N}\mathsf{Tr}\left\{\sum_{k=0}^{N-1}\mathbf{E}\left\{\varepsilon_k(\Theta)\varepsilon_k^T(\Theta)\right\}\right\}\right\},$$

with

$$\varepsilon_k(\Theta) \triangleq y_k - G(q,\Theta)u_k = \left(G(q) - G(q,\Theta)\right)u_k + \nu_k.$$

Using Parseval's Theorem (in the form given in equation (2.3)), and neglecting terms independent of Θ then allows $\Theta_\star$ to be re–expressed as

$$\Theta_\star = \underset{\Theta\in\mathbb{R}^{m\times np}}{\arg\min}\left\{\frac{1}{2\pi}\int_{-\pi}^{\pi}\mathsf{Tr}\left\{\Phi_{\tilde{u}}(\omega,\Theta)\right\}\,\mathrm{d}\omega\right\}, \tag{5.43}$$

where

$$\Phi_{\tilde{u}}(\omega,\Theta) = \left(G(e^{\mathrm{j}\omega}) - G(e^{\mathrm{j}\omega},\Theta)\right)\Phi_u(\omega)\left(G(e^{\mathrm{j}\omega}) - G(e^{\mathrm{j}\omega},\Theta)\right)^\star.$$

In the special case of white input excitation where $\Phi_u = I_n$, this gives the asymptotic estimate as

$$\Theta_\star = \underset{\Theta\in\mathbb{R}^{m\times np}}{\arg\min}\left\{\frac{1}{2\pi}\sum_{k=1}^{m}\sum_{\ell=1}^{n}\int_{-\pi}^{\pi}|G_{k\ell}(e^{\mathrm{j}\omega}) - G_{k\ell}(e^{\mathrm{j}\omega},\Theta)|^2\,\mathrm{d}\omega\right\}.$$

This criterion is satisfied by the solution

$$\Theta_\star = \left[\theta_0^T,\cdots,\theta_{p-1}^T\right],$$

with

$$\theta_k^T = \begin{bmatrix}\langle G,\mathcal{B}_k^{11}\rangle & \cdots & \langle G,\mathcal{B}_k^{1n}\rangle\\ \vdots & \ddots & \vdots\\ \langle G,\mathcal{B}_k^{m1}\rangle & \cdots & \langle G,\mathcal{B}_k^{mn}\rangle\end{bmatrix},$$

so that the result of Theorem 5.6.1 implies

$$\left|G_{ij}(e^{\mathrm{j}\omega}) - G_{ij}(e^{\mathrm{j}\omega},\Theta_\star)\right| < \sum_{\ell=0}^{r^{ij}-1}\left|\frac{\alpha_\ell^{ij}}{e^{\mathrm{j}\omega}-\gamma_\ell^{ij}}\right|\prod_{k=0}^{p-1}\left|\frac{\gamma_\ell^{ij}-\xi_k}{1-\overline{\xi_k}\gamma_\ell^{ij}}\right|, \tag{5.44}$$

which provides quantification (which applies with probability one) of the asymptotic undermodelling induced error in the case of white input excitation.

The case of non–white input is more difficult. Nevertheless, for such cases it is still possible to derive frequency dependent upper bounds on the undermodelling error similar to that possible for the white input case (5.44), but that are

not as tight or as explicit. To present them, it is necessary to recall that since $\{\mathcal{B}_k^{ij}\}$ is complete in $H_2^{m\times n}(\mathbb{T})$ then any $G(e^{\mathrm{j}\omega}) \in H_2^{m\times n}(\mathbb{T})$ may be expanded as

$$G(e^{\mathrm{j}\omega},\Theta) = \sum_{\ell=0}^{p-1}\sum_{i=1}^{m}\sum_{j=1}^{n}\theta_\ell^{ij}\mathcal{B}_\ell^{ij}(e^{\mathrm{j}\omega}) + \sum_{\ell=p}^{\infty}\sum_{i=1}^{m}\sum_{j=1}^{n}\theta_\ell^{ij}\mathcal{B}_\ell^{ij}(e^{\mathrm{j}\omega}) \tag{5.45}$$

where $\theta_\ell^{ij} \triangleq < G, \mathcal{B}_\ell^{ij} >$ are the generalized Fourier coefficients. Appealing to Lemma 5.3.1, equation (5.45) can be written as

$$G(e^{\mathrm{j}\omega}) = \sum_{\ell=0}^{p-1}\theta_\ell\mathcal{B}_\ell(e^{\mathrm{j}\omega}) + \sum_{\ell=p}^{\infty}\theta_\ell\mathcal{B}_\ell(e^{\mathrm{j}\omega}). \tag{5.46}$$

where $\theta_\ell \in \mathbb{R}^{m\times n}$ are defined in equation (5.17). When $G(z)$ is finite dimensional, a simple argument using Cauchy's Residue Theorem shows that these parameters decay exponentially as $|\theta_k^{ij}| = o(\eta^k)$ as $k \to \infty$ for some $|\eta| < 1$.

In terms of these decaying parameters, and a possibly coloured input spectrum $\Phi_u(\omega)$, a frequency dependent bound on the undermodelling induced estimation error may then be given as follows.

Lemma 5.6.1. *Let K_1 be defined as*

$$K_1 \triangleq \sqrt{\frac{1+|\xi_{max}|}{1-|\xi_{max}|}}, \tag{5.47}$$

where ξ_{max} denotes the pole with maximum module in the set $\{\xi_k\}$. Then, with probability one:

$$\begin{aligned}&\left|[G(e^{\mathrm{j}\omega}) - G(e^{\mathrm{j}\omega},\Theta_0)]_{i,j}\right| \leq \\ &\leq \quad K_1\left(\sum_{\ell=p}^{\infty}\max_{i,j}\left|\theta_l^{ij}\right|\right)\left(\sup_\omega \gamma_p(\omega)\sqrt{mn}\frac{\sup_\omega \overline{\sigma}[\Phi_u(\omega)]}{\inf_\omega \underline{\sigma}[\Phi_u(\omega)]} + 1\right).\end{aligned}$$

Proof: See Appendix 5.B. ■

The chief use for this result is in the derivation of the main result of this chapter (Theorem 5.7.1) where it is used to show that as the model order grows, a particular component of the estimation variance expression decays to zero uniformly in frequency.

A less explicit characterization of the nature of the undermodelling induced estimation error is given by the following Lemma.

Lemma 5.6.2. *The frequency response is on average (over frequency) under–estimated in the sense that*

$$\int_{-\pi}^{\pi}\mathsf{Tr}\left\{G(e^{\mathrm{j}\omega},\Theta_\star)\Phi_u(\omega)G^\star(e^{\mathrm{j}\omega},\Theta_\star)\right\}\,\mathrm{d}\omega \leq \int_{-\pi}^{\pi}\mathsf{Tr}\left\{G(e^{\mathrm{j}\omega})\Phi_u(\omega)G^\star(e^{\mathrm{j}\omega})\right\}\,\mathrm{d}\omega.$$

Proof: The proof is analogous to the one corresponding to the single-input, single-output case given in [Nin96], and can be found in Appendix 5.B. ■

5.7 Noise Induced Error

In this section, the main result (Theorem 5.7.1) of the chapter, regarding the quantification of the component of the frequency response estimation error that is induced by the measurement noise, is derived. The result is the extension to general multivariable model structures with fixed poles of several asymptotic variance expressions that have been derived in the literature in the single-input, single-output context [LY85, Wah91b, Wah94b, VHB95, NHG97a], or in the multivariable framework but only for FIR model structures [YL84]. The result is built on earlier work that applies only to single-input, single-output systems [NHG97a], as summarized in Section 4.4.

Here again, as in the SISO case, a quantification of the noise induced error is given by the covariance matrix of the transfer matrix estimate $G(e^{j\omega}, \widehat{\Theta})$. However, since it is desiderable to be able to quantify the error in each individual component $G_{ij}(e^{j\omega}, \Theta)$ of the matrix $G(e^{j\omega}, \Theta)$ it is more convenient to use the methods in [YL84, Zhu89, Zhu90] and deal with a vectorized form of $G(e^{j\omega}, \Theta)$. This vectorized form is obtained by stacking the columns of $G(e^{j\omega}, \Theta)$ on top of each other, that is by applying the vec-operator (see Appendix A for the definition and some properties) to the matrix $G(e^{j\omega}, \Theta)$ to obtain

$$g(e^{j\omega}, \Theta) \triangleq \mathsf{vec}\, G(e^{j\omega}, \Theta).$$

Analyzing the distribution of the estimate $g(e^{j\omega}, \widehat{\Theta})$ then provides information about the noise induced error in each component of $G(e^{j\omega}, \Theta)$.

We start the analysis by re-defining the parameter matrix in such a way that the vectorized transfer matrix is still a linear function of the parameters, so that the linear regressor form of the model (5.18) is preserved and therefore a closed form solution for the least squares estimation can also be obtained for this case. Considering the linear regressor model in equation (5.18), and using properties of the Kronecker products and vec-operator (see Appendix A) we can write

$$\begin{aligned} y_k &= \Theta^T \phi_k + \nu_k, \\ &= \mathsf{vec}\, \left\{\Theta^T \phi_k\right\} + \nu_k, \\ &= \left(\phi_k^T \otimes I_m\right) \mathsf{vec}\, \Theta^T + \nu_k, \\ &= \psi_k^T \eta + \nu_k, \end{aligned} \tag{5.48}$$

where

$$\eta \triangleq \mathsf{vec}\, \Theta^T \tag{5.49}$$

is the new parameter vector, and

$$\psi_k \triangleq \phi_k \otimes I_m \tag{5.50}$$

is the new regressor matrix. It is clear that the new parameter vector η linearly parameterizes the vectorized transfer matrix $g(e^{\mathrm{j}\omega}, \Theta)$, since

$$\begin{aligned} g(e^{\mathrm{j}\omega}, \Theta) &= \mathsf{vec}\,\{\Theta^T \Gamma_p(e^{\mathrm{j}\omega})\}, \\ &= \left(\Gamma_p^T(e^{\mathrm{j}\omega}) \otimes I_m\right) \mathsf{vec}\,\Theta^T, \\ &= \widetilde{\Gamma}_p^T(e^{\mathrm{j}\omega})\ \eta, \end{aligned} \tag{5.51}$$

where the definition

$$\widetilde{\Gamma}_p(e^{\mathrm{j}\omega}) \triangleq \Gamma_p(e^{\mathrm{j}\omega}) \otimes I_m$$

has been made.

It is not difficult to prove (see Lemma 5.C.1 in Appendix 5.C) that the least squares estimate $\widehat{\eta}$ of η, given by the well known expression

$$\widehat{\eta} = \left(\frac{1}{N}\sum_{k=0}^{N-1} \psi_k \psi_k^T\right)^{-1} \left(\frac{1}{N}\sum_{k=0}^{N-1} \psi_k y_k\right), \tag{5.52}$$

and the least squares estimate $\widehat{\Theta}$ of Θ in equation (5.26), are related according to

$$\widehat{\eta} = \mathsf{vec}\,\widehat{\Theta}^T.$$

The vectorized transfer matrix estimate is then given by

$$g(e^{\mathrm{j}\omega}, \widehat{\eta}) = \widetilde{\Gamma}_p^T(e^{\mathrm{j}\omega})\ \widehat{\eta}.$$

We undertake now the derivation of the main result of the chapter regarding the asymptotic (in data-length and model order) distribution of the (vectorized) transfer matrix estimate. Fundamental in this analysis will be some results concerning the asymptotic properties of what we will call *Block Toeplitz-like matrices*. These results are presented in the following subsection.

5.7.1 Convergence of Block Toeplitz-like Matrices

In this subsection and the following ones we consider positive definite real matrices $M_p(F, W)$ of dimensions $nmp \times nmp$, defined by two real symmetric positive definite matrix valued functions $F(\omega)$ and $W(\omega)$ (of dimensions $n \times n$ and $m \times m$ respectively) as

$$M_p(F, W) \triangleq \frac{1}{2\pi}\int_{-\pi}^{\pi} \Gamma_p(\omega) F(\omega) \Gamma_p^\star(\omega) \otimes W(\omega)\,\mathrm{d}\omega. \tag{5.53}$$

Here, for the sake of notational simplicity, we have used the notation $\Gamma_p(\omega)$ as a shorthand for $\Gamma_p(e^{\mathrm{j}\omega})$ as defined in (5.21), viz

$$\Gamma_p(\omega) \equiv \Gamma_p(e^{\mathrm{j}\omega}) \triangleq \left[I_n\mathcal{B}_0(e^{\mathrm{j}\omega}), I_n\mathcal{B}_1(e^{\mathrm{j}\omega}), \cdots, I_n\mathcal{B}_{p-1}(e^{\mathrm{j}\omega})\right]^T,$$

where $\{\mathcal{B}_k(z)\}$ are orthonormal basis functions in $H_2(\mathbb{T})$.

We refer to $M_p(F,W)$ in (5.53) as a *Block Toeplitz-like Matrix*, since for the case of $\{\mathcal{B}_k(z)\}$ being the standard FIR (or trigonometric) bases $\left\{\mathcal{B}_k(z) = z^{-k}\right\}$, $M_p(F,W)$ has effectively a block Toeplitz structure[5] [GS58]. To see this, let us consider the explicit expression of $M_p(F,W)$ for this case. Using properties of the Kronecker product (see Appendix A) we can write

$$\begin{aligned} M_p(F,W) &\triangleq \frac{1}{2\pi}\int_{-\pi}^{\pi} \Gamma_p(\omega)F(\omega)\Gamma_p^\star(\omega) \otimes W(\omega)\,\mathrm{d}\omega, \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \left(\mathsf{B}_p(e^{\mathrm{j}\omega}) \otimes I_n\right) F(\omega) \left(\mathsf{B}_p^\star(e^{\mathrm{j}\omega}) \otimes I_n\right) \otimes W(\omega)\,\mathrm{d}\omega, \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \mathsf{B}_p(e^{\mathrm{j}\omega})\mathsf{B}_p^\star(e^{\mathrm{j}\omega}) \otimes (F(\omega) \otimes W(\omega))\,\mathrm{d}\omega. \end{aligned}$$

That $M_p(F,W)$ has a block Toeplitz structure now follows from the fact that for the case of FIR basis, the matrix $\left(\mathsf{B}_p(e^{\mathrm{j}\omega})\mathsf{B}_p^\star(e^{\mathrm{j}\omega})\right)$ has itself a Toeplitz structure, since it is given by

$$\mathsf{B}_p(e^{\mathrm{j}\omega})\mathsf{B}_p^\star(e^{\mathrm{j}\omega}) = \begin{bmatrix} 1 & e^{\mathrm{j}\omega} & e^{\mathrm{j}2\omega} & \cdots & e^{\mathrm{j}(p-1)\omega} \\ e^{-\mathrm{j}\omega} & 1 & e^{\mathrm{j}\omega} & \cdots & e^{\mathrm{j}(p-2)\omega} \\ e^{-\mathrm{j}2\omega} & e^{-\mathrm{j}\omega} & 1 & \cdots & e^{\mathrm{j}(p-3)\omega} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ e^{-\mathrm{j}(p-1)\omega} & e^{-\mathrm{j}(p-2)\omega} & e^{-\mathrm{j}(p-3)\omega} & \cdots & 1 \end{bmatrix}.$$

For arbitrary orthonormal bases $\{\mathcal{B}_k(z)\}$ the matrix form (5.53) will in general not have a Toeplitz structure. However, some asymptotic properties of Toeplitz matrices [GS58, HN77, HW89] will also hold for the form (5.53), what justifies the name *Toeplitz-like* we are using. In particular, the following Lemma provides a result concerning the convergence of a quadratic form of the block Toeplitz-like matrix (5.53) as the model order p tends to infinity, for the case of $\{\mathcal{B}_k(z)\}$ being the OBFP introduced in Section 3.5.

Lemma 5.7.1. *Let $M_p(F,W)$ be the $(mnp \times mnp)$ block Toeplitz-like matrix defined in equation (5.53), where $F(\omega)$, and $W(\omega)$ (of dimensions $(n \times n)$ and $(m \times m)$ respectively) are assumed to be positive definite, Lipschitz continuous in $[-\pi,\pi]$, and with finite dimensional spectral factorizations.*

[5] A matrix is said to be block Toeplitz if its i,j-th block entry depends only on the difference $(i-j)$ of the indices.

It is also assumed that all the poles $\{\xi_k\}$ of the basis functions are chosen in the open unit disc $\mathbb{D}$. Then the following limit result holds

$$\lim_{p\to\infty} \frac{(\Gamma_p^\star(\mu)\otimes I_m)M_p(F,W)(\Gamma_p(\omega)\otimes I_m)}{\gamma_p^{1/2}(\omega)\gamma_p^{1/2}(\mu)} = \begin{cases} F(\omega)\otimes W(\omega) & ; \quad \mu=\omega \\ 0 & ; \quad \mu\neq\omega \end{cases}$$

where

$$\gamma_p(\omega) \triangleq K_p(\omega,\omega) = \sum_{k=0}^{p-1} \left|\mathcal{B}_k(e^{\mathrm{j}\omega})\right|^2 .$$

Proof: See Appendix 5.D. ■

Lemma 5.7.2. *Let $M_p(\cdot,\cdot)$ be the $(mnp\times mnp)$ block Toeplitz-like matrix defined in equation (5.53), and let $W(\omega)>0, X(\omega)>0, Z(\omega)>0$ and $U(\omega)>0$ be spectral densities (of dimensions $(n\times n)$, $(m\times m)$, $(n\times n)$ and $(m\times m)$, respectively) with finite dimensional spectral factorizations. Then there exists $|\eta|<1$ and $K<\infty$ such that*

$$\left|[M_p(W,X)M_p(Z,U)]_{smn+y,tmn+x} - [M_p(WZ,XU)]_{smn+y,tmn+x}\right| \leq \\ \leq mnK^2(\eta^{p-s}+\eta^s)(\eta^{p-t}+\eta^t)$$

Proof: See Appendix 5.D. ■

5.7.2 Asymptotic Distribution of the (Vectorized) Tranfer Matrix Estimate

Based on the convergence results of the block Toeplitz-like matrices of the previous subsection, we are now able to analyze the asymptotic (in data-length and model order) distribution of the vectorized transfer matrix estimate. The result is given in the following theorem.

Theorem 5.7.1. *Let $\Phi_u(\omega)$ and $\Phi_\nu(\omega)$ be the input and measurement noise spectral densities, respectively, and let $\Phi_u(\omega)$ have a finite dimensional spectral factorisation. If the poles $\{\xi_k\}$ of the bases (3.14) are all chosen within the open unit disc $\mathbb{D}$, then as $N\to\infty$ and $p\to\infty$*

$$\left(\sqrt{N}\begin{bmatrix} \gamma_p(\omega_1) & 0 \\ 0 & \gamma_p(\omega_2)\end{bmatrix}^{-1/2} \otimes I_{mn}\right)\begin{bmatrix} g(e^{\mathrm{j}\omega_1},\widehat{\eta}) - g(e^{\mathrm{j}\omega_1},\eta_0) \\ g(e^{\mathrm{j}\omega_2},\widehat{\eta}) - g(e^{\mathrm{j}\omega_2},\eta_0)\end{bmatrix} \xrightarrow{dist} \\ \xrightarrow{dist} \mathcal{N}\left(0,\Lambda(\omega_1,\omega_2)\right)$$

where

$$\Lambda(\omega_1,\omega_2) = \begin{bmatrix} \Phi_u^{-1}(\omega_1)\otimes\Phi_\nu(\omega_1) & 0 \\ 0 & \Phi_u^{-1}(\omega_2)\otimes\Phi_\nu(\omega_2)\end{bmatrix}$$

if $\omega_1 \neq \omega_2$, *and*

$$\gamma_p(\omega) \triangleq K_p(\omega,\omega) = \sum_{k=0}^{p-1} |\mathcal{B}_k(e^{\mathrm{j}\omega})|^2.$$

Proof: By appealing to the results in [Lju87] as summarized in Theorems 2.2.1 and 2.2.2, it is possible to conclude that under the given assumptions

$$\sqrt{N}\,(\widehat{\eta} - \eta_\star) \xrightarrow{\text{dist}} \mathcal{N}\,(0, P_p) \qquad \text{as} \quad N \to \infty,$$

where

$$\begin{aligned}
P_p &\triangleq R_p^{-1} Q_p R_p^{-1}, \\
R_p &\triangleq \mathsf{E}\left\{V_N''(\eta_\star)\right\}, \\
Q_p &\triangleq \lim_{N\to\infty} N\mathsf{E}\left\{V_N'(\eta_\star)\left(V_N'(\eta_\star)\right)^T\right\}, \\
V_N(\eta) &\triangleq \frac{1}{N}\sum_{k=0}^{N-1} \mathsf{Tr}\left\{(y_k - (\phi_k^T \otimes I_m)\eta)(y_k - (\phi_k^T \otimes I_m)\eta)^T\right\}.
\end{aligned}$$

Therefore, since

$$g(e^{\mathrm{j}\omega}, \widehat{\eta}) = \widetilde{\Gamma}_p^T(e^{\mathrm{j}\omega})\widehat{\eta},$$

where $\widetilde{\Gamma}_p(\omega) \triangleq \Gamma_p(\omega) \otimes I_m$, it is possible to conclude that, as $N \to \infty$,

$$\left(\sqrt{N}\begin{bmatrix} \gamma_p(\omega_1) & 0 \\ 0 & \gamma_p(\omega_2) \end{bmatrix}^{-1/2} \otimes I_{mn}\right)\begin{bmatrix} g(e^{\mathrm{j}\omega_1}, \widehat{\eta}) - g(e^{\mathrm{j}\omega_1}, \eta_\star) \\ g(e^{\mathrm{j}\omega_2}, \widehat{\eta}) - g(e^{\mathrm{j}\omega_2}, \eta_\star) \end{bmatrix} \xrightarrow{\text{dist}}$$

$$\xrightarrow{\text{dist}} \mathcal{N}\,(0, \Lambda_p(\omega_1,\omega_2)),$$

where

$$\begin{aligned}
\Lambda_p(\omega_1,\omega_2) &\triangleq \left(\begin{bmatrix} \gamma_p(\omega_1) & 0 \\ 0 & \gamma_p(\omega_2) \end{bmatrix}^{-1/2} \otimes I_{mn}\right)\begin{bmatrix} \widetilde{\Gamma}_p^T(\omega_1) \\ \widetilde{\Gamma}_p^T(\omega_2) \end{bmatrix} \times \\
&\times P_p \begin{bmatrix} \widetilde{\Gamma}_p^T(\omega_1) \\ \widetilde{\Gamma}_p^T(\omega_2) \end{bmatrix}^\star \left(\begin{bmatrix} \gamma_p(\omega_1) & 0 \\ 0 & \gamma_p(\omega_2) \end{bmatrix}^{-1/2} \otimes I_{mn}\right).
\end{aligned}$$

After some matrix manipulation, $\Lambda_p(\omega_1,\omega_2)$ can be written as

$$\Lambda_p(\omega_1,\omega_2) = \begin{bmatrix} \dfrac{\widetilde{\Gamma}_p^\star(\omega_1)P_p\widetilde{\Gamma}_p(\omega_1)}{\gamma_p(\omega_1)} & \dfrac{\widetilde{\Gamma}_p^\star(\omega_1)P_p\widetilde{\Gamma}_p(\omega_2)}{\gamma_p^{1/2}(\omega_1)\gamma_p^{1/2}(\omega_2)} \\ \dfrac{\widetilde{\Gamma}_p^\star(\omega_2)P_p\widetilde{\Gamma}_p(\omega_1)}{\gamma_p^{1/2}(\omega_1)\gamma_p^{1/2}(\omega_2)} & \dfrac{\widetilde{\Gamma}_p^\star(\omega_2)P_p\widetilde{\Gamma}_p(\omega_2)}{\gamma_p(\omega_2)} \end{bmatrix},$$

so that it becomes clear that in order to analyze its asymptotic behavior when the model order p tends to infinity, we need only to study the asymptotic behavior of the term

$$\frac{\widetilde{\Gamma}_p^{\star}(\omega_1) P_p \widetilde{\Gamma}_p(\omega_2)}{\gamma_p^{1/2}(\omega_1)\gamma_p^{1/2}(\omega_2)}. \tag{5.54}$$

We first provide a frequency domain expression for the covariance matrix P_p. Considering the definition of the matrix R_p, simple algebra gives

$$R_p = \lim_{N\to\infty} \frac{1}{N} \sum_{k=0}^{N-1} \mathbf{E}\left\{\phi_k \phi_k^T \otimes I_m\right\}.$$

Using now Parseval's Theorem, and the definition in (5.53), the following frequency domain expression for R_p is obtained

$$R_p = \frac{1}{2\pi}\int_{-\pi}^{\pi} \Gamma_p(e^{\mathrm{j}\omega})\Phi_u(\omega)\Gamma_p^{\star}(e^{\mathrm{j}\omega}) \otimes I_m \,\mathrm{d}\omega = M_p(\Phi_u, I_m). \tag{5.55}$$

The derivation of a frequency domain expression for Q_p is more difficult. By Lemma 5.E.4 and the definition (5.53) Q_p may be expressed as

$$Q_p = \underbrace{\frac{1}{2\pi}\int_{-\pi}^{\pi} \Gamma_p(\omega)\Phi_u(\omega)\Gamma_p^{\star}(\omega) \otimes \Phi_\nu(\omega)\,\mathrm{d}\omega}_{=M_p(\Phi_u,\Phi_\nu)} + \Delta_p = M_p(\Phi_u,\Phi_\nu) + \Delta_p,$$

where

$$\begin{aligned} \Delta_p &\triangleq \frac{1}{2\pi}\int_{-\pi}^{\pi} \Gamma_p(\omega)\Phi_u(\omega)\Gamma_p^{\star}(\omega) \otimes \widetilde{G}_p(\omega)\Phi_u(\omega)\widetilde{G}_p^{\star}(\omega)\,\mathrm{d}\omega, \\ \widetilde{G}_p(\omega) &\triangleq G(e^{j\omega}) - G(e^{j\omega}, \eta_\star). \end{aligned}$$

Finally, matrix P_p can then be written as

$$\begin{aligned} P_p &= M_p^{-1}(\Phi_u, I_m)\left(M_p(\Phi_u,\Phi_\nu) + \Delta_p\right) M_p^{-1}(\Phi_u, I_m), \\ &= M_p^{-1}(\Phi_u, I_m) M_p(\Phi_u,\Phi_\nu) M_p^{-1}(\Phi_u, I_m) + M_p^{-1}(\Phi_u, I_m)\Delta_p M_p^{-1}(\Phi_u, I_m). \end{aligned}$$

With this expression for P_p the term in (5.54) becomes

$$\begin{aligned} \frac{\widetilde{\Gamma}_p^{\star}(\omega_2) P_p \widetilde{\Gamma}_p(\omega_1)}{\gamma_p^{1/2}(\omega_1)\gamma_p^{1/2}(\omega_2)} &= \frac{\widetilde{\Gamma}_p^{\star}(\omega_1) M_p^{-1}(\Phi_u, I_m) M_p(\Phi_u,\Phi_\nu) M_p^{-1}(\Phi_u, I_m) \widetilde{\Gamma}_p(\omega_2)}{\gamma_p^{1/2}(\omega_1)\gamma_p^{1/2}(\omega_2)} + \\ &+ \frac{\widetilde{\Gamma}_p^{\star}(\omega_1) M_p^{-1}(\Phi_u, I_m)\Delta_p M_p^{-1}(\Phi_u, I_m) \widetilde{\Gamma}_p(\omega_2)}{\gamma_p^{1/2}(\omega_1)\gamma_p^{1/2}(\omega_2)}. \end{aligned} \tag{5.56}$$

We analyze first the asymptotic behavior of the second term on the RHS of equation (5.56). For the i,j-th component we have[6]

$$\left|\left[\frac{\widetilde{\Gamma}_p^\star(\omega_1)M_p^{-1}(\Phi_u,I_m)\Delta_p M_p^{-1}(\Phi_u,I_m)\widetilde{\Gamma}_p(\omega_2)}{\gamma_p^{1/2}(\omega_1)\gamma_p^{1/2}(\omega_2)}\right]_{i,j}\right| \leq$$
$$\leq \left\|\frac{\widetilde{\Gamma}_p^\star(\omega_1)M_p^{-1}(\Phi_u,I_m)\Delta_p M_p^{-1}(\Phi_u,I_m)\widetilde{\Gamma}_p(\omega_2)}{\gamma_p^{1/2}(\omega_1)\gamma_p^{1/2}(\omega_2)}\right\|_2$$
$$\leq \left\|M_p^{-1}(\Phi_u,I_m)\right\|_2^2 \|\Delta_p\|_2$$

where in passing to the second line use was made of the fact [GV89] that for a given matrix $A=(a_{i,j})$

$$\max_{i,j}|a_{i,j}| \leq \|A\|_2, \tag{5.57}$$

and in passing to the last line of the fact that, by Lemma 5.E.6,

$$\|\widetilde{\Gamma}_p(\omega)\|_2 = \sqrt{\gamma_p(\omega)}.$$

Considering now that by Lemma 5.E.8, $\left\|M_p^{-1}(\Phi_u,I_m)\right\|_2$ is bounded above, and that $\|\Delta_p\|_2 \to 0$ as $p\to\infty$ (see Lemma 5.E.9), we can conclude that

$$\lim_{p\to\infty}\frac{\widetilde{\Gamma}_p^\star(\omega_1)M_p^{-1}(\Phi_u,I_m)\Delta_p M_p^{-1}(\Phi_u,I_m)\widetilde{\Gamma}_p(\omega_2)}{\gamma_p^{1/2}(\omega_1)\gamma_p^{1/2}(\omega_2)} = 0. \tag{5.58}$$

Let us analyze now the asymptotic behavior of the first term on the RHS of equation (5.56). We can write

$$\frac{\widetilde{\Gamma}_p^\star(\omega_1)M_p^{-1}(\Phi_u,I_m)M_p(\Phi_u,\Phi_\nu)M_p^{-1}(\Phi_u,I_m)\widetilde{\Gamma}_p(\omega_2)}{\gamma_p^{1/2}(\omega_1)\gamma_p^{1/2}(\omega_2)} =$$
$$= \frac{\widetilde{\Gamma}_p^\star(\omega_1)M_p(\Phi_u^{-1},\Phi_\nu)\widetilde{\Gamma}_p(\omega_2)}{\gamma_p^{1/2}(\omega_1)\gamma_p^{1/2}(\omega_2)} \tag{5.59}$$
$$+\frac{\widetilde{\Gamma}_p^\star(\omega_1)M_p^{-1}(\Phi_u,I_m)\left[M_p(I_n,\Phi_\nu)-M_p(\Phi_u,I_m)M_p(\Phi_u^{-1},\Phi_\nu)\right]\widetilde{\Gamma}_p(\omega_2)}{\gamma_p^{1/2}(\omega_1)\gamma_p^{1/2}(\omega_2)} \tag{5.60}$$
$$+\frac{\widetilde{\Gamma}_p^\star(\omega_1)M_p^{-1}(\Phi_u,I_m)\left[M_p(\Phi_u,\Phi_\nu)-M_p(I_n,\Phi_\nu)M_p(\Phi_u,I_m)\right]M_p^{-1}(\Phi_u,I_m)\widetilde{\Gamma}_p(\omega_2)}{\gamma_p^{1/2}(\omega_1)\gamma_p^{1/2}(\omega_2)} \tag{5.61}$$

[6]Here the symbol $\|\cdot\|_2$ stands for the matrix induced 2-norm or spectral norm (i.e. the maximum singular value), not to be confussed with the H_2-norm induced by the inner product denoted by $\|\cdot\|$ or $\|\cdot\|_{H_2}$.

We will prove that the last two terms on the RHS of the above equation (terms (5.60) and (5.61)) both tend to zero as p tends to infinity, and that the first term (5.59) tends, for $\omega_1 = \omega_2 = \omega$, to $\Phi_u^{-1}(\omega) \otimes \Phi_\nu(\omega)$ when p tends to infinity.

To proceed with this, let us consider first the term (5.60). By construction, the elements of the matrix $\widetilde{\Gamma}_p(\omega)$ are bounded in magnitude by some finite number K_1 (as defined in equation (5.47)), i.e.

$$\left| \left[\widetilde{\Gamma}_p(\omega) \right]_{i,j} \right| \leq K_1.$$

To see this, recall that the elements of this matrix are either equal to zero, or equal to $\mathcal{B}_k(e^{\mathrm{j}\omega})$, and that by Lemma 5.E.2, $|\mathcal{B}_k(e^{\mathrm{j}\omega})| \leq K_1$. Furthermore, by appealing to the relation (5.57), Lemma 5.E.6, and Lemma 5.E.8 we can write

$$\begin{aligned}
\left| \left[\frac{\widetilde{\Gamma}_p^\star(\omega_1) M_p^{-1}(\Phi_u, I_m)}{\gamma_p^{1/2}(\omega_1)} \right]_{i,j} \right| &\leq \left\| \frac{\widetilde{\Gamma}_p^\star(\omega_1) M_p^{-1}(\Phi_u, I_m)}{\gamma_p^{1/2}(\omega_1)} \right\|_2 \\
&\leq \left\| M_p^{-1}(\Phi_u, I_m) \right\|_2 \\
&\leq \left(\inf_\omega \underline{\sigma}(\Phi_u(\omega)) \right)^{-1} \triangleq K_2.
\end{aligned}$$

In this case, using Lemma 5.7.2 gives that for some $|\eta| < 1$

$$\begin{aligned}
&\left| \left[\frac{\widetilde{\Gamma}_p^\star(\omega_1) M_p^{-1}(\Phi_u, I_m)}{\gamma_p^{1/2}(\omega_1)} [M_p(I_n, \Phi_\nu) - M_p(\Phi_u, I_m) M_p(\Phi_u^{-1}, \Phi_\nu)] \widetilde{\Gamma}_p(\omega_2) \right]_{i,j} \right| \leq \\
&\leq \sum_{s=0}^{p-1} \sum_{t=0}^{p-1} \sum_{x=1}^{mn} \sum_{y=1}^{mn} \left| \left[\frac{\widetilde{\Gamma}_p^\star(\omega_1) M_p(\Phi_u, I_m)^{-1}}{\gamma_p^{1/2}(\omega_1)} \right]_{i,tmn+x} \right| \times \\
&\quad \times \left| [M_p(I_n, \Phi_\nu)]_{smn+y,tmn+x} - \left[M_p(\Phi_u, I_m) M_p(\Phi_u^{-1}, \Phi_\nu) \right]_{smn+y,tmn+x} \right| \times \\
&\quad \times \left| \left[\widetilde{\Gamma}_p(\omega_2) \right]_{smn+y,j} \right| \\
&\leq K^2 K_1 K_2 \sum_{x=1}^{mn} \sum_{y=1}^{mn} \sum_{s=0}^{p-1} \sum_{t=0}^{p-1} mn(\eta^{p-s} + \eta^s)(\eta^{p-t} + \eta^t) \\
&= K^2 K_1 K_2 (mn)^3 \left(\frac{1+\eta}{1-\eta} \right)^2 (1 - \eta^p)^2 < \infty.
\end{aligned}$$

But by Lemma 5.E.3 we have

$$\gamma_p(\omega) \geq \frac{1}{2} \sum_{k=0}^{p-1} (1 - |\xi_k|)$$

so that under the conditions of the theorem we can conclude that for any choice of ω_1 and ω_2

$$\lim_{p\to\infty} \frac{\widetilde{\Gamma}_p^{\star}(\omega_1) M_p^{-1}(\Phi_u, I_m)[M_p(I_n, \Phi_\nu) - M_p(\Phi_u, I_m) M_p(\Phi_u^{-1}, \Phi_\nu)]\widetilde{\Gamma}_p(\omega_2)}{\gamma_p^{1/2}(\omega_1)\gamma_p^{1/2}(\omega_2)} = 0$$

Using an identical argument for the term (5.61) we can conclude that

$$\lim_{p\to\infty} \frac{\widetilde{\Gamma}_p^{\star}(\omega_1) M_p^{-1}(\Phi_u, I_m)\left[M_p(\Phi_u, \Phi_\nu) - M_p(I_n, \Phi_\nu) M_p(\Phi_u, I_m)\right] M_p^{-1}(\Phi_u, I_m)\widetilde{\Gamma}_p(\omega_2)}{\gamma_p^{1/2}(\omega_1)\gamma_p^{1/2}(\omega_2)} = 0$$

Finally, by applying Lemma 5.7.1 to the term (5.59) we have

$$\lim_{p\to\infty} \frac{\widetilde{\Gamma}_p^{\star}(\omega_1) M_p(\Phi_u^{-1}, \Phi_\nu)\widetilde{\Gamma}_p(\omega_2)}{\gamma_p^{1/2}(\omega_1)\gamma_p^{1/2}(\omega_2)} = \begin{cases} \Phi_u^{-1}(\omega_1) \otimes \Phi_\nu(\omega_1) & ; \quad \omega_1 = \omega_2 \\ 0 & ; \quad \omega_1 \neq \omega_2 \end{cases}$$

what completes the proof. ■

As a direct consequence of the previous theorem we have the following corollary.

Corollary 5.7.2. *Under the same conditions of the previous theorem, but with the strengthened requirement on the sequence $\{e_k\}$ of having bounded eighth moments, then*

$$\lim_{p\to\infty} \lim_{N\to\infty} \frac{N}{\gamma_p(\omega)} \mathsf{Cov}\left\{g(e^{\mathrm{j}\omega}, \widehat{\eta})\right\} = \Phi_u^{-1}(\omega) \otimes \Phi_\nu(\omega).$$

Proof: The result follows by appealing to the results in Chapter 9 of [Lju87] (as summarized in Theorems 2.2.1 and 2.2.2) and using the same arguments as in the proof of Theorem 5.7.1. ■

Theorem 5.7.1 and its corollary provide a generalization to the multivariable and arbitrary fixed pole setting of the asymptotic results available for single-input, single-output systems [LY85, Wah91b, Wah94b, VHB95, NHG96], and for multivariable systems with FIR model structures [YL84].

Apart from this generalizing aspect, the previous theorem and corollary also have significant utility in providing an approximate expression for the covariance matrix of the (vectorized) transfer matrix estimate. For large model order and data-length, this covariance matrix can be approximated by

$$\mathsf{Cov}\left\{g(e^{\mathrm{j}\omega}, \widehat{\eta})\right\} \approx \frac{\gamma_p(\omega)}{N}\, \Phi_u^{-1}(\omega) \otimes \Phi_\nu(\omega). \tag{5.62}$$

This approximate expression implies the following component-wise quantification of the noise induced error in the frequency response estimate

$$\mathsf{E}\left\{|G_{i\ell}(e^{\mathrm{j}\omega}) - G_{i\ell}(e^{\mathrm{j}\omega}, \eta_\star)|^2\right\} \approx \frac{\gamma_p(\omega)}{N} \left[\Phi_u^{-1}(\omega)\right]_{\ell\ell} \left[\Phi_\nu(\omega)\right]_{ii}.$$

The generalizing aspect above mentioned can be demonstrated explicitly by noting that for the particular case of FIR model structures the factor $\gamma_p(\omega)$ is given by

$$\gamma_p(\omega) \triangleq \sum_{k=0}^{p-1} |\mathcal{B}_k(e^{\mathrm{j}\omega})|^2 = \sum_{k=0}^{p-1} |e^{\mathrm{j}\omega k}|^2 = p,$$

so that the covariance expression reduces to

$$\mathrm{Cov}\left\{g(e^{\mathrm{j}\omega}, \widehat{\eta})\right\} \approx \frac{p}{N}\, \Phi_u^{-1}(\omega) \otimes \Phi_\nu(\omega),$$

which is the same expression as that derived in [YL84] where only the FIR model structure case is studied.

A significant aspect of the approximate covariance expression (5.62) is that it explicitly shows how the choice of the fixed pole locations affects the noise induced estimation error. Specifically, equation (5.62) expresses the already known principle that the noise induce error is proportional to the 'noise-to-signal' ratio and the model order, and inversely proportional to the number of observed data. In addition, it also shows a new phenomenom, viz, that the choice of the poles affects the noise induced error via the frequency dependent factor $\gamma_p(\omega)$ (See figure 4.2, where the factor $\gamma_p(\omega)$ is plotted for model order $p = 4$ and for various pole choices. It can be noted from this diagram that for the case of all the poles at the origin, which corresponds to an FIR model structure, then $\gamma_p(\omega) = p = 4$.).

5.8 Bias/Variance Trade-off

The results in Theorems 5.6.1 and 5.7.1 provide a complete characterization of the estimation accuracy and show, as in the SISO case, two different phenomena of bias/variance trade-off: the well known trade-off with respect to model order selection, and an until now unrecognized phenomenom of trade-off with respect to the choice of the poles of the basis functions. The comments we made in Section 4.5 apply also here component-wise.

5.9 Intermezzo: Brief Overview of Subspace-Based Identification Methods

For the purposes of comparison with the multivariable identification technique we propose in this chapter, a brief overview of a class of methods that also provide MIMO estimates in closed form is presented in this section. The methods are known as Subspace Identification Methods.

5.9.1 Introduction

State-space models of the type introduced in Subsubsection 2.1.1.3 are very convenient to represent multivariable systems for the purposes of control design and simulation. This is so since many control problems can be solved more elegantly in the state-space domain, while the solution to these problems is sometimes more involved, or not so elegant with other representations (such as transfer functions or matrix fraction descriptions (MFD) [Kai80]). State-space models have been successfully used to provide accurate descriptions of many industrial processes. In addition, or rather as a consequence of the above comments, most of the available CACSD[7] software packages (e.g., the *Control System Toolbox* [GLLT92] for use with MATLAB [8]) can handle this type of models, although most of them also allow some other system representations such as transfer function models.

Even though traditional PEM can be used for the estimation of state-space models, typically the parameters in this model structure appear in a nonlinear fashion so that the minimum of the cost function cannot be computed analytically. Instead, the estimation will usually involve a computationally intensive (possibly nonlinear and non convex) iterative optimization procedure, with the inherent problems associated with these techniques, such as existence of local minima of the cost function, initialization problems and no guaranteed convergence. Another difficulty of the use of prediction error methods for the estimation of state space models is the large number of parameters that need to be estimated (for example consider the case of a fully parameterized state space model [McK94, McK95]). The number of parameters could be reduced by resorting to a canonical parameterization (see [Kai80]); however, as pointed out in [Vib94, Vd94a], the problem of finding a numerically reliable canonical parameterization for multivariable systems is not trivial and is, to a large extent, unsolved.

In recent years considerable amount of research has been devoted to the development of new identification methods that are able to deliver reliable state-space models of multivariable dynamical systems directly from input-output data, and that require only a modest computational complexity without the need of iterative optimization procedures. These techniques have become collectively known as Subspace-based State-Space System IDentification (4SID) methods (see for instance the recent book [Vd96] for a unified description of the different subspace algorithms, and the survey paper [Vib95]), and have their origin in state-space realization theory as developed in the sixties [HK66, Kun78]. The main computational tools employed by subspace methods are QR and singular-value decompositions [GV89].

All 4SID methods involve at some stage the computation of the extended

[7] CACSD stands for Computer Aided Control System Design.

[8] MATLAB is a registered trademark of The MathWorks, Inc.

observability matrix from input-output data. In [Vib94, Vib95], subspace methods are grouped into the following two sub-classes:

- *Realization-based 4SID Methods:* in which an explicit estimate of the impulse response coefficients is obtained as a previous step of the estimation of the extended observability matrix [HK66, ZM74, Kun78, LS91, Liu92].
- *Direct 4SID Methods:* in which the estimate of the extended observability matrix in obtained directly from the input-output data [DVM$^+$88, Ver91, VD91b, VD92, VOWL93, MR93, Ver94, OV94, Vd91a, Vd93, Vd94a, Vd94b, Vd95a, Vd95b, CV97, Lar90, Lar94].

The various algorithms, with particular names attached to them, are then classified according to the way the observability matrix is estimated from input-output data, and how it is used to compute the system matrices.

In the following Subsection the fundamental concepts of *realization theory* will be reviewed and the basic realization-based 4SID methods will be presented.

5.9.2 Realization-based 4SID Methods

We consider the state-space description introduced in Subsubsection 2.1.1.3 (equations (2.8) and (2.9)), which is repeated here for convenience

$$x_{k+1} = Ax_k + Bu_k, \tag{5.63}$$
$$y_k = Cx_k + Du_k, \tag{5.64}$$

where $x_k \in \mathbb{R}^{n_x}$, $y_k \in \mathbb{R}^m$, $u_k \in \mathbb{R}^n$ are respectively the state, the output and the input vectors at time k, and $A \in \mathbb{R}^{n_x \times n_x}$, $B \in \mathbb{R}^{n_x \times n}$, $C \in \mathbb{R}^{m \times n_x}$, and $D \in \mathbb{R}^{m \times n}$ are the system matrices. It is assumed that the pair (A, B) is controllable and the pair (C, A) is observable (in the sense given by Definitions 2.1.4 and 2.1.5, respectively), which implies that the realization in minimal with McMillan degree n_x.

As already mentioned in Chapter 2, the realization (A, B, C, D) uniquely defines the input-output properties of the system via

$$y_k = \sum_{\ell=0}^{\infty} g(\ell)\, u_{k-\ell},$$

where

$$g(\ell) = \begin{cases} D, & \ell = 0 \\ CA^{\ell-1}B, & \ell > 0 \end{cases}$$

are the impulse response coefficients.

Classical realization theory [HK66] deals with the estimation of the system matrices (A, B, C, D) from the impulse response coefficients $\{g(\ell)\}$. Most of the realization algorithms are based on the fact that the impulse response block Hankel matrix $\mathcal{H}_{ij}$ (constructed from the impulse response coefficients as in equation (2.12)), can be factorized as

$$\mathcal{H}_{ij} = \mathcal{O}_i \mathcal{C}_j, \tag{5.65}$$

where $\mathcal{O}_i$ and $\mathcal{C}_j$ are respectively the extended observability and controllability matrices, and that these matrices always have rank n_x. Then, if any full rank factorization of $\mathcal{H}_{ij}$ of the form

$$\mathcal{H}_{ij} = \widehat{\mathcal{O}}_i \widehat{\mathcal{C}}_j, \tag{5.66}$$

is available, the matrices $\widehat{\mathcal{O}}_i$ and $\widehat{\mathcal{C}}_j$ in this factorization can be interpreted as the extended observability and controllability matrices for some realization $(\widehat{A}, \widehat{B}, \widehat{C}, \widehat{D})$ of the system (which will be equivalent to the realization (A, B, C, D) in (5.63)-(5.64), modulo a similarity transformation). The fact that $\mathcal{H}_{ij}$ has rank n_x can be used to infer the unknown system order. The problem then is the computation of the system matrices from the factorization (5.66) of the Hankel matrix. Considering the definitions of the extended observability and controllability matrices in equations (2.10) and (2.11) respectively, it is clear that the matrices $\widehat{C}$ and $\widehat{B}$ can be recovered by taking the first block row of $\widehat{\mathcal{O}}_i$, and the first block column of $\widehat{\mathcal{C}}_j$, respectively. Furthermore, the matrix $\widehat{D}$ is readily available from the impulse response coefficient $g(0)$.

Regarding the matrix $\widehat{A}$, there are several ways to compute it from the factorization (5.66) of $\mathcal{H}_{ij}$. Before presenting them we introduce some notation. For a given block matrix $\mathcal{H}$, $\overline{\overline{\mathcal{H}}}$ will denote the matrix obtained by deleting the first block row of $\mathcal{H}$, while $\underline{\underline{\mathcal{H}}}$ will denote the matrix obtained by deleting the last block row of $\mathcal{H}$.

One of the methods for computing $\widehat{A}$, proposed by Zeiger and McEwen in [ZM74], follows from the observation that

$$\overline{\overline{\mathcal{H}_{ij}}} = \widehat{\mathcal{O}}_i \widehat{A} \widehat{\mathcal{C}}_j,$$

so that $\widehat{A}$ can be computed as

$$\widehat{A} = \widehat{\mathcal{O}}_i^{\dagger} \overline{\overline{\mathcal{H}_{ij}}} \widehat{\mathcal{C}}_j^{\ddagger},$$

where $(\cdot)^{\ddagger}$ denotes the right pseudo-inverse[9].

[9] The right pseudo-inverse of the (non square) matrix A is defined as $A^{\ddagger} \triangleq A^T (A A^T)^{-1}$. Not to be confused with the left (Moore-Penrose) pseudo-inverse defined as $A^{\dagger} \triangleq (A^T A)^{-1} A^T$.

Another possibility, proposed by Kung in [Kun78] (see also [JP85]), is to note that the observability matrix $\widehat{\mathcal{O}}_i$ enjoys the so-called *shift-invariant* property, reflected by the identity

$$\overline{\overline{\widehat{\mathcal{O}}_i}} = \underline{\underline{\widehat{\mathcal{O}}_i}} \widehat{A}.$$

The matrix $\widehat{A}$ can then be computed by solving (in the least squares sense) this equation to obtain

$$\widehat{A} = \underline{\underline{\widehat{\mathcal{O}}_i}}^{\dagger} \overline{\overline{\widehat{\mathcal{O}}_i}}.$$

A dual result can be obtained by exploiting the shift invariant property of the controllability matrix $\widehat{\mathcal{C}}_j$.

This completes the basic deterministic realization-based algorithm. In practice, however, it is more desiderable to start the algorithm directly from input-output measurements. This raises the question of how the impulse response coefficients can be estimated from input-output data. Several methods have been proposed in the literature for achieving this. For instance, in [LS92] a method is proposed where the response of the system to impulse input signals is directly measured. Another possibility is to use correlation analysis [Lju87, SS89] to estimate the input-output cross-covariance function from which a finite number of impulse response coefficients can be estimated.

A fundamental aspect of the realization algorithms not considered yet is the way the factorization (5.66) of the (estimated) impulse response Hankel matrix is performed, as well as the rank determination needed to estimate the system order. This issue is important for the case in which the impulse response is corrupted by noise (or it has been estimated from noisy data) because the rank determination of a matrix is very sensitive to perturbations on its entries [GV89]. Both the Zeiger-McEwen [ZM74] and the Kung [Kun78] algorithms employ the singular-value decomposition (SVD) [GV89] to factorize the impulse response Hankel matrix.

The SVD of $\mathcal{H}_{ij}$ is defined as:

$$\mathcal{H}_{ij} = [U_1 U_2] \begin{bmatrix} \Sigma_1 & 0 \\ 0 & \Sigma_2 \end{bmatrix} \begin{bmatrix} V_1^T \\ V_2^T \end{bmatrix}, \tag{5.67}$$

where $U = [U_1 U_2]$ and $V = [V_1 V_2]$ are orthogonal matrices whose columns are respectively the *left* and *right singular vectors* of $\mathcal{H}_{ij}$, and where

$$\Sigma = \begin{bmatrix} \Sigma_1 & 0 \\ 0 & \Sigma_2 \end{bmatrix}$$

is a diagonal matrix containing the *singular values* of $\mathcal{H}_{ij}$ in non-increasing order, with Σ_1 containing the n_x largest, and Σ_2 the $(\min(i,j) - n_x)$ smallest.

For the case in which the impulse response is not corrupted by noise, $\mathcal{H}_{ij}$ has rank n_x and then $\Sigma_2 = 0$. In this case equation (5.67) can be written as

$$\mathcal{H}_{ij} = U_1 \Sigma_1 V_1^T = \left(U_1 \Sigma_1^{1/2}\right) \left(\Sigma_1^{1/2} V_1^T\right),$$

which provides the factorization (5.66) by choosing

$$\begin{aligned} \widehat{\mathcal{O}}_i &= U_1 \Sigma_1^{1/2}, \\ \widehat{\mathcal{C}}_j &= \Sigma_1^{1/2} V_1^T. \end{aligned}$$

When the available measurements of the impulse response are noisy, the matrix $\mathcal{H}_{ij}$ will in general have full rank and therefore a rank reduction step will be necessary. This rank reduction is provided by the SVD, but the user must decide which is the number of significant singular values or equivalently which is the system order. Once the system order is given, a low-rank approximation of $\mathcal{H}_{ij}$ can be computed as

$$\widehat{\mathcal{H}}_{ij} = \left(U_1 \Sigma_1^{1/2}\right) \left(\Sigma_1^{1/2} V_1^T\right),$$

but now $\widehat{\mathcal{H}}_{ij}$ will in general not be a Hankel matrix, neither the estimates of $\widehat{\mathcal{O}}_i$ and $\widehat{\mathcal{C}}_j$ will have the shift invariant property. As a consequence, the algorithms will provide only approximate estimates of the system matrices. Some results concerning the quantification of the error induced by the noise corruption of the impulse response data are given in [Kun78] and [DD87].

5.9.3 Direct 4SID Algorithms

In the direct 4SID algorithms, the impulse response Hankel matrix is not required for the estimation of the extended observability matrix that can be computed directly from the input-output data. This represents an advantange with respect to the realization-based algorithms, since in practice the impulse response coefficients are difficult to measure.

We describe the basic direct 4SID algorithms corresponding to two different cases, namely:

- **Deterministic Case**, where the unknown system is assumed to have only deterministic inputs (u_k).

- **Combined Deterministic-Stochastic Case**, where the unknown system is assumed to have both deterministic (u_k) and stochastic (w_k and v_k) inputs.

5.9.3.1 Deterministic Case

We consider that the unknown system has the state-space representation given in (5.63)-(5.64). The identification objective is to estimate the model order n_x, and the system matrices A, B, C and D in (5.63)-(5.64) from N samples of the inputs u_k and the outputs y_k. Before presenting the basic deterministic 4SID algorithm we introduce some notation adapted from [Vd96]. The input block Hankel matrix is defined as

$$U^0_{2i,j} \triangleq \left[\begin{array}{cccc} u_0 & u_1 & \cdots & u_{j-1} \\ u_1 & u_2 & \cdots & u_j \\ \vdots & \vdots & \cdots & \vdots \\ u_{i-1} & u_i & \cdots & u_{i+j-2} \\ \hline u_i & u_{i+1} & \cdots & u_{i+j-1} \\ u_{i+1} & u_{i+2} & \cdots & u_{i+j} \\ \vdots & \vdots & \cdots & \vdots \\ u_{2i-1} & u_{2i} & \cdots & u_{2i+j-2} \end{array}\right] \triangleq \left[\begin{array}{c} U^0_{i,j} \\ \hline U^i_{i,j} \end{array}\right] \triangleq \left[\begin{array}{c} U_p \\ \hline U_f \end{array}\right] \tag{5.68}$$

where:

- The superscript in the matrices $U^0_{2i,j}, U^0_{i,j}$ and $U^i_{i,j}$ denotes the subscript of the first element of the first row, the first subscript denotes the number of block rows, and the second subscript, the number of (block) columns of the block Hankel matrix.

- The input Hankel matrix $U^0_{2i,j}$ is partitioned off into two parts of i block rows each. Somewhat arbitrarily, the upper part $U_p \triangleq U^0_{i,j}$ will be called the *past inputs* (with the subscript 'p' standing for 'past'), while the bottom part $U_f \triangleq U^i_{i,j}$ will be called the *future inputs* (with the subscript 'f' standing for 'future').

- The number of block rows i has to be larger than the expected order of the system to be identified.

- Typically, the number of columns j is such that all the available data is used. That is, if N is the total number of data, then $j = N - 2i + 2$.

The output block Hankel matrices $Y^0_{2i,j}$, Y_p and Y_f are defined in similar way. The state sequence matrix is defined as:

$$X^0_j \triangleq [x_0, x_1, \cdots, x_{j-1}], \tag{5.69}$$

where the superscript denotes the subscript of the first element of the state sequence, and the subscript denotes the number of columns. To be consistent

with the definitions of the past and future inputs and outputs we define the past state sequence as

$$X_p \triangleq X_j^0,$$

and the future state sequence as

$$X_f \triangleq X_j^i.$$

The block lower triangular Toeplitz matrix H_i is defined as

$$H_i \triangleq \begin{bmatrix} D & 0 & 0 & \cdots & 0 \\ CB & D & 0 & \cdots & 0 \\ CAB & CB & D & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ CA^{i-2}B & CA^{i-3}B & CA^{i-4}B & \cdots & D \end{bmatrix},$$

where the subscript indicates the number of block rows of the Toeplitz matrix.

With these definitions, the following matrix input-output relations can be derived from the state-space representation (5.63)-(5.64)

$$Y_{2i,j}^0 = \mathcal{O}_{2i} X_j^0 + H_{2i} U_{2i,j}^0, \tag{5.70}$$
$$Y_p = \mathcal{O}_i X_p + H_i U_p, \tag{5.71}$$
$$Y_f = \mathcal{O}_i X_f + H_i U_f. \tag{5.72}$$

Furthermore, the following 'state equation' relating the past and future state sequences can also be derived from (5.63)-(5.64)

$$X_f = A^i X_p + \mathcal{C}_i^{rd} U_p, \tag{5.73}$$

where $\mathcal{C}_i^{rd}$ is the *reversed extended (deterministic) controllability matrix* defined as

$$\mathcal{C}_i^{rd} \triangleq \left[A^{i-1}B, A^{i-2}B, \cdots, AB, B\right],$$

with the subscript i denoting the number of block columns, the superscript r standing for 'reversed' and the superscript d standing for 'deterministic'.

From equations (5.73) and (5.71), the future state sequence matrix X_f can be written as a linear combination of the past input U_p and past output Y_p Hankel matrices. To see this, we compute X_p from (5.71) and substitute into (5.73), to obtain

$$\begin{aligned} X_f &= A^i \mathcal{O}_i^\dagger \left(Y_p - H_i U_p\right) + \mathcal{C}_i^{rd} U_p, \\ &= \left(\mathcal{C}_i^{rd} - A^i \mathcal{O}_i^\dagger H_i\right) U_p + A^i \mathcal{O}_i^\dagger Y_p, \\ &= \begin{bmatrix} \left(\mathcal{C}_i^{rd} - A^i \mathcal{O}_i^\dagger H_i\right) & A^i \mathcal{O}_i^\dagger \end{bmatrix} \begin{bmatrix} U_p \\ Y_p \end{bmatrix}, \\ &= L_{p \to f} W_p, \end{aligned} \tag{5.74}$$

where in passing to the last line we have defined the block Hankel matrix of past inputs and outputs $W_p \triangleq W_{i,j}^0 \triangleq \begin{bmatrix} U_p^T & Y_p^T \end{bmatrix}^T$ (with the subscripts and superscript in $W_{i,j}^0$ having the same meaning as in $U_{i,j}^0$), and the matrix $L_{p\to f}$ relating the past inputs and outputs with the future states

$$L_{p\to f} \triangleq \begin{bmatrix} \left(\mathcal{C}_i^{rd} - A^i \mathcal{O}_i^\dagger H_i\right) & A^i \mathcal{O}_i^\dagger \end{bmatrix}.$$

From these equations it is possible to compute the extended observability matrix $\mathcal{O}_i$ and the future state sequence matrix X_f based only on the input-output data. The main tool employed is the singular value decomposition (SVD). The basic deterministic algorithm proceeds as follows.

Algorithm 5.9.1 (Deterministic Algorithm).

1. Perform the full SVD of the Hankel matrix of future inputs U_f

$$U_f = \Phi \begin{bmatrix} \Sigma & 0 \end{bmatrix} \begin{bmatrix} \Psi_1^T \\ \Psi_2^T \end{bmatrix} = \Phi \Sigma \Psi_1^T.$$

Now notice that since $\Psi = \begin{bmatrix} \Psi_1 & \Psi_2 \end{bmatrix}$ is an orthogonal matrix, then $\Psi^T \Psi = I$, so that

$$\Psi_1^T \Psi_2 = 0,$$

and then

$$U_f \Psi_2 = 0.$$

We can then define the orthogonal complement of U_f as

$$U_f^\perp \triangleq \Psi_2.$$

Multiplying the input-output equation (5.72) from the right by $U_f^\perp$ then gives

$$Y_f U_f^\perp = \mathcal{O}_i X_f U_f^\perp. \tag{5.75}$$

Substituting the expression for X_f in (5.74), in the previous equation we obtain

$$Y_f U_f^\perp = \mathcal{O}_i L_{p\to f} W_p U_f^\perp,$$

so that we can write

$$\mathcal{O}_i L_{p\to f} = Y_f U_f^\perp \left(W_p U_f^\perp\right)^\ddagger.$$

Finally, multiplying both sides of the above equation from the right by W_p yields

$$\mathcal{O}_i X_f = Y_f U_f^{\perp} \left(W_p U_f^{\perp}\right)^{\ddagger} W_p. \tag{5.76}$$

Now the matrices $\mathcal{O}_i$ and X_f can be obtained by performing the SVD of the matrix on the right hand side of the equation above. Note that this matrix depends only on the available input-output data. It can be proved that the matrix on the right-hand side of equation (5.76) is equivalent to the *oblique projection* of the row space[10] of the future outputs Y_f along the row space of the future inputs U_f onto the row space of the past inputs and outputs W_p, defined as

$$\mathcal{P}_i \triangleq \left(Y_f - Y_f U_f^T \left(U_f U_f^T\right)^{\dagger} U_f\right) \left(W_p - W_p U_f^T \left(U_f U_f^T\right)^{\dagger} U_f\right)^{\dagger} W_p.$$

2. Perform the SVD of the matrix $\mathcal{P}_i = Y_f U_f^{\perp} \left(W_p U_f^{\perp}\right)^{\ddagger} W_p$, to obtain

$$\mathcal{P}_i = \mathcal{O}_i X_f = [U_1 U_2] \begin{bmatrix} \Sigma_1 & 0 \\ 0 & \Sigma_2 \end{bmatrix} \begin{bmatrix} V_1^T \\ V_2^T \end{bmatrix}. \tag{5.77}$$

In the noise free case, $\Sigma_2 = 0$, and then

$$\mathcal{P}_i = Y_f U_f^{\perp} \left(W_p U_f^{\perp}\right)^{\ddagger} W_p = \mathcal{O}_i X_f = U_1 \Sigma_1 V_1^T = \left(U_1 \Sigma_1^{1/2}\right) \left(\Sigma_1^{1/2} V_1^T\right).$$

3. The order n_x of the system (5.63)-(5.64) is given by the number of singular values different from zero in (5.77), i.e. by $n_x = \mathsf{rank}\,(\Sigma_1)$.

4. The extended observability matrix is given by

$$\mathcal{O}_i = \left(U_1 \Sigma_1^{1/2}\right).$$

5. The future state sequence is given by

$$X_f = \left(\Sigma_1^{1/2} V_1^T\right).$$

6. **System Matrices:** The system matrices can be computed basically in two different ways. One of the methods uses the extended observability matrix $\mathcal{O}_i$, and the other one uses the future state sequence X_f. We describe here the first method.

[10] The *row (column) space* of a matrix A is the space spanned by the rows (columns) of A.

- **Matrix** A can be computed by exploiting the shift-invariant property of $\mathcal{O}_i$, as in the realization algorithm by Kung [Kun78], as described in Subsection 5.9.2, i.e.

$$A = \underline{\underline{\mathcal{O}_i}}^{\dagger} \, \overline{\overline{\mathcal{O}_i}}.$$

- **Matrix** C can be computed by taking the first block row of $\mathcal{O}_i$.
- **Matrices** B **and** D**.** With matrices A and C known, matrices B and D can be computed from the input-output equation (5.72) as follows. Compute a full row rank matrix $\mathcal{O}_i^{\perp}$ such that $\mathcal{O}_i^{\perp}\mathcal{O}_i = 0$. A natural candidate for $\mathcal{O}_i^{\perp}$ is the matrix U_2^T in (5.77). This is so since matrix $U = \begin{bmatrix} U_1 & U_2 \end{bmatrix}$ is an orthogonal matrix and then $U_2^T U_1 = 0$, which implies $U_2^T \mathcal{O}_i = 0$.

  Multiplying the input-output equation (5.72) from the left by $\mathcal{O}_i^{\perp}$, and from the right by $U_f^{\dagger}$, we obtain

$$\mathcal{O}_i^{\perp} Y_f U_f^{\dagger} = \mathcal{O}_i^{\perp} H_i, \tag{5.78}$$

  which is a set of linear equations in the matrices B and D (provided that matrices A and C are known). The system is typically overdetermined (for the case of noisy data) and it can be solved for B and D in the least squares sense (see [DVM+88], or [Vd96] for details).

This concludes the basic deterministic 4SID algorithm. At several points in the algorithm some matrices are assumed to have full rank or to be nonsingular. In order for this to be ensured, the following assumptions have to be made [Vd96]

i. The input u_k is persistently exciting of order $2i$, in the sense given by Definition 2.1.3.

ii. The intersection of the row space of U_f and the row space of X_p is empty.

♦

5.9.3.2 Combined Deterministic-Stochastic Case

We consider now that the unknown system has both stochastic (w_k and v_k) and deterministic (u_k) inputs. A state-space representation of the combined system was introduced in Subsubsection 2.1.1.3, in equations (2.5), (2.6) and (2.7). For the reader's convenience, we rewrite these equations here.

$$x_{k+1} = Ax_k + Bu_k + w_k, \tag{5.79}$$
$$y_k = Cx_k + Du_k + v_k, \tag{5.80}$$

$$\mathbf{E}\left\{ \begin{pmatrix} w_k \\ v_k \end{pmatrix} \begin{pmatrix} w_s^T & v_s^T \end{pmatrix} \right\} = \begin{pmatrix} Q & S \\ S^T & R \end{pmatrix} \delta_{ks} \tag{5.81}$$

The identification objective for this case is to estimate the system order n_x, the system matrices A, B, C and D, and the second order statistics of the process and measurement noise defined by the covariance matrices Q, S and R, from N samples of the inputs u_k and outputs y_k.

Before presenting the algorithm we need to introduce some notation. The system (5.79)-(5.80) can be regarded as the superposition of two subsystems: one reflecting the influence of the deterministic input (u_k), which will be called the 'deterministic subsystem', and the other one reflecting the influence of the stochastic inputs (the noise sequences w_k and v_k), which will be called the 'stochastic subsystem'. The state variable x_k and the output y_k can accordingly be written as the superposition of a deterministic and a stochastic component, as follows

$$x_k = x_k^d + x_k^s, \tag{5.82}$$
$$y_k = y_k^d + y_k^s, \tag{5.83}$$

where the superscript d denotes 'deterministic' and the superscript s denotes 'stochastic'. The deterministic subsystem is then given by

$$\begin{aligned} x_{k+1}^d &= Ax_k^d + Bu_k, \\ y_k^d &= Cx_k^d + Du_k, \end{aligned}$$

while the stochastic subsystem by

$$\begin{aligned} x_{k+1}^s &= Ax_k^s + w_k, \\ y_k^s &= Cx_k^s + v_k. \end{aligned}$$

It is assumed that the pair (A, C) is observable, and that the pair $\left(A, \begin{bmatrix} B & Q^{1/2} \end{bmatrix}\right)$ is controllable, which implies that all the modes of the system are excited either by the deterministic input u_k or by the process noise w_k.

The processes w_k and v_k are assumed to be zero mean white noise vector sequences independent of x_k^s, which implies that

$$\begin{aligned} \mathbf{E}\left\{x_k^s v_k^T\right\} &= 0, \\ \mathbf{E}\left\{x_k^s w_k^T\right\} &= 0. \end{aligned}$$

It is also assumed that the process $\{x_k^s\}$ is stationary, so that

$$\begin{aligned} \mathbf{E}\left\{x_k^s\right\} &= 0, \\ \mathbf{E}\left\{x_k^s (x_k^s)^T\right\} &\triangleq \Sigma_x, \end{aligned}$$

where the stochastic state covariance matrix Σ_x is independent of the time k. It is not difficult to show that Σ_x satisfies the following Lyapunov equation [Vd94a, Vd96]

$$\Sigma_x = A\Sigma_x A^T + Q. \tag{5.84}$$

The stochastic output covariance matrices are defined as

$$\Lambda_i \triangleq \mathbf{E}\left\{y_{k+i}^s (y_k^s)^T\right\}, \tag{5.85}$$

and the cross-covariance matrix between stochastic outputs and states as

$$G \triangleq \mathbf{E}\left\{x_{k+1}^s (y_k^s)^T\right\}. \tag{5.86}$$

It is straightforward to show that the following identities hold:

$$\Lambda_0 \triangleq \mathbf{E}\left\{y_k^s (y_k^s)^T\right\} = C\Sigma_x C^T + R, \tag{5.87}$$
$$G = A\Sigma_x C^T + S, \tag{5.88}$$

and, for $i = 1, 2, \cdots$,

$$\Lambda_i = CA^{i-1}G, \tag{5.89}$$
$$\Lambda_{-i} = G^T (A^{i-1})^T C^T. \tag{5.90}$$

The *reversed extended stochastic controllability matrix* $\mathcal{C}_i^{rs}$ is defined as

$$\mathcal{C}_i^{rs} \triangleq \left[A^{i-1}G, \quad A^{i-2}G, \quad \cdots, \quad AG, \quad G\right],$$

where the subscript i denotes the number of block columns. The following block Toeplitz matrices can be constructed from the output covariance matrices Λ_i

$$M_i \triangleq \begin{bmatrix} \Lambda_i & \Lambda_{i-1} & \Lambda_{i-2} & \cdots & \Lambda_1 \\ \Lambda_{i+1} & \Lambda_i & \Lambda_{i-1} & \cdots & \Lambda_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \Lambda_{2i-1} & \Lambda_{2i-2} & \Lambda_{2i-3} & \cdots & \Lambda_i \end{bmatrix} = \mathcal{O}_i \, \mathcal{C}_i^{rs}, \tag{5.91}$$

$$N_i \triangleq \begin{bmatrix} \Lambda_0 & \Lambda_{-1} & \Lambda_{-2} & \cdots & \Lambda_{1-i} \\ \Lambda_1 & \Lambda_0 & \Lambda_{-1} & \cdots & \Lambda_{2-i} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \Lambda_{i-1} & \Lambda_{i-2} & \Lambda_{i-3} & \cdots & \Lambda_0 \end{bmatrix}. \tag{5.92}$$

Consistently with equation (5.82) are defined the deterministic $X_j^{i^d}$ and the stochastic $X_j^{i^s}$ state sequence matrices as

$$X_j^{i^d} \triangleq \left[x_i^d, \quad x_{i+1}^d, \quad \cdots, \quad x_{i+j-1}^d\right],$$
$$X_j^{i^s} \triangleq \left[x_i^s, \quad x_{i+1}^s, \quad \cdots, \quad x_{i+j-1}^s\right].$$

It is obvious that these matrices satisfy

$$X_j^i = X_j^{i^d} + X_j^{i^s}.$$

Similarly as we did for the deterministic case in Subsubsection 5.9.3.1, we define the past and future deterministic and stochastic state sequences as

$$\begin{aligned} X_p^d \triangleq X_j^{0^d} \quad , \quad X_f^d \triangleq X_j^{i^d}, \\ X_p^s \triangleq X_j^{0^s} \quad , \quad X_f^s \triangleq X_j^{i^s}, \end{aligned}$$

and the past and future deterministic and stochastic block Hankel output matrices as

$$\begin{aligned} Y_p^d \triangleq Y_{i,j}^{0^d} \quad , \quad Y_f^d \triangleq Y_{i,j}^{i^d}, \\ Y_p^s \triangleq Y_{i,j}^{0^s} \quad , \quad Y_f^s \triangleq Y_{i,j}^{i^s}. \end{aligned}$$

The matrix input-output equations for this case, analogous to equations (5.71) to (5.73) in the deterministic problem, become

$$\begin{aligned} Y_p &= \mathcal{O}_i X_p^d + H_i U_p + Y_p^s, \\ Y_f &= \mathcal{O}_i X_f^d + H_i U_f + Y_f^s, \\ X_f^d &= A^i X_p^d + \mathcal{C}_i^{rd} U_p. \end{aligned}$$

The algorithm for this combined deterministic-stochastic case is based on the computation of the extended observability matrix $\mathcal{O}_i$ and a non-steady state Kalman filter estimate of the state sequences (we will denote this estimate by $\widehat{X}_j^i$) from input-output data. The system matrices are then recovered from $\mathcal{O}_i$ and $\widehat{X}_j^i$.

The **non-steady state Kalman filter estimate** $\widehat{x}_k$ of the state x_k, given the input and output measurements $u_0, u_1, \cdots, u_{k-1}$ and $y_0, y_1, \cdots, y_{k-1}$, is given by the following recursive expressions (see [Vd94a, Vd96] for the proof of this result)

$$\widehat{x}_k = A\widehat{x}_{k-1} + K_{k-1}\left(y_{k-1} - C\widehat{x}_{k-1} - Du_{k-1}\right), \tag{5.93}$$

$$K_{k-1} = \left(G - AP_{k-1}C^T\right)\left(\Lambda_0 - CP_{k-1}C^T\right)^{-1}, \tag{5.94}$$

$$P_k = AP_{k-1}A^T + \left(G - AP_{k-1}C^T\right)\left(\Lambda_0 - CP_{k-1}C^T\right)^{-1}\left(G - AP_{k-1}C^T\right)^T, \tag{5.95}$$

with initial state estimate $\widehat{x}_0 = 0$, and initial covariance of the state estimate P_0. The Kalman filter estimate can be explicitly written as

$$\widehat{x}_k = \left[A^k - \Omega_k\mathcal{O}_k \;\middle|\; \mathcal{C}_k^{rd} - \Omega_k H_k \;\middle|\; \Omega_k\right] \left[\begin{array}{c} \widehat{x}_0 \\ \hline u_0 \\ \vdots \\ u_{k-1} \\ \hline y_0 \\ \vdots \\ y_{k-1} \end{array}\right], \tag{5.96}$$

where the matrix Ω_k is defined as

$$\Omega_k \triangleq \left(\mathcal{C}_k^{rs} - A^k P_0 \mathcal{O}_k^T\right)\left(N_k - \mathcal{O}_k P_0 \mathcal{O}_k^T\right)^{-1}.$$

Equation (5.96) shows that the Kalman filter state estimate $\widehat{x}_k$ is a linear combination of the past inputs and outputs $u_0, u_1, \cdots, u_{k-1}, y_0, y_1, \cdots, y_{k-1}$, and the initial state estimate $\widehat{x}_0$. We can then compute the Kalman filter state estimate sequence matrix as

$$\begin{aligned}\widehat{X}_j^i &\triangleq \left[\widehat{x}_i, \ \widehat{x}_{i+1}, \ \cdots, \ \widehat{x}_{i+j-1}\right], \\ &= \left[A^i - \Omega_i \mathcal{O}_i \ \middle| \ \left[\mathcal{C}_i^{rd} - \Omega_i H_i \ \middle| \ \Omega_i\right]\right]\begin{bmatrix}\widehat{X}_0 \\ W_p\end{bmatrix},\end{aligned}$$

where $\widehat{X}_0$ is the sequence of initial states. This equation can be interpreted as a bank of Kalman filters working in parallel in each of the columns of the block Hankel matrix of past inputs and outputs W_p.

We are now able to present the basic combined deterministic-stochastic algorithm. The algorithm proceeds as follows:

Algorithm 5.9.2 (Combined Deterministic-Stochastic Algorithm).

1. From the input-output data compute the *oblique projection* of the row space of Y_f along the row space of U_f on the row space of W_p

   $$\mathcal{P}_i \triangleq \left(Y_f - Y_f U_f^T \left(U_f U_f^T\right)^\dagger U_f\right)\left(W_p - W_p U_f^T \left(U_f U_f^T\right)^\dagger U_f\right)^\dagger W_p.$$

   In [Vd94a, Vd96] it is proved that in the limit when the number of measurements $N \to \infty$ (and consequently $j \to \infty$), the matrix $\mathcal{P}_i$ equals the product of the extended observability matrix $\mathcal{O}_i$ and the Kalman filter state sequence estimate $\widehat{X}_j^i$, i.e.

   $$\mathcal{P}_i = \mathcal{O}_i \widehat{X}_j^i,$$

   where the Kalman filter is run from particular values of the initial estimates $\widehat{X}_0$ and P_0, which are computed from the input-output data. The interested reader is referred to [Vd94a, Vd96] for the details.

2. Perform the SVD of the matrix $\mathcal{P}_i$ to obtain

   $$\mathcal{P}_i = \mathcal{O}_i \widehat{X}_j^i = [U_1 U_2]\begin{bmatrix}\Sigma_1 & 0 \\ 0 & 0\end{bmatrix}\begin{bmatrix}V_1^T \\ V_2^T\end{bmatrix} = \left(U_1 \Sigma_1^{1/2}\right)\left(\Sigma_1^{1/2} V_1^T\right). \qquad (5.97)$$

3. The system order n_x is equal to the number of singular values different from zero in (5.97). That is $n_x = \mathsf{rank}\,(\Sigma_1)$.

4. The extended observability matrix is given by

$$\mathcal{O}_i = U_1 \Sigma_1^{1/2}.$$

5. The Kalman filter state estimate sequence matrix is given by

$$\widehat{X}_j^i = \Sigma_1^{1/2} V_1^T.$$

6. **System matrices:** As in the purely deterministic case, there are several ways to compute the system matrices from the extended observability matrix and the Kalman filter state estimate sequence matrix. We don't analyze this topic here and refer the interested reader to [Vd96].

♦

The N4SID[11] algorithms [Vd94a], MOESP[12] algorithms [Ver94], and CVA[13] algorithms [Lar90, Lar94] are variations of Algorithm 5.9.2, where the matrix $\mathcal{P}_i$ is weighted with different matrices for each case, before the SVD is performed (see [Vd94a, Vd96] for details).

5.10 Simulation Examples

In this section, the proposed method for identification of multivariable systems using OBFP is illustrated with a brief simulation study.

An underlying continuous time multivariable system, with $m = 2$ outputs and $n = 2$ inputs and with transfer matrix representation given by:

$$G(s) = \begin{bmatrix} \dfrac{0.1e^{-s}}{(s+1)(s+0.1)} & \dfrac{1}{(s+2)(s+0.5)} \\ \dfrac{0.21}{(s+0.7)(s+0.3)} & \dfrac{0.32e^{-s}}{(s+0.4)(s+0.8)} \end{bmatrix} \tag{5.98}$$

is considered. It is assumed that there is available an observation of $N = 2000$ samples spaced 1 second apart of the outputs $\{y_k^1\}, \{y_k^2\}$ of $G(s)$ when the inputs $\{u_k^1\}, \{u_k^2\}$ are unit amplitude square waves of fundamental frequencies 0.02 Hz and 0.05 Hz respectively. The output data is corrupted with stationary and white Gaussian distributed noise of variances $\sigma_{e^1}^2 = \sigma_{e^2}^2 = 0.001$ and with colouring filter $H(q) = I$. Based on this observed data, the identification objective is to estimate the dynamics of the zero order hold equivalent [ÅW84]

[11] N4SID stands for "Numerical algorithms for Subspace State Space System IDentification".
[12] MOESP stands for "Multivariable Output-Error State sPace".
[13] CVA stands for "Canonical Variate Analysis".

discrete time system

$$G(z) = \text{ZOH}\left\{G(s)\right\} = \begin{bmatrix} \dfrac{(0.0355z + 0.0247)}{z(z - 0.9048)(z - 0.3679)} & \dfrac{(0.2364z + 0.1038)}{(z - 0.6065)(z - 0.1353)} \\ \dfrac{(0.0760z + 0.0545)}{(z - 0.7408)(z - 0.4966)} & \dfrac{(0.1087z + 0.0729)}{z(z - 0.6703)(z - 0.4493)} \end{bmatrix}. \tag{5.99}$$

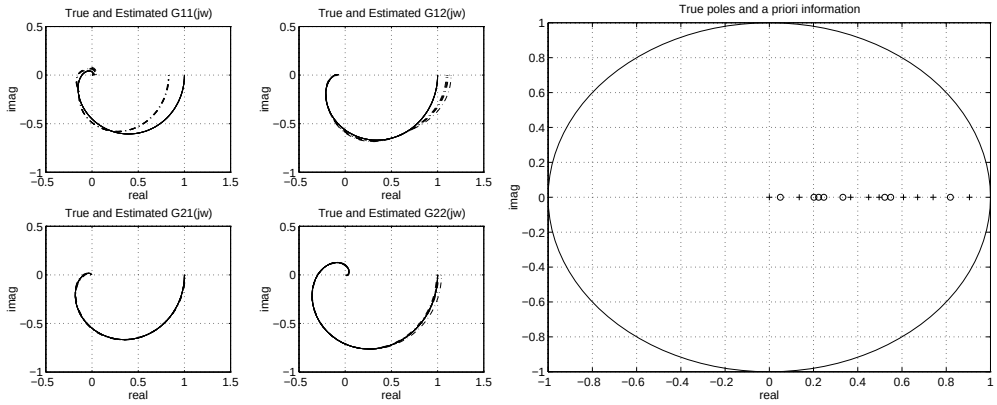

Figure 5.3: *Left hand figure shows the true (solid line) and estimated (dash–dotted line) frequency response. The estimate was obtained using a fixed denominator structure and a least squares criterion with* $N = 2000$ *samples. The four plots shown are the four possible responses from the 2 inputs to the 2 outputs. The right hand figure shows the true poles as crosses and the a-priori guesses as circles.*

In the left hand diagram of Figure 5.3 we show the results as true and estimated Nyquist plots (for each of the four scalar entries of the estimated transfer function matrix) when using the least squares estimation methods of Section 5.4, and the fixed denominator model structure of (5.16)-(3.14). Five estimates based on five different noise realizations are shown as dash–dotted lines while the true frequency responses are shown as solid lines. The poles in the structure (5.16)-(3.14) where chosen at

$$\{\xi_k\} = \{0.2231, 0.8187, 0.0498, 0.3329, 0.2466, 0.5488, 0.5220, 0.2019\}.$$

These choices correspond to an eighth order model and 32 scalar parameters being estimated. The choice of the pole locations is of limited accuracy as illustrated in the right hand diagram of Figure 5.3 where the true pole locations are shown as crosses, and the above choices are shown as circles.

Another estimation approach was also investigated whereby a state space model structure was found using a standard prediction error identification

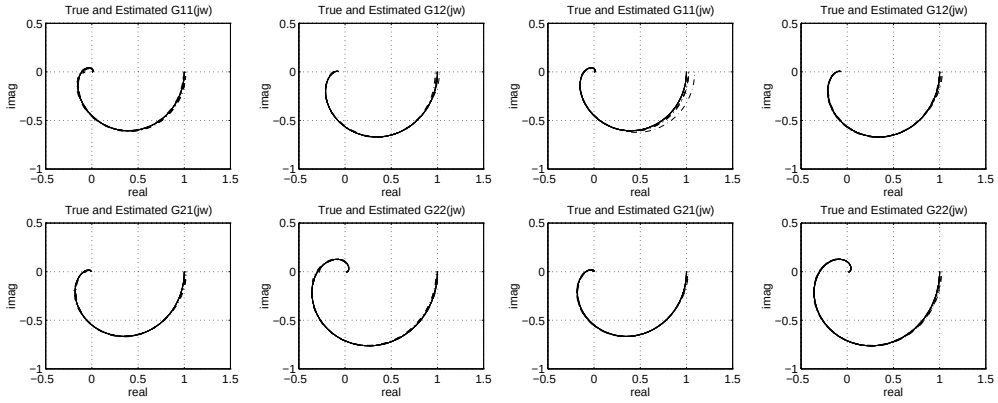

Figure 5.4: *Left hand figure shows estimation results over five different noise realizations when using an 8th order state-space model structure and a prediction error estimation method. The right hand figure shows the results using the same five noisy data records and the same model structure, but using an N4SID method to obtain the estimates. In both cases the true response is the solid line and the estimates are the dash–dot lines.* ($N = 2000$ *samples)*

algorithm, implemented with The Mathworks's System Identification Toolbox [Lju95] along the lines suggested in [Lju91]. In this case, an eighth order model was used to capture the eight different poles in (5.98). This implies (after a suitable canonical form for the state space model structure is selected) the estimation of 48 parameters. The same data used to provide the estimates shown in Figure 5.3 were used to derive the prediction error method estimates.

The results are shown in the left hand diagram of Figure 5.4. As can be seen, the results are more accurate than for the fixed denominator structure, but as reference to Table 5.1 shows, they involve two orders of magnitude more computation. On the other hand, the derivation of the state–space model structure estimate does not require approximate prior knowledge of the location of system poles.

Finally, a sixth order state-space model was estimated using the Subspace State Space System Identification (N4SID) algorithm proposed in [Vd94a] (see the combined deterministic-stochastic algorithm in Section 5.9). The results are shown in the left hand diagram of Figure 5.4. The results are as accurate as for the prediction error method estimates, but as reference to Table 5.1 shows, they involve an order of magnitude less computation.

To complete the simulation example, this set of identification experiments was repeated with the only change being that the amount of observed data was decreased by 80% to 400 samples. The results are shown in Figures 5.5–5.6. The key point to notice is that the variability of the parameter estimates increases only slightly (as compared to the 2000 sample case) when using the

Method	# of Parameters	Flops	Order	Addit. Inform.
OBFP	32	4.59×10^6	8	Guesses for Poles
PEM	48	1.42×10^8	8	Obs. Indices: $[4\ 4]$
N4SID	72 (D=0; X0=0)	3.92×10^7	6	Aux Order: 10

Table 5.1: *Computational Load of different Identification Methods (N=2000 samples)*

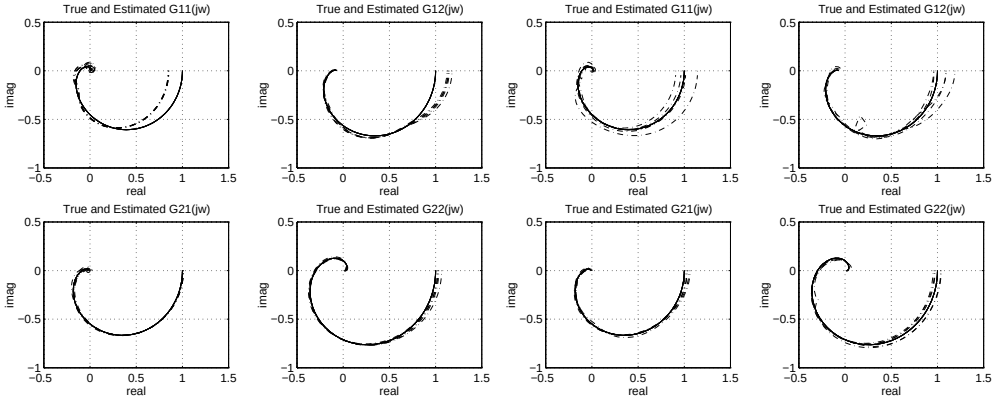

Figure 5.5: *Same experiment as shown in left hand diagrams of Figures 5.3 and 5.4 but with only 20% the amount of observed data (400 samples). Fixed denominator model structure on the left and state–space structure estimated using prediction error method on the right.*

fixed denominator structure, but increases very markedly when estimating a state-space model structure with a general prediction error method or with a subspace method. This deterioration of performance for relatively small amount of observed data comes to no surprise for the case of subspace methods which provide accurate estimates only when the number of data is large (when $N \rightarrow \infty$).

Together, these examples suggest that if prior knowledge of pole positions is available, and specially if only short data records are available or if computational load is a serious concern, then estimation using a fixed denominator model structure (re-parameterized using Orthonormal Bases with Fixed Poles) can yield improved results when compared to state-space structure estimates using prediction error or subspace methods. However, if no prior information is available, (and if the calculation of confidence regions valid for finite data is not required) then it is likely that Subspace Methods will yield better results.

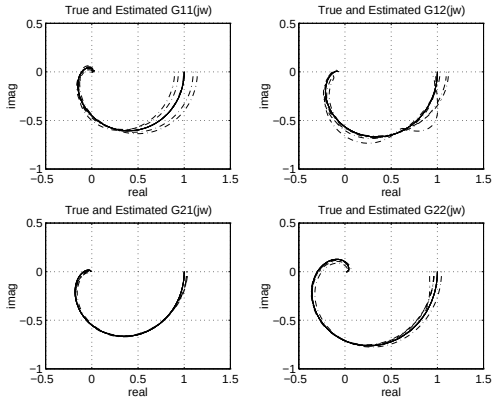

Figure 5.6: *Same experiment as shown in right hand diagram of Figure 5.4 but with only 20% the amount of observed data (400 samples). State–space model structure estimated by N4SID methods.*

5.11 Conclusions

In this chapter we have studied the problem of identification, from input-output data in the time domain, of discrete-time linear multivariable systems using fixed denominator model structures and least squares techniques. The estimation accuracy has been analyzed by quantifying the undermodelling error and the noise induced error. Fundamental in this analysis has been the reparameterization of the fixed denominator model structure using rational orthonormal bases with the same fixed poles. The main contributions of the chapter have been the extension to the multivariable setting, and to the case of using general orthonormal bases with fixed poles, of some single-input, single-output FIR results concerning the asymptotic (in model order and data-length) distribution of the frequency domain estimation error. Fundamental in the derivations were some results on convergence properties of block Toeplitz-like matrices that were also established in the chapter. As in the SISO case, a new phenomenom of bias/variance trade-off with respect to the choice of the poles in the model structure could also be recognized in the multivariable setting of this chapter.

The numerical robustness of the proposed identification scheme was also analyzed, and it was shown that the estimation with an orthonormal structure does not necessarily guarantee a better numerical conditioning when compared with the estimation using an equivalent (with the same fixed poles) non-orthonormal structure, the exception being the case of white input in which the estimation with an orthonormal structure is perfectly numerically conditioned.

For the purposes of comparison with the orthonormal basis-based estimation method proposed here, a brief overview of Subspace Methods, which also provide closed form estimates, was presented in this chapter. The simulation

results showed that both methods have a similar performance, specially when prior information about the system dynamics is available and a relatively small amount of data is observed.

Appendices

5.A Proofs for Lemmas 5.3.1 and 5.5.1

Proof of Lemma 5.3.1 From the definition of the bases $\left\{\mathcal{B}_\ell^{ij}(z)\right\}$ in equation (3.28) we can write

$$\mathcal{B}_\ell^{ij}(z) = \mathcal{B}_\ell(z) \begin{bmatrix} 0 & \dots & 0 & \dots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \dots & 1 & \dots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & \dots & 0 \end{bmatrix} \begin{matrix} \\ \\ \leftarrow i \\ \\ \\ \end{matrix}$$
$$\begin{matrix} & & \uparrow & & \\ & & j & & \end{matrix}$$

so that

$$\theta_\ell^{ij}\mathcal{B}_\ell^{ij}(z) = \mathcal{B}_\ell(z) \begin{bmatrix} 0 & \dots & 0 & \dots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \dots & \theta_\ell^{ij} & \dots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & \dots & 0 \end{bmatrix} \begin{matrix} \\ \\ \leftarrow i \\ \\ \\ \end{matrix}$$
$$\begin{matrix} & & \uparrow & & \\ & & j & & \end{matrix}$$

Then

$$\sum_{i=1}^{m}\sum_{j=1}^{n}\theta_\ell^{ij}\mathcal{B}_\ell^{ij}(z) = \begin{bmatrix} \theta_\ell^{11} & \dots & \theta_\ell^{1n} \\ \vdots & \ddots & \vdots \\ \theta_\ell^{m1} & \dots & \theta_\ell^{mn} \end{bmatrix} \mathcal{B}_\ell(z) = \theta_\ell^T \ \mathcal{B}_\ell(z)$$

and the result follows straightforwardly. ■

Proof of Lemma 5.5.1 Using Parseval's Theorem, $\widetilde{R}_p$ can be given the following frequency domain representation

$$\widetilde{R}_p = \frac{1}{2\pi}\int_{-\pi}^{\pi}\Gamma_p(e^{\mathrm{j}\omega})\Phi_u(\omega)\Gamma_p^\star(e^{\mathrm{j}\omega})\,\mathrm{d}\omega, \tag{5.A.1}$$

where $\Gamma_p(e^{\mathrm{j}\omega})$ is defined in equations (5.20) and (5.21). Then, appealing to the orthonormality of the bases we can write

$$\begin{aligned}\widetilde{R}_p &\geq \inf_{\omega} \underline{\sigma}[\Phi_u(\omega)] \frac{1}{2\pi} \int_{-\pi}^{\pi} \Gamma_p(e^{\mathrm{j}\omega}) \Gamma_p^{\star}(e^{\mathrm{j}\omega})\, \mathrm{d}\omega \\ &= \inf_{\omega} \underline{\sigma}[\Phi_u(\omega)] I_{np}.\end{aligned}$$

Using a similar argument for the upper bound we obtain

$$\begin{aligned}\widetilde{R}_p &\leq \sup_{\omega} \overline{\sigma}[\Phi_u(\omega)] \frac{1}{2\pi} \int_{-\pi}^{\pi} \Gamma_p(e^{\mathrm{j}\omega}) \Gamma_p^{\star}(e^{\mathrm{j}\omega})\, \mathrm{d}\omega \\ &= \sup_{\omega} \overline{\sigma}[\Phi_u(\omega)] I_{np}.\end{aligned}$$

The result now follows straightforwardly. ■

5.B Proofs for Lemmas 5.6.1 and 5.6.2

Proof of Lemma 5.6.1 Let the true system be represented by

$$y_k = G(q)\, u_k + \nu_k,$$

where

$$G(q) = \sum_{k=0}^{\infty} \theta_k \mathcal{B}_k(q),$$

and let $\Theta_0 \triangleq [\theta_0^T, \theta_1^T, \cdots, \theta_{p-1}^T]$ be the 'true' truncated parameter matrix, and $\widehat{\Theta}$ the least squares estimate as given in equation (5.26). Then the error between the parameter estimate and the true truncated parameter matrix is given by

$$\widetilde{\Theta} \triangleq \widehat{\Theta} - \Theta_0.$$

Considering equation (5.26) in the limit when N tends to infinity, we can write

$$\overline{\mathbf{E}}\left\{\phi_k \phi_k^T\right\} \widehat{\Theta} = \overline{\mathbf{E}}\left\{\phi_k y_k^T\right\}, \tag{5.B.2}$$

where the notation

$$\overline{\mathbf{E}}\{x_k\} \triangleq \lim_{N\to\infty} \frac{1}{N} \sum_{k=0}^{N-1} \mathbf{E}\{x_k\}$$

due to Ljung [Lju87] has been used. Now, since

$$y_k = \Theta_0^T \phi_k + \sum_{\ell=p}^{\infty} \theta_\ell^T \mathcal{B}_\ell(q) u_k = \Theta_0^T \phi_k + z_k,$$

where $z_k \triangleq [G(q) - G(q,\Theta_0)]u_k = \sum_{\ell=p}^{\infty} \theta_\ell^T \mathcal{B}_\ell(q) u_k$, then

$$\overline{\mathbf{E}}\left\{\phi_k y_k^T\right\} = \overline{\mathbf{E}}\left\{\phi_k \phi_k^T\right\}\Theta_0 + \overline{\mathbf{E}}\left\{\phi_k z_k^T\right\}.$$

Equation (5.B.2) can then be written as

$$\overline{\mathbf{E}}\left\{\phi_k \phi_k^T\right\}\widehat{\Theta} = \overline{\mathbf{E}}\left\{\phi_k \phi_k^T\right\}\Theta_0 + \overline{\mathbf{E}}\left\{\phi_k z_k^T\right\},$$

so that the error between the parameter estimate and the true truncated parameter matrix is given by

$$\widetilde{\Theta} = \widetilde{R}_p^{-1}\overline{\mathbf{E}}\left\{\phi_k z_k^T\right\}, \tag{5.B.3}$$

where the definition $\widetilde{R}_p \triangleq \overline{\mathbf{E}}\left\{\phi_k \phi_k^T\right\}$ has been used.

Therefore, the error in the transfer matrix estimate is given by

$$G(e^{\mathrm{j}\omega}) - G(e^{\mathrm{j}\omega},\widehat{\Theta}) = \overline{\mathbf{E}}\left\{z_k \phi_k^T\right\}\widetilde{R}_p^{-1}\Gamma_p(e^{\mathrm{j}\omega}) + \sum_{\ell=p}^{\infty}\theta_\ell^T \mathcal{B}_\ell(e^{\mathrm{j}\omega}),$$

and then

$$\begin{aligned}
&\left|\left[G(e^{\mathrm{j}\omega}) - G(e^{\mathrm{j}\omega},\widehat{\Theta})\right]_{i,j}\right| = \\
&= \left|\left[\overline{\mathbf{E}}\left\{z_k \phi_k^T\right\}\widetilde{R}_p^{-1}\Gamma_p(e^{\mathrm{j}\omega}) + \sum_{\ell=p}^{\infty}\theta_\ell^T \mathcal{B}_\ell(e^{\mathrm{j}\omega})\right]_{i,j}\right|, \\
&\leq \left|\left[\overline{\mathbf{E}}\left\{z_k \phi_k^T\right\}\widetilde{R}_p^{-1}\Gamma_p(e^{\mathrm{j}\omega})\right]_{i,j}\right| + \sum_{\ell=p}^{\infty}\left|\theta_\ell^{ij}\right|\left|\mathcal{B}_\ell(e^{\mathrm{j}\omega})\right|, \\
&\leq \left\|\overline{\mathbf{E}}\left\{z_k \phi_k^T\right\}\right\|_2 \left\|\widetilde{R}_p^{-1}\right\|_2 \left\|\Gamma_p(e^{\mathrm{j}\omega})\right\|_2 + \sum_{\ell=p}^{\infty}\left|\theta_\ell^{ij}\right|\left|\mathcal{B}_\ell(e^{\mathrm{j}\omega})\right|, \\
&\leq \frac{\sqrt{\gamma_p(\omega)}\left\|\overline{\mathbf{E}}\left\{z_k \phi_k^T\right\}\right\|_2}{\inf_\omega \underline{\sigma}(\Phi_u(\omega))} + \sum_{\ell=p}^{\infty}\left|\theta_\ell^{ij}\right|\left|\mathcal{B}_\ell(e^{\mathrm{j}\omega})\right|.
\end{aligned} \tag{5.B.4}$$

Now, by Parseval's Theorem, we can write

$$\overline{\mathbf{E}}\left\{z_k \phi_k^T\right\} = \frac{1}{2\pi}\int_{-\pi}^{\pi}\sum_{\ell=p}^{\infty}\theta_\ell^T \mathcal{B}_\ell(e^{\mathrm{j}\omega})\Phi_u(\omega)\Gamma_p^\star((e^{\mathrm{j}\omega})\mathrm{d}\omega,$$

so that

$$\begin{aligned}
\left\|\overline{\mathbf{E}}\left\{z_k\phi_k^T\right\}\right\|_2 &\leq \sum_{\ell=p}^{\infty}\left\|\theta_\ell^T\right\|_2\left\|\frac{1}{2\pi}\int_{-\pi}^{\pi}\mathcal{B}_\ell(e^{\mathrm{j}\omega})\Phi_u(\omega)\Gamma_p^\star((e^{\mathrm{j}\omega})\mathrm{d}\omega\right\|_2, \\
&\leq \sum_{\ell=p}^{\infty}\left\|\theta_\ell^T\right\|_2\frac{1}{2\pi}\int_{-\pi}^{\pi}\left|\mathcal{B}_\ell(e^{\mathrm{j}\omega})\right|\left\|\Phi_u(\omega)\right\|_2\left\|\Gamma_p^\star((e^{\mathrm{j}\omega})\right\|_2\mathrm{d}\omega, \\
&\leq K_1\sup_\omega\overline{\sigma}(\Phi_u(\omega))\sum_{\ell=p}^{\infty}\left\|\theta_\ell^T\right\|_2\frac{1}{2\pi}\int_{-\pi}^{\pi}\sqrt{\gamma_p(\omega)}\mathrm{d}\omega, \\
&\leq K_1\sqrt{mn}\sup_\omega\sqrt{\gamma_p(\omega)}\sup_\omega\overline{\sigma}(\Phi_u(\omega))\sum_{\ell=p}^{\infty}\max_{i,j}\left|\theta_\ell^{ij}\right|,
\end{aligned}$$

with the constant K_1 defined in the statement of the lemma. Substituting back in equation (5.B.4) then yields the result. ■

Proof of Lemma 5.6.2 Equation (5.43) can be interpreted as the norm minimization problem

$$\Theta_\star = \arg\min_{\Theta\in\mathbb{R}^{np\times m}}\left\{\left\|G(e^{\mathrm{j}\omega}) - G(e^{\mathrm{j}\omega},\Theta)\right\|_u\right\}$$

where $\|\cdot\|_u$ stands for the norm on the Hilbert space $H_2^{m\times n}(\mathbb{T})$ induced by the inner product

$$\langle F, W\rangle_u = \frac{1}{2\pi}\int_{-\pi}^{\pi}\mathsf{Tr}\left\{F(e^{\mathrm{j}\omega})\Phi_u(\omega)W^\star(e^{\mathrm{j}\omega})\right\}\mathrm{d}\omega.$$

Now, since $G(e^{\mathrm{j}\omega},\Theta)$ is in the subspace of $(H_2^{m\times n}(\mathbb{T}),\|\cdot\|_u)$ spanned by the basis functions $\{\mathcal{B}_\ell(e^{\mathrm{j}\omega})\}$ [14], then by the Projection Theorem

$$\left\langle G(e^{\mathrm{j}\omega}) - G(e^{\mathrm{j}\omega},\Theta_\star), G(e^{\mathrm{j}\omega},\Theta_\star)\right\rangle_u = 0$$

so that

$$\left\langle G(e^{\mathrm{j}\omega}), G(e^{\mathrm{j}\omega},\Theta_\star)\right\rangle_u = \left\|G(e^{\mathrm{j}\omega},\Theta_\star)\right\|_u^2.$$

Use of Cauchy-Schwarz inequality then gives the result. ■

5.C Relation between the estimates $\widehat{\eta}$ and $\widehat{\Theta}$

Lemma 5.C.1. *Let Θ be defined as in equation (5.19) and η as in equation (5.49), then the least squares estimate $\widehat{\eta}$ of η is related to the least squares estimate $\widehat{\Theta}$ of Θ according to*

$$\widehat{\eta} = \mathsf{vec}\ \widehat{\Theta}^T$$

[14] To be precise, by the matrix bases $\{\mathcal{B}_\ell^{ij}(e^{\mathrm{j}\omega})\}$.

Proof: The least squares estimate $\widehat{\eta}$ of η is given by

$$\widehat{\eta} = \left(\sum_{k=0}^{N-1} \psi_k \psi_k^T \right)^{-1} \sum_{k=0}^{N-1} \psi_k y_k \tag{5.C.5}$$

Considering that $\psi_k \triangleq (\phi_k \otimes I_m)$ we have

$$\psi_k \psi_k^T = (\phi_k \otimes I_m)(\phi_k^T \otimes I_m) = \phi_k \phi_k^T \otimes I_m$$

and then

$$\sum_{k=0}^{N-1} \psi_k \psi_k^T = \sum_{k=0}^{N-1} (\phi_k \phi_k^T \otimes I_m) = \left(\sum_{k=0}^{N-1} \phi_k \phi_k^T \right) \otimes I_m$$

so that

$$\left(\sum_{k=0}^{N-1} \psi_k \psi_k^T \right)^{-1} = \left(\sum_{k=0}^{N-1} \phi_k \phi_k^T \right)^{-1} \otimes I_m$$

Furthermore

$$\sum_{k=0}^{N-1} \psi_k y_k = \sum_{k=0}^{N-1} (\phi_k \otimes I_m) y_k$$

Equation (5.C.5) can then be written as

$$\widehat{\eta} = \left(\left(\sum_{k=0}^{N-1} \phi_k \phi_k^T \right)^{-1} \otimes I_m \right) \sum_{k=0}^{N-1} (\phi_k \otimes I_m) y_k \tag{5.C.6}$$

On the other hand, the least squares estimate $\widehat{\Theta}$ of Θ is given by

$$\widehat{\Theta}^T = \left(\sum_{k=0}^{N-1} \phi_k y_k^T \right)^T \left(\sum_{k=0}^{N-1} \phi_k \phi_k^T \right)^{-1},$$

so that

$$\mathsf{vec}\ \widehat{\Theta}^T = \left(\left(\sum_{k=0}^{N-1} \phi_k \phi_k^T \right)^{-1} \otimes I_m \right) \mathsf{vec}\ \left\{ \sum_{k=0}^{N-1} y_k \phi_k^T \right\} \tag{5.C.7}$$

But

$$\begin{aligned} \mathsf{vec}\ \left\{ \sum_{k=0}^{N-1} y_k \phi_k^T \right\} &= \sum_{k=0}^{N-1} \mathsf{vec}\ \left\{ y_k \phi_k^T \right\}, \\ &= \sum_{k=0}^{N-1} (\phi_k \otimes I_m)\, \mathsf{vec}\ \ y_k, \\ &= \sum_{k=0}^{N-1} (\phi_k \otimes I_m)\, y_k. \end{aligned}$$

In this case equation (5.C.7) can be written as

$$\text{vec } \widehat{\Theta}^T = \left(\left(\sum_{k=0}^{N-1} \phi_k \phi_k^T \right)^{-1} \otimes I_m \right) \sum_{k=0}^{N-1} (\phi_k \otimes I_m) y_k. \tag{5.C.8}$$

The result then follows by noting that the right hand sides of equations (5.C.6) and (5.C.8) are equal. ■

5.D Proof of Convergence of Block Toeplitz-like Matrices

Proof of Lemma 5.7.1 Let $\mathsf{B}_p(\omega)$ denote

$$\mathsf{B}_p(\omega) \equiv \mathsf{B}_p(e^{\mathrm{j}\omega}) \triangleq \left[\mathcal{B}_0(e^{\mathrm{j}\omega}), \mathcal{B}_1(e^{\mathrm{j}\omega}), \cdots, \mathcal{B}_{p-1}(e^{\mathrm{j}\omega}) \right]^T. \tag{5.D.9}$$

Then

$$\begin{aligned} \Gamma_p(\omega) &= \mathsf{B}_p(\omega) \otimes I_n, \\ \widetilde{\Gamma}_p(\omega) &= \mathsf{B}_p(\omega) \otimes I_n \otimes I_m = \mathsf{B}_p(\omega) \otimes I_{nm}, \end{aligned}$$

so that

$$\begin{aligned} \Gamma_p^\star(\mu)\Gamma_p(\sigma) &= (\mathsf{B}_p^\star(\mu) \otimes I_n)(\mathsf{B}_p(\sigma) \otimes I_n), \\ &= \mathsf{B}_p^\star(\mu)\mathsf{B}_p(\sigma) \otimes I_n, \\ &= \sum_{k=0}^{p-1} \overline{\mathcal{B}_k(e^{\mathrm{j}\mu})} \mathcal{B}_k(e^{\mathrm{j}\sigma}) \otimes I_n, \\ &= K_p(\sigma, \mu) \otimes I_n = K_p(\sigma, \mu) I_n, \end{aligned}$$

and similarly

$$\Gamma_p^\star(\sigma)\Gamma_p(\omega) = K_p(\omega, \sigma) I_n,$$

where $K_p(\cdot, \cdot)$ is the Reproducing Kernel associated with the orthonormal system $\{\mathcal{B}_k(z)\}$. Now we can write

$$\begin{aligned} &\widetilde{\Gamma}_p^\star(\mu) M_p(F, W) \widetilde{\Gamma}_p(\omega) = \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} (\Gamma_p^\star(\mu) \otimes I_m) \left(\Gamma_p(\sigma) F(\sigma) \Gamma_p^\star(\sigma) \otimes W(\sigma) \right) (\Gamma_p(\omega) \otimes I_m) \mathrm{d}\sigma \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \Gamma_p^\star(\mu)\Gamma_p(\sigma) F(\sigma) \Gamma_p^\star(\sigma)\Gamma_p(\omega) \otimes W(\sigma) \mathrm{d}\sigma \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} K_p(\sigma, \mu) K_p(\omega, \sigma) \left(F(\sigma) \otimes W(\sigma) \right) \mathrm{d}\sigma \end{aligned} \tag{5.D.10}$$

Considering the case $\mu = \omega$ the previous expression becomes

$$\widetilde{\Gamma}_p^\star(\mu) M_p(F,W) \widetilde{\Gamma}_p(\omega) = \frac{1}{2\pi} \int_{-\pi}^{\pi} |K_p(\omega,\sigma)|^2 \, (F(\sigma) \otimes W(\sigma)) \, \mathrm{d}\sigma$$

Therefore, using Lemma 4.C.1, we have

$$\begin{aligned} &\frac{\widetilde{\Gamma}_p^\star(\mu) M_p(F,W) \widetilde{\Gamma}_p(\omega)}{\gamma_p(\omega)} - F(\omega) \otimes W(\omega) = \\ &= \frac{1}{\gamma_p(\omega)} \left(\widetilde{\Gamma}_p^\star(\mu) M_p(F,W) \widetilde{\Gamma}_p(\omega) - \gamma_p(\omega) F(\omega) \otimes W(\omega) \right) \\ &= \frac{1}{2\pi\gamma_p(\omega)} \int_{-\pi}^{\pi} |K_p(\omega,\sigma)|^2 \, (F(\sigma) \otimes W(\sigma) - F(\omega) \otimes W(\omega)) \, \mathrm{d}\sigma \end{aligned}$$

Considering now the i,j-th component of the above matrix difference, we have

$$\begin{aligned} &\left| \left[\frac{\widetilde{\Gamma}_p^\star(\mu) M_p(F,W) \widetilde{\Gamma}_p(\omega)}{\gamma_p(\omega)} - F(\omega) \otimes W(\omega) \right]_{i,j} \right| \leq \\ &\leq \frac{1}{2\pi\gamma_p(\omega)} \int_{-\pi}^{\pi} |K_p(\omega,\sigma)|^2 \left| [F(\sigma) \otimes W(\sigma)]_{i,j} - [F(\omega) \otimes W(\omega)]_{i,j} \right| \mathrm{d}\sigma \\ &= \frac{1}{2\pi\gamma_p(\omega)} \int_{\sigma\in\Omega} |K_p(\omega,\sigma)|^2 \left| [F(\sigma) \otimes W(\sigma)]_{i,j} - [F(\omega) \otimes W(\omega)]_{i,j} \right| \mathrm{d}\sigma + \\ &\quad + \frac{1}{2\pi\gamma_p(\omega)} \int_{\sigma\notin\Omega} |K_p(\omega,\sigma)|^2 \left| [F(\sigma) \otimes W(\sigma)]_{i,j} - [F(\omega) \otimes W(\omega)]_{i,j} \right| \mathrm{d}\sigma \end{aligned} \tag{5.D.11}$$

where $\Omega = [\omega - \delta, \omega + \delta]$, for some $\delta > 0$. Now, since $F(\omega)$ and $W(\omega)$ are assumed (component-wise) continuous, then for an arbitrary $\varepsilon > 0$ there is a δ sufficiently small such that

$$\left| [F(\sigma) \otimes W(\sigma)]_{i,j} - [F(\omega) \otimes W(\omega)]_{i,j} \right| < \varepsilon$$

for $\sigma \in \Omega$. Therefore, using this and Lemma 4.C.1 the first term on the RHS of equation (5.D.11) can be bounded as

$$\begin{aligned} \frac{1}{2\pi\gamma_p(\omega)} \int_{\sigma\in\Omega} |K_p(\omega,\sigma)|^2 \left| [F(\sigma) \otimes W(\sigma)]_{i,j} - [F(\omega) \otimes W(\omega)]_{i,j} \right| \mathrm{d}\sigma \; &\leq \\ \leq \frac{\varepsilon}{2\pi\gamma_p(\omega)} \int_{-\pi}^{\pi} |K_p(\omega,\sigma)|^2 \, \mathrm{d}\sigma = \varepsilon \end{aligned}$$

On the other hand, since $F(\omega)$ and $W(\omega)$ are continuous on compact $[-\pi,\pi]$, then $[F(\omega) \otimes W(\omega)]_{i,j}$ is bounded by some constant $M/2 < \infty$. Therefore, the

second term in the RHS of equation (5.D.11) can be bounded as

$$\frac{1}{2\pi\gamma_p(\omega)}\int_{\sigma\notin\Omega}|K_p(\omega,\sigma)|^2\left|[F(\sigma)\otimes W(\sigma)]_{i,j}-[F(\omega)\otimes W(\omega)]_{i,j}\right|\mathrm{d}\sigma \leq \leq \frac{M}{2\pi\gamma_p(\omega)}\int_{\sigma\notin\Omega}|K_p(\omega,\sigma)|^2\,\mathrm{d}\sigma$$

We finally have

$$\left|\left[\frac{\widetilde{\Gamma}_p^\star(\mu)M_p(F,W)\widetilde{\Gamma}_p(\omega)}{\gamma_p(\omega)}-F(\omega)\otimes W(\omega)\right]_{i,j}\right| \leq \leq \varepsilon+\frac{M}{2\pi\gamma_p(\omega)}\int_{\sigma\notin\Omega}|K_p(\omega,\sigma)|^2\,\mathrm{d}\sigma$$

Using Lemma 4.C.1 and the fact that ε is arbitrary then gives the result for $\mu=\omega$.

■

Proof of Lemma 5.7.2 The $(smn+y, tmn+x)$-th element of matrix $M_p(W,X)$ is given by

$$\begin{aligned}[M_p(W,X)]_{smn+y,tmn+x} &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\mathcal{B}_s\overline{\mathcal{B}_t}[W(\omega)\otimes X(\omega)]_{y,x}\mathrm{d}\omega \\ &= \langle\mathcal{B}_s,\mathcal{B}_t[W\otimes X]_{y,x}\rangle\,.\end{aligned}$$

Therefore

$$\begin{aligned}&[M_p(W,X)M_p(Z,U)]_{smn+y,tmn+x} = \\ &= \sum_{k=0}^{p-1}\sum_{r=1}^{mn}[M_p(W,X)]_{smn+y,kmn+r}\,[M_p(Z,U)]_{kmn+r,tmn+x} \\ &= \sum_{k=0}^{p-1}\sum_{r=1}^{mn}\langle\mathcal{B}_s[W\otimes X]_{y,r},\mathcal{B}_k\rangle\,\overline{\langle\mathcal{B}_t[Z\otimes U]_{r,x},\mathcal{B}_k\rangle} \qquad (5.D.12)\end{aligned}$$

On the other hand

$$\begin{aligned}[M_p(WZ,XU)]_{smn+y,tmn+x} &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\mathcal{B}_s\overline{\mathcal{B}_t}\,[WZ\otimes XU]_{y,x}\,\mathrm{d}\omega \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\mathcal{B}_s\overline{\mathcal{B}_t}\,[(W\otimes X)(Z\otimes U)]_{y,x}\,\mathrm{d}\omega\end{aligned}$$

Considering that

$$[(W\otimes X)(Z\otimes U)]_{y,x}=\sum_{r=1}^{mn}[W\otimes X]_{y,r}[Z\otimes U]_{r,x}$$

we can write

$$\begin{aligned}[M_p(WZ,XU)]_{smn+y,tmn+x} &= \sum_{r=1}^{mn}\frac{1}{2\pi}\int_{-\pi}^{\pi}\mathcal{B}_s\overline{\mathcal{B}_t}[W\otimes X]_{y,r}[Z\otimes U]_{r,x}\mathrm{d}\omega \\ &= \sum_{r=1}^{mn}\langle\mathcal{B}_s[W\otimes X]_{y,r},\mathcal{B}_t[Z\otimes U]_{r,x}\rangle \qquad (5.D.13)\end{aligned}$$

Now, from equations (5.D.12) and (5.D.13), and Lemma 5.E.10 we have

$$\begin{aligned}&\left|[M_p(W,X)M_p(Z,U)]_{smn+y,tmn+x}-[M_p(WZ,XU)]_{smn+y,tmn+x}\right| = \\ &= \left|\sum_{r=1}^{mn}\left[\sum_{k=0}^{p-1}\langle\mathcal{B}_s[W\otimes X]_{y,r},\mathcal{B}_k\rangle\,\overline{\langle\mathcal{B}_t[Z\otimes U]_{r,x},\mathcal{B}_k\rangle}-\langle\mathcal{B}_s[W\otimes X]_{y,r},\mathcal{B}_t[Z\otimes U]_{r,x}\rangle\right]\right| \\ &\le \sum_{r=1}^{mn}\left|\sum_{k=0}^{p-1}\langle\mathcal{B}_s[W\otimes X]_{y,r},\mathcal{B}_k\rangle\,\overline{\langle\mathcal{B}_t[Z\otimes U]_{r,x},\mathcal{B}_k\rangle}-\langle\mathcal{B}_s[W\otimes X]_{y,r},\mathcal{B}_t[Z\otimes U]_{r,x}\rangle\right| \\ &\le \sum_{r=1}^{mn}\left\|\mathcal{B}_s[W\otimes X]_{y,r}-\widehat{V}_{y,r}\right\|_{H_2}\left\|\mathcal{B}_t[Z\otimes U]_{r,x}-\widehat{J}_{r,x}\right\|_{H_2} \qquad (5.D.14)\end{aligned}$$

where

$$\widehat{V}_{y,r}(\omega)=\sum_{k=0}^{p-1}\langle\mathcal{B}_s[W\otimes X]_{y,r},\mathcal{B}_k\rangle\,\mathcal{B}_k(\omega)$$

and

$$\widehat{J}_{r,x}(\omega)=\sum_{k=0}^{p-1}\langle\mathcal{B}_t[Z\otimes U]_{r,x},\mathcal{B}_k\rangle\,\mathcal{B}_k(\omega)$$

But by Lemma 5.E.11, for some $K<\infty$ and $|\eta|<1$ we have

$$\begin{aligned}\left\|\mathcal{B}_s[W\otimes X]_{y,r}-\widehat{V}_{y,r}\right\|_{H_2} &\le K(\eta^{p-s}+\eta^s) \\ \left\|\mathcal{B}_t[Z\otimes U]_{r,x}-\widehat{J}_{r,x}\right\|_{H_2} &\le K(\eta^{p-t}+\eta^t).\end{aligned}$$

Substituting this into (5.D.14) then gives the result. ■

5.E Technical Lemmas

Lemma 5.E.1. Invariance of LSE under linear re-parameterization. *Let the parameterized model be given by*

$$y_k=G(q,\Theta)u_k+\nu_k=\Theta^T\phi_k+\nu_k \qquad (5.E.15)$$

where

$$G(q,\Theta) \triangleq \Theta^T(\mathsf{B}_p(q)\otimes I_n) \quad (5.E.16)$$

$$\mathsf{B}_p(q) \triangleq [\mathcal{B}_0(q),\mathcal{B}_1(q),\cdots,\mathcal{B}_{p-1}(q)]^T \quad (5.E.17)$$

$$\phi_k = (\mathsf{B}_p(q)\otimes I_n)u_k. \quad (5.E.18)$$

Let

$$G(q,\beta) \triangleq \beta^T(\mathsf{A}_p(q)\otimes I_n) \quad (5.E.19)$$

$$\mathsf{A}_p(q) \triangleq J\mathsf{B}_p(q), \quad (5.E.20)$$

be a linear re-parameterization of the transfer matrix of the system, where J is a nonsingular matrix of appropriate dimensions. Then, the least squares estimates of the parameter matrices Θ and β are related according to

$$\widehat{\beta} = (J^T\otimes I_n)^{-1}\widehat{\Theta},$$

so that the corresponding transfer matrix estimates are the same, i.e.

$$G(q,\widehat{\beta}) = G(q,\widehat{\Theta}).$$

Proof: The re-parameterization (5.E.19)-(5.E.20) leads to the linear regressor form

$$y_k = G(q,\beta)u_k + \nu_k = \beta^T\psi_k + \nu_k, \quad (5.E.21)$$

where the new regressor matrix ψ_k is related to the original one, ϕ_k, according to

$$\psi_k = (J\otimes I_n)\phi_k. \quad (5.E.22)$$

Adopting the vectorized notation

$$\begin{aligned} Y^T &= (y_0, y_1, \cdots, y_{N-1}), \\ V^T &= (\nu_0, \nu_1, \cdots, \nu_{N-1}), \\ \Phi^T &= (\phi_0, \phi_1, \cdots, \phi_{N-1}), \\ \Psi^T &= (\psi_0, \psi_1, \cdots, \psi_{N-1}), \end{aligned}$$

the model for the N point observed data record can be written as:

$$Y = \Phi\Theta + V, \quad (5.E.23)$$

or in re-parameterized form

$$Y = \Psi\beta + V. \quad (5.E.24)$$

Now, the least squares estimate $\widehat{\beta}$ of the parameter matrix β is given by

$$\widehat{\beta} = \left(\Psi^T\Psi\right)^{-1}\Psi^T Y.$$

Based on equation (5.E.22), simple algebra gives

$$\Psi = \Phi(J^T \otimes I_n),$$

so that $\widehat{\beta}$ can be written as

$$\begin{aligned}
\widehat{\beta} &= \left(\Psi^T\Psi\right)^{-1}\Psi^T Y, \\
&= \left((J\otimes I_n)\Phi^T\Phi(J\otimes I_n)^T\right)^{-1}(J\otimes I_n)\Phi^T Y, \\
&= (J\otimes I_n)^{-T}\left(\Phi^T\Phi\right)^{-1}(J\otimes I_n)^{-1}(J\otimes I_n)\Phi^T Y, \\
&= (J\otimes I_n)^{-T}\left(\Phi^T\Phi\right)^{-1}\Phi^T Y, \\
&= (J\otimes I_n)^{-T}\widehat{\Theta}.
\end{aligned}$$

It remains now to prove that the transfer matrix estimates with both parameterizations are the same. We have

$$\begin{aligned}
G(q,\widehat{\beta}) &= \widehat{\beta}^T(\mathsf{A}_p(q)\otimes I_n), \\
&= \widehat{\Theta}^T(J\otimes I_n)^{-1}(J\otimes I_n)(\mathsf{B}_p(q)\otimes I_n), \\
&= \widehat{\Theta}^T(\mathsf{B}_p(q)\otimes I_n), \\
&= G(q,\widehat{\Theta}).
\end{aligned}$$

■

Lemma 5.E.2. *Let $\{\mathcal{B}_k\}$ be a set of OBFP defined as in equation (3.14) from the set of poles $\{\xi_k\}$, and let ξ_{max} denote the pole with maximum module in this set. Then*

$$\sqrt{\frac{1-|\xi_{max}|}{1+|\xi_{max}|}} \le |\mathcal{B}_k(e^{\mathrm{j}\omega})| \le \sqrt{\frac{1+|\xi_{max}|}{1-|\xi_{max}|}} \triangleq K_1 < \infty$$

Proof:

$$\begin{aligned}
\left|\mathcal{B}_k(e^{\mathrm{j}\omega})\right| &= \left|\left(\frac{\sqrt{1-|\xi_k|^2}}{e^{\mathrm{j}\omega}-\xi_k}\right)\prod_{i=0}^{k-1}\left(\frac{1-\overline{\xi_i}e^{\mathrm{j}\omega}}{e^{\mathrm{j}\omega}-\xi_i}\right)\right| \\
&= \left|\left(\frac{\sqrt{1-|\xi_k|^2}}{e^{\mathrm{j}\omega}-\xi_k}\right)\right| = \frac{\sqrt{1-|\xi_k|^2}}{|e^{\mathrm{j}\omega}-\xi_k|}
\end{aligned}$$

Now, since by assumption is $|\xi_k| \le |\xi_{max}| < 1$, we have

$$(1-|\xi_{max}|) \le (1-|\xi_k|) \le \left|e^{\mathrm{j}\omega}-\xi_k\right| \le (1+|\xi_k|) \le (1+|\xi_{max}|)$$

and the result follows straightforwardly. ■

Lemma 5.E.3. *Let $\gamma_p(\omega)$ be defined as*

$$\gamma_p(\omega) \triangleq K_p(\omega,\omega) = \sum_{k=0}^{p-1} \left|\mathcal{B}_k(e^{\mathrm{j}\omega})\right|^2$$

and let ξ_{max} denote the pole with maximum module in the set $\{\xi_k\}$ of poles of the basis functions $\{\mathcal{B}_k\}$. Then the following inequalities hold

$$\frac{p}{2}\left(1-|\xi_{max}|\right) \leq \frac{1}{2}\sum_{k=0}^{p-1}(1-|\xi_k|) \leq \gamma_p(\omega) \leq p\,\frac{1+|\xi_{max}|}{1-|\xi_{max}|}$$

Proof: The result is a direct consequence of Lemma 5.E.2. ■

Lemma 5.E.4. *Let Q_p and $\widetilde{G}_p(\omega)$ be defined as*

$$\begin{aligned} Q_p &\triangleq \lim_{N\to\infty} N\mathbf{E}\left\{V_N'(\theta_0)\left(V_N'(\theta_0)\right)^T\right\}, \\ \widetilde{G}_p(\omega) &\triangleq G(e^{\mathrm{j}\omega}) - G(e^{\mathrm{j}\omega},\theta_0). \end{aligned}$$

Then

$$Q_p = \frac{1}{2\pi}\int_{-\pi}^{\pi} \Gamma_p(\omega)\Phi_u(\omega)\Gamma_p^\star(\omega) \otimes \Phi_\nu(\omega)\,\mathrm{d}\omega + \Delta_p$$

where

$$\Delta_p \triangleq \frac{1}{2\pi}\int_{-\pi}^{\pi} \Gamma_p(\omega)\Phi_u(\omega)\Gamma_p^\star(\omega) \otimes \widetilde{G}_p(\omega)\Phi_u(\omega)\widetilde{G}_p^\star(\omega)\,\mathrm{d}\omega.$$

Proof: By the definition of Q_p

$$Q_p = \lim_{N\to\infty}\frac{1}{N}\sum_{k=0}^{N-1}\sum_{\ell=0}^{N-1}(\phi_k\otimes I_m)\mathbf{E}\left\{\varepsilon_k\varepsilon_\ell^T\right\}(\phi_\ell^T\otimes I_m)$$

where ϕ_k is defined in (5.22) and

$$\varepsilon_k = \underbrace{[G(q)-G_p(q,\theta_0)]u_k}_{z_k} + \nu_k.$$

Therefore, using Lemma 5.E.5

$$\begin{aligned} Q_p &= \lim_{N\to\infty}\frac{1}{N}\sum_{k=0}^{N-1}\sum_{\ell=0}^{N-1}(\phi_k\otimes I_m)\mathbf{E}\left\{\nu_k\nu_\ell^T\right\}(\phi_\ell^T\otimes I_m) + \\ &\quad + \lim_{N\to\infty}\frac{1}{N}\sum_{k=0}^{N-1}\sum_{\ell=0}^{N-1}(\phi_k\otimes I_m)\mathbf{E}\left\{z_k z_\ell^T\right\}(\phi_\ell^T\otimes I_m) \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\Gamma_p(\omega)\Phi_u(\omega)\Gamma_p^\star(\omega)\otimes\Phi_\nu(\omega)\,\mathrm{d}\omega + \\ &\quad + \underbrace{\frac{1}{2\pi}\int_{-\pi}^{\pi}\Gamma_p(\omega)\Phi_u(\omega)\Gamma_p^\star(\omega)\otimes\widetilde{G}_p(\omega)\Phi_u(\omega)\widetilde{G}_p^\star(\omega)\,\mathrm{d}\omega}_{\Delta_p}. \end{aligned}$$

■

Lemma 5.E.5.

$$\lim_{N\to\infty} \frac{1}{N} \sum_{k=0}^{N-1}\sum_{\ell=0}^{N-1} (\phi_k \otimes I_m)\mathbf{E}\left\{\nu_k \nu_\ell^T\right\}(\phi_\ell^T \otimes I_m) =$$
$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} \Gamma_p(\omega)\Phi_u(\omega)\Gamma_p^\star(\omega) \otimes \Phi_\nu(\omega)\, \mathrm{d}\omega.$$

Proof: Since $\{\nu_k\}$ is a stationary process with an associated spectral density $\Phi_\nu(\omega)$ then

$$\mathbf{E}\left\{\nu_k \nu_\ell^T\right\} \triangleq R_\nu(k-\ell) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \Phi_\nu(\omega) e^{-\mathrm{j}\omega(k-\ell)}\, \mathrm{d}\omega.$$

Therefore,

$$\begin{aligned}
&\frac{1}{N} \sum_{k=0}^{N-1}\sum_{\ell=0}^{N-1} (\phi_k \otimes I_m)\mathbf{E}\left\{\nu_k \nu_\ell^T\right\}(\phi_\ell^T \otimes I_m) = \\
&= \frac{1}{2\pi N}\int_{-\pi}^{\pi} \sum_{k=0}^{N-1}\sum_{\ell=0}^{N-1} (\phi_k \otimes I_m)\Phi_\nu(\phi_\ell^T \otimes I_m) e^{-\mathrm{j}\omega(k-\ell)}\, \mathrm{d}\omega \\
&= \frac{1}{2\pi N}\int_{-\pi}^{\pi} \sum_{k=0}^{N-1}\sum_{\ell=0}^{N-1} \left(\phi_k\phi_\ell^T \otimes \Phi_\nu(\omega)\right) e^{-\mathrm{j}\omega(k-\ell)}\, \mathrm{d}\omega \\
&= \frac{1}{2\pi N}\int_{-\pi}^{\pi} \left(\sum_{k=0}^{N-1}\sum_{\ell=0}^{N-1} \phi_k\phi_\ell^T e^{-\mathrm{j}\omega(k-\ell)}\right) \otimes \Phi_\nu(\omega)\, \mathrm{d}\omega \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} \sum_{\tau=1-N}^{N-1} \underbrace{\left(\frac{1}{N}\sum_{k=0}^{N-1} \phi_k\phi_{k-\tau}^T\right)}_{R_{\phi_N}(\tau)} e^{-\mathrm{j}\omega\tau} \otimes \Phi_\nu(\omega)\, \mathrm{d}\omega \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} \left(\sum_{\tau=1-N}^{N-1} R_{\phi_N}(\tau) e^{-\mathrm{j}\omega\tau}\right) \otimes \Phi_\nu(\omega)\, \mathrm{d}\omega \qquad (5.\mathrm{E}.25)
\end{aligned}$$

where use has been made of the change of variable $k-\ell=\tau$ and it has been assumed that $u_k = 0$ for $k < 0$. Since u_k is a quasi-stationary signal, then also $\phi_k \triangleq \Gamma_p(q)u_k$ is quasi-stationary, and then its covariance matrix is given by

$$R_\phi(\tau) \triangleq \lim_{N\to\infty} \frac{1}{N}\sum_{k=0}^{N-1} \phi_k\phi_{k-\tau}^T = \lim_{N\to\infty} R_{\phi_N}(\tau)$$

and the spectral density by

$$\Phi_\phi(\omega) \triangleq \sum_{\tau=-\infty}^{\infty} R_\phi(\tau) e^{-\mathrm{j}\omega\tau}$$

Now, the result follows by taking the limits when N tends to infinity in equation (5.E.25), and considering that

$$\Phi_\phi(\omega) = \Gamma_p(\omega)\Phi_u(\omega)\Gamma_p^\star(\omega)$$

■

Lemma 5.E.6. *Let* $\Gamma_p(\omega)$ *and* $\widetilde{\Gamma}_p(\omega)$ *be defined as*

$$\begin{aligned}\Gamma_p(\omega) &\triangleq \left[\mathcal{B}_0(e^{\mathrm{j}\omega}), \mathcal{B}_1(e^{\mathrm{j}\omega}), \cdots, \mathcal{B}_{p-1}(e^{\mathrm{j}\omega})\right]^T \otimes I_n \\ \widetilde{\Gamma}_p(\omega) &\triangleq \Gamma_p(\omega) \otimes I_m\end{aligned}$$

respectively. Then

$$\|\Gamma_p(\omega)\|_2 = \|\widetilde{\Gamma}_p(\omega)\|_2 = \sqrt{\gamma_p(\omega)}$$

Proof: Let $\mathsf{B}_p(\omega)$ be defined as in equation (5.D.9), so that $\Gamma_p(\omega)$ and $\widetilde{\Gamma}_p(\omega)$ can be written as

$$\begin{aligned}\Gamma_p(\omega) &= \mathsf{B}_p(\omega) \otimes I_n \\ \widetilde{\Gamma}_p(\omega) &= \mathsf{B}_p(\omega) \otimes I_n \otimes I_m = \mathsf{B}_p(\omega) \otimes I_{nm}\end{aligned}$$

respectively. Then

$$\begin{aligned}\|\Gamma_p(\omega)\|_2 &\triangleq \sup_{x\neq 0} \frac{\|\Gamma_p(\omega)x\|_2}{\|x\|_2} \\ &= \sup_{x\neq 0} \frac{\sqrt{x^\star \Gamma_p^\star(\omega)\Gamma_p(\omega)x}}{\sqrt{x^\star x}} \\ &= \sup_{x\neq 0} \frac{\sqrt{x^\star (\mathsf{B}_p^\star(\omega)\otimes I_n)(\mathsf{B}_p(\omega)\otimes I_n)x}}{\sqrt{x^\star x}} \\ &= \sup_{x\neq 0} \frac{\sqrt{x^\star (\mathsf{B}_p^\star(\omega)\mathsf{B}_p(\omega)\otimes I_n)x}}{\sqrt{x^\star x}} = \sqrt{\gamma_p(\omega)}\end{aligned}$$

where the last equality follows considering that $\mathsf{B}_p^\star(\omega)\mathsf{B}_p(\omega) = \gamma_p(\omega)$. In similar way it can be proved that $\|\widetilde{\Gamma}_p(\omega)\|_2 = \sqrt{\gamma_p(\omega)}$. ■

Lemma 5.E.7. *Let R_p be defined as in equation (5.55), and let $\lambda(R_p)$ denote the eigenvalues of R_p, and $\underline{\sigma}(\Phi_u(\omega))$ and $\overline{\sigma}(\Phi_u(\omega))$ denote the minimum and maximum singular values of $\Phi_u(\omega)$, respectively. Then*

$$\inf_{\omega} \underline{\sigma}(\Phi_u(\omega)) \leq \lambda(R_p) \leq \sup_{\omega} \overline{\sigma}(\Phi_u(\omega)) \tag{5.E.26}$$

and

$$\left(\sup_{\omega} \overline{\sigma}(\Phi_u(\omega))\right)^{-1} \leq \lambda(R_p^{-1}) \leq \left(\inf_{\omega} \underline{\sigma}(\Phi_u(\omega))\right)^{-1}. \tag{5.E.27}$$

Proof: For any μ, R_p can be written as

$$\begin{aligned} R_p &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \Gamma_p(e^{\mathrm{j}\omega})\left(\mu I_n - (\mu I_n - \Phi_u(\omega))\right)\Gamma_p^{\star}(e^{\mathrm{j}\omega}) \otimes I_m \,\mathrm{d}\omega \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \Gamma_p(e^{\mathrm{j}\omega})\mu I_n \Gamma_p^{\star}(e^{\mathrm{j}\omega}) \otimes I_m \,\mathrm{d}\omega - \\ &- \frac{1}{2\pi}\int_{-\pi}^{\pi} \Gamma_p(e^{\mathrm{j}\omega})\left(\mu I_n - \Phi_u(\omega)\right)\Gamma_p^{\star}(e^{\mathrm{j}\omega}) \otimes I_m \,\mathrm{d}\omega \end{aligned}$$

Now choose $\mu = \sup_{\omega} \overline{\sigma}(\Phi_u(\omega))$. Since $\Phi_u(\omega) \geq 0$, then $(\mu I_n - \Phi_u(\omega)) \geq 0$, and then

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \Gamma_p(e^{\mathrm{j}\omega})\left(\mu I_n - \Phi_u(\omega)\right)\Gamma_p^{\star}(e^{\mathrm{j}\omega}) \otimes I_m \,\mathrm{d}\omega \geq 0$$

so that we can write

$$\begin{aligned} R_p &\leq \frac{1}{2\pi}\int_{-\pi}^{\pi} \Gamma_p(e^{\mathrm{j}\omega})\mu I_n \Gamma_p^{\star}(e^{\mathrm{j}\omega}) \otimes I_m \,\mathrm{d}\omega \\ &= \sup_{\omega} \overline{\sigma}(\Phi_u(\omega)) \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} \Gamma_p(e^{\mathrm{j}\omega})\Gamma_p^{\star}(e^{\mathrm{j}\omega}) \,\mathrm{d}\omega\right) \otimes I_m \\ &= \sup_{\omega} \overline{\sigma}(\Phi_u(\omega))\ I_{mnp} \end{aligned}$$

where in passing to the last line the following identity was used

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \Gamma_p(e^{\mathrm{j}\omega})\Gamma_p^{\star}(e^{\mathrm{j}\omega}) \,\mathrm{d}\omega = I_{np}$$

On the other hand, R_p can be written as

$$\begin{aligned} R_p &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \Gamma_p(e^{\mathrm{j}\omega})\left(\mu I_n + (\Phi_u(\omega) - \mu I_n)\right)\Gamma_p^{\star}(e^{\mathrm{j}\omega}) \otimes I_m \,\mathrm{d}\omega \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \Gamma_p(e^{\mathrm{j}\omega})\mu I_n \Gamma_p^{\star}(e^{\mathrm{j}\omega}) \otimes I_m \,\mathrm{d}\omega + \\ &+ \frac{1}{2\pi}\int_{-\pi}^{\pi} \Gamma_p(e^{\mathrm{j}\omega})\left(\Phi_u(\omega) - \mu I_n\right)\Gamma_p^{\star}(e^{\mathrm{j}\omega}) \otimes I_m \,\mathrm{d}\omega \end{aligned}$$

Now choose $\mu = \inf_\omega \underline{\sigma}(\Phi_u(\omega))$. Since $\Phi_u(\omega) \geq 0$, then $(\Phi_u(\omega) - \mu I_n) \geq 0$, and then

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \Gamma_p(e^{\mathrm{j}\omega})\,(\Phi_u(\omega) - \mu I_n)\,\Gamma_p^\star(e^{\mathrm{j}\omega}) \otimes I_m \,\mathrm{d}\omega \geq 0$$

so that we can write

$$\begin{aligned} R_p &\geq \frac{1}{2\pi}\int_{-\pi}^{\pi} \Gamma_p(e^{\mathrm{j}\omega})\mu I_n \Gamma_p^\star(e^{\mathrm{j}\omega}) \otimes I_m \,\mathrm{d}\omega \\ &= \inf_\omega \underline{\sigma}(\Phi_u(\omega)) \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} \Gamma_p(e^{\mathrm{j}\omega})\Gamma_p^\star(e^{\mathrm{j}\omega})\,\mathrm{d}\omega\right) \otimes I_m \\ &= \inf_\omega \underline{\sigma}(\Phi_u(\omega))\; I_{mnp} \end{aligned}$$

Finally, we have

$$\inf_\omega \underline{\sigma}(\Phi_u(\omega))\; I_{mnp} \leq R_p \leq \sup_\omega \overline{\sigma}(\Phi_u(\omega))\; I_{mnp} \tag{5.E.28}$$

so that

$$\left(\sup_\omega \overline{\sigma}(\Phi_u(\omega))\right)^{-1} I_{mnp} \leq R_p^{-1} \leq \left(\inf_\omega \underline{\sigma}(\Phi_u(\omega))\right)^{-1}\; I_{mnp} \tag{5.E.29}$$

and the result follows straightforwardly. ■

Lemma 5.E.8. *Let R_p be defined as in equation (5.55). Then*

$$\left\|R_p^{-1}\right\|_2 = \left\|M_p^{-1}(\Phi_u, I_m)\right\|_2 \leq \left(\inf_\omega \underline{\sigma}(\Phi_u(\omega))\right)^{-1}$$

Proof: The result follows directly from Lemma 5.E.7 considering that the 2-norm equals the maximum singular value. ■

Lemma 5.E.9. *Let Δ_p be defined as*

$$\Delta_p \triangleq \frac{1}{2\pi}\int_{-\pi}^{\pi} \Gamma_p(\omega)\Phi_u(\omega)\Gamma_p^\star(\omega) \otimes \widetilde{G}_p(\omega)\Phi_u(\omega)\widetilde{G}_p^\star(\omega)\,\mathrm{d}\omega$$

Then

$$\lim_{p\to\infty} \|\Delta_p\|_2 = 0$$

Proof:

$$\begin{aligned}
\|\Delta_p\|_2 &\leq \frac{1}{2\pi}\int_{-\pi}^{\pi}\left\|\Gamma_p(\omega)\Phi_u(\omega)\Gamma_p^\star(\omega)\otimes\widetilde{G}_p(\omega)\Phi_u(\omega)\widetilde{G}_p^\star(\omega)\right\|_2 \mathrm{d}\omega\\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi}\left\|\Gamma_p(\omega)\Phi_u(\omega)\Gamma_p^\star(\omega)\right\|_2\left\|\widetilde{G}_p(\omega)\Phi_u(\omega)\widetilde{G}_p^\star(\omega)\right\|_2 \mathrm{d}\omega\\
&\leq \frac{1}{2\pi}\int_{-\pi}^{\pi}\|\Gamma_p(\omega)\|_2^2\,\|\Phi_u(\omega)\|_2^2\left\|\widetilde{G}_p(\omega)\right\|_2^2 \mathrm{d}\omega\\
&\leq \frac{1}{2\pi}\int_{-\pi}^{\pi}\gamma_p(\omega)\,\|\Phi_u(\omega)\|_2^2\left\|\widetilde{G}_p(\omega)\right\|_F^2 \mathrm{d}\omega\\
&\leq \left(\sup_\omega\,\overline{\sigma}(\Phi_u(\omega))\right)^2\left(\sup_\omega\,\gamma_p(\omega)\right)\,\frac{1}{2\pi}\int_{-\pi}^{\pi}\left\|\widetilde{G}_p(\omega)\right\|_F^2 \mathrm{d}\omega\\
&= \left(\sup_\omega\,\overline{\sigma}(\Phi_u(\omega))\right)^2\left(\sup_\omega\,\gamma_p(\omega)\right)\,\left\|\widetilde{G}_p(\omega)\right\|_{H_2}^2
\end{aligned}$$

Here, $\|\cdot\|_F$ stands for the Frobenius[15] matrix norm, while $\|\cdot\|_{H_2}$ stands for the H_2-norm[16] induced by the inner product. Finally, the result follows under the assumption that $\left\|\widetilde{G}_p(\omega)\right\|_{H_2}^2$ decays to zero faster than $1/\gamma_p(\omega)$, so that

$$\lim_{p\to\infty}\left(\sup_\omega\,\gamma_p(\omega)\right)\,\left\|\widetilde{G}_p(\omega)\right\|_{H_2}^2 = 0.$$

■

Lemma 5.E.10. [NHG96] *For $f, g \in L_2(\mathbb{T})$, let $\widehat{f}$ and $\widehat{g}$ be defined as*

$$\begin{aligned}
\widehat{f} &\triangleq \sum_{k=0}^{p-1}\langle f, \mathcal{B}_k\rangle\,\mathcal{B}_k\\
\widehat{g} &\triangleq \sum_{k=0}^{p-1}\langle g, \mathcal{B}_k\rangle\,\mathcal{B}_k
\end{aligned}$$

respectively. Then

$$\left|\sum_{k=0}^{p-1}\langle f, \mathcal{B}_k\rangle\,\overline{\langle g, \mathcal{B}_k\rangle} - \langle f, g\rangle\right| \leq \left\|f-\widehat{f}\right\|_{L_2}\|g-\widehat{g}\|_{L_2}.$$

Proof: See Lemma C.3 in [NHG96]. ■

[15] The Frobenius norm of the matrix $A = (a_{ij})_{m\times n}$ is defined as $\|A\|_F^2 \triangleq \sum_{i=1}^m\sum_{j=1}^n |a_{ij}|^2$.

[16] The H_2-norm induced by the inner product is given by $\|G(\omega)\|_{H_2}^2 \triangleq < G, G > = \frac{1}{2\pi}\int_{-\pi}^{\pi}\mathsf{Tr}\left\{G(e^{\mathrm{j}\omega})G^\star(e^{\mathrm{j}\omega})\right\}\,\mathrm{d}\omega = \frac{1}{2\pi}\int_{-\pi}^{\pi}\|G(e^{\mathrm{j}\omega})\|_F^2\,\mathrm{d}\omega$.

Lemma 5.E.11. [NHG96] *For $f \in L_2(\mathbb{T})$, let $\widehat{h}$ be defined as*

$$\widehat{h} \triangleq \sum_{k=0}^{p-1} \langle \mathcal{B}_m f, \mathcal{B}_k \rangle \mathcal{B}_k.$$

Then if f has a finite dimensional spectral factorization there exists $K < \infty$, $|\eta| < 1$ such that for $m \leq p$

$$\left\| \mathcal{B}_m f - \widehat{h} \right\|_{L_2} \leq K(\eta^{p-m} + \eta^m)$$

Proof: See Lemma C.4 in [NHG96]. ■

6

Adaptive Tracking Performance Analysis using Orthonormal Bases

A fundamental problem in control and signal processing is to track the time-varying parameters of a system or the properties of a signal as they vary with time. In recent years, considerable effort has been devoted to the derivation and analysis of a variety of adaptation algorithms for their use in recursive identification, adaptive filtering and control. In this chapter, a frequency domain analysis of the tracking performance of several adaptive algorithms for the recursive identification of time-varying linear systems will be carried out for the case in which the system is represented by a fixed denominator model structure. As done in the preceding chapters, to facilitate the analysis these model structures will be re-parameterized using the orthonormal bases with fixed poles introduced in Section 3.5. We concentrate on the study of the trade-off between disturbance rejection and tracking ability. This trade-off will be illustrated by showing how the quality of the adaptive estimation is influenced by such things as input and noise spectral densities, step size of the adaptive algorithms, and the choice of the fixed pole locations in the model structure.

6.1 Introduction

The derivation of a mathematical model for a system from observation data is the topic of System Identification. When dealing with time-invariant systems, the identification can be carried out off-line, which means that the parameter estimation is performed once the whole set of data is collected from the system. This cannot be done when the system properties vary with time. In this case the identification has to be performed on-line, which basically means that the estimate of the unknown parameters is updated each time a new data becomes available. This leads us to the field of Recursive Identification, where a consirable amount of research has been carried out in recent years (see for instance the books [You84, GS84, WS85, CG91]). In this context, identification algorithms

have to **adapt** themselves to **track** the changes in the system, and for this reason they are called **adaptive** algorithms or (adaptive) **tracking** algorithms.

At this point, a question that arises is what characteristics of the algorithms should be considered to evaluate their performance. In general, one is interested in the following aspects of the algorithm

- Stability: exponential stability of the algorithm is necesary to guarantee boundedness of the tracking error, and is a pre-requisite for the practical applications and the analysis. The general conditions for stability of the algorithms we consider here have been established by Guo and Ljung in [GL95a], and therefore this issue is not studied in this chapter.

- Transient Response: the dominating dynamics governing the behaviour of the adjusted parameters when there is an abrupt change in the true system, or after an initialization.

- Steady-State Response: the variability of the estimated parameters with respect to the 'true' parameters, after a long period of time, when the system is assumed to be time-invariant (or slowly time-varying).

We concentrate in this chapter only in the analysis of the last two issues, viz. the transient response and the steady-state response of the algorithms.

A second question that arises is how to quantify the quality of the estimation. For instance, Guo and Ljung in [GL95b, GL95a] consider a general family of tracking algorithms, and quantify the quality of the estimation in the parameter space. The quantification is done in terms of the covariance matrix of the parameter tracking error (the so-called Mean Squares Error of the parameter estimate). Since the exact expression for this error is, in general, very complicated, the authors propose a simple expression that provides a good approximation when the adaptation rate of the algorithm is small. In this chapter, we use these results but, following the suggestions in [EJLW92, RJL96, Joh95, Joh93] about the utility of analysis of adaptive algorithms in the frequency domain, the quality of the estimate is analyzed here in terms of the covariance matrix of the frequency response estimation error rather than in terms of the covariance matrix of the parameter error. These frequency domain expressions are still too complicated to be of any practical utility. Drawing inspiration from the work in [GL89, LG90, LY85, Gun88] we derive more tractable expressions by considering high model orders.

Recent work in the areas of Adaptive Filtering [Wil95, Wil93a, WZ96] and Recursive Identification [GW90], suggesting the use of new model structures -as an alternative to the popular FIR structures- in adaptive algorithms has inspired us to use the fixed denominator model structures we consider in this chapter. Specifically, the fixed poles adaptive filters (FPAF) proposed in [Wil93a],

the Gamma filters in [Pdd93], the adaptive IIR filters in [WZ96], and the recursively identified Laguerre models in [GW90], can all be regarded as fixed denominator model structures. All these model structures can be represented in a unified way by re-parameterizing them using the OBFP introduced in Chapter 3 (Section 3.5). As mentioned several times in this thesis, these fixed denominator model structures can be regarded as generalizations of the FIR structure, where the poles need not all be fixed at the origin. As pointed out in [Wil95, WZ96, GW90], the use of these structures can lead to many improvements in terms of estimation accuracy, and implementation and computational complexity, while still retaining the desirable convergence properties enjoyed by adaptive FIR schemes.

This chapter provides a frequency domain analysis of the tracking performance of some standard adaptation algorithms when used to track time-varying systems represented by these 'generalized FIR' structures. Both a transient, and a steady state analysis of the behavior of the frequency response estimation error are performed. The analysis shows that there is an important design compromise between tracking ability and noise sensitivity of the algorithms, and it makes explicit how this trade-off is influenced by input and noise spectral densities, choice of step size (or adaptation rate) of the algorithm, and (what is a main focus here) the choice of the fixed pole positions in the model structure.

The material in this chapter is closely related with the work of Gunnarsson and Ljung [GL89, LG90, Gun88] who studied adaptive FIR algorithms in the frequency domain. Specifically, we borrow from [GL89, Gun88] the main idea of this chapter which is to simplify error expressions by considering large model order, and small adaptation rate. The results of this chapter specialise to some of those in [GL89, Gun88] when all the poles of the orthonormal bases are chosen at the origin.

These ideas, together with the use of recent results by Guo and Ljung [GL95b] that provide approximations to the parameter covariance for a general class of adaptive algorithms under mild assumptions, constitute the base upon which our results are constructed. An important tool employed in the derivation of these results is the re-parameterization of the fixed denominator model structure into the orthonormal form studied in Chapter 3 (Section 3.5), in order to facilitate the theoretical analysis.

6.2 Motivation

A result derived by Gunnarsson and Ljung [GL89, Gun88] that has proved to be of great utility in the intuitive understanding and design [EJLW92, RJL96, Joh93, Joh95] of certain adaptive tracking schemes, is that the variability of a recursively computed p-th order FIR transfer function estimate $G(e^{j\omega}, \widehat{\theta})$ may

be approximated (for time invariant systems) by the simple expression

$$\mathrm{Var}\{G(e^{j\omega},\widehat{\theta})\} \approx p\frac{\mu\sigma_\nu^2\kappa}{[\Phi_u(\omega)]^r} \tag{6.1}$$

where κ is some constant, μ is the step size, σ_ν^2 is the white measurement noise variance, $\Phi_u(\omega)$ is the input excitation spectral density, and $r = 1$ for Recursive Least Squares (RLS), $r = 1/2$ for Kalman Filtering and $r = 0$ for the Least Mean Square (LMS) algorithm.

In assessing the utility of the idea of implementing adaptive filters with arbitrary fixed poles, a natural question arises as to how (6.1) should be modified from the FIR case so as to describe $\mathrm{Var}\{G(e^{j\omega},\widehat{\theta})\}$. The most obvious course is to conclude that these new 'fixed denominator' model structures are really just the old FIR ones with an input $\{u_k\}$ pre-filtered by an all-pole filter $F(q) = 1/D(q)$ where $D(q) = \prod_{n=0}^{p-1}(q-\xi_n)$ with $\{\xi_n\}$ being the user chosen guesses as to the true pole locations. This would imply that the variability of the FIR 'numerator' part is then given by the expression (6.1) with the substitution

$$\Phi_u(\omega) \mapsto |F(e^{j\omega})|^2\Phi_u(e^{j\omega}) = \frac{\Phi_u(\omega)}{|D(e^{j\omega})|^2} \tag{6.2}$$

made. The frequency domain variability of the whole model structure, being the FIR numerator part divided by the frequency response of the fixed denominator part, should then be (6.1) with the substitution (6.2) and then divided by $|D(e^{j\omega})|^2$. Clearly the $|D(e^{j\omega})|^2$ terms will cancel, and the conclusion will ensue that the variability of $\mathrm{Var}\{G(e^{j\omega},\widehat{\theta})\}$ is invariant to the choice of fixed pole location.

This can be tested on a simple example wherein the true system is

$$G(q) = \frac{0.1548q + 0.0939}{(q-0.6065)(q-0.3679)}$$

and a $p = 10$'th order model is fitted using RLS when the input $\{u_k\}$ has spectral density $\Phi_u(\omega) = 10(1.25-\cos\omega)^{-1}$, the output measurements $\{y_k\}$ are corrupted by white noise of variance $\sigma_\nu^2 = 0.01$, and the algorithm is run for $N = 2000$ data samples. In this case, the true variability can be estimated by the sample average over 200 Monte-Carlo simulations with different input and noise realizations. This can then be compared to the approximation (6.1). For the case of all the $\{\xi_n\}$ being at the origin (so that a true FIR structure is employed), then the results of such a comparison are shown in the top plot of figure 6.1 - the approximation (6.1) being the dash-dotted line, and the Monte-Carlo estimate of true variability being the solid line. The agreement is excellent.

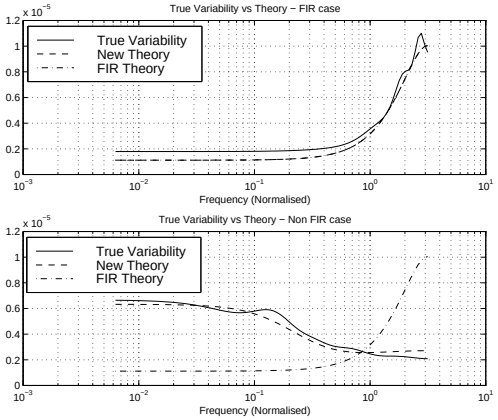

Figure 6.1: *Comparison of (solid line) true variability to (dash-dotted line) FIR based theoretical approximation (6.1). Top plot is case of all poles at origin (FIR case), bottom plot is case of all poles away from origin. The dashed line is the improved new approximation presented in this chapter. In the top plot, the dash-dotted line showing the pre-existing approximation (6.1) obscures the new approximation because the new and old approximations are identical for the FIR case.*

However, if all the poles are chosen away from the origin, five at $\xi_k = 0.2$ and five at $\xi_k = 0.8$ then when examining the theoretical prediction (6.1) and the true variability as shown in the bottom plot of figure 6.1, the agreement between the two has disappeared.

Clearly, the previous heuristic reasoning that tried to adapt the FIR result (6.1) to a situation for which it is prima-facie applicable is flawed, and this indicates the need for custom analysis of fixed denominator model structure estimation.

Such is the purpose of this chapter, and pertaining to this note in the bottom diagram of figure 6.1 the good agreement between the true variability and the dashed line, which is a plot of the new 'extended' approximation presented in this chapter. This new approximation is the old one (6.1) with the model order term p replaced with a frequency dependent function $\gamma_p(\omega)$. However, in order to develop the ideas that lead to this and other enhanced approximations, a more formal problem definition is required.

6.3 Problem Formulation

We consider the (recursive) identification of linear slowly time-varying systems from observed input-output data. We assume that the underlying model relat-

ing the observed input $\{u_k\}$ and output $\{y_k\}$ sequences is given by

$$y_k = G_k(q)u_k + \nu_k \tag{6.3}$$

where $\{\nu_k\}$ is a zero mean white noise process with variance $\mathbf{E}\{\nu_k^2\} = \sigma_\nu^2 < \infty$ and where $G_k(q)$ represents the transfer function of the time-varying linear system at time k

$$G_k(q) = \sum_{n=0}^{\infty} g_k(n)q^{-n}$$

with square summable impulse response sequence $\{g_k(n)\} \in \ell_2$. In order for this representation to make sense we have of course to assume that the time variation is slow so that $G_k(q)$ can be given the usual interpretation of transfer function describing the input–output properties of the system.

It is assumed that the input signal $\{u_k\}$ is a bounded, deterministic, and quasi-stationary sequence in the sense of [Lju87], as summarized in Chapter 2 (page 6).

We also assume that the input sequence $\{u_k\}$ is weakly uncorrelated with the noise $\{\nu_k\}$ in the sense that $|\mathbf{E}\{u_k\nu_{k-\tau}\}| \to 0$ as $\tau \to \infty$, and that the input spectral density has a finite dimensional spectral factorization and it is positive definite $\Phi_u(\omega) > 0$ for almost all ω, what guarantees that the input signal is persistently exciting of any order [Reg95] (c.f. Proposition 2.1.1).

Our interest is on the (recursive) estimation of the (assumed unknown) time varying dynamics $G_k(q)$ by means of the observations of inputs $\{u_k\}$ and outputs $\{y_k\}$. Of the many approaches available in the literature to solve this estimation problem [SK95, GS84, Lju87, CG91], a common choice that facilitates the analysis of the adaptive algorithms, is to express the model (6.3) in linear regression form

$$y_k = \phi_k^T\theta_k + \nu_k \tag{6.4}$$

where the 'regression vector' ϕ_k depends in general on past outputs and inputs, and $\theta_k \in \mathbb{R}^p$ is a vector of p parameters in a model structure $G(q,\theta_k)$ that attempts to describe the true dynamics $G_k(q)$. An estimate of $G_k(q)$ can then be computed as $G(q,\widehat{\theta}_k)$ where the parameter estimate $\widehat{\theta}_k$ is obtained recursively. In this context, several well-known adaptive algorithms such as Least Mean Squares (LMS) (see [WS85] and the references therein), Recursive Least Squares (RLS) [GS84, You84], and the Kalman Filter (KF) [Kal60, KB61, Guo90] can be used for the estimation.

We restrict the analysis in this chapter to the three above mentioned adaptation algorithms, namely the LMS, RLS, and KF algorithms, which can be represented in a unified way by the general adaptation law [GL89]

$$\widehat{\theta}_{k+1} = \widehat{\theta}_k + L_k(y_k - \phi_k^T\widehat{\theta}_k), \tag{6.5}$$

where L_k is a gain vector that is computed in different ways, according to the particular algorithm. For instance, when the gain vector is given by

$$L_k = \mu\phi_k, \quad \mu \in (0,1), \tag{6.6}$$

then the update law (6.5) is the 'gradient-type' algorithm known as *Least Mean Squares* [WS85]. The case

$$L_k = P_k\phi_k \tag{6.7}$$

where P_k satisfies

$$P_k = \frac{1}{\lambda}\left\{P_{k-1} - \frac{P_{k-1}\phi_k\phi_k^T P_{k-1}}{\lambda + \phi_k^T P_{k-1}\phi_k}\right\}; \quad \lambda \in (0,1) \tag{6.8}$$

initialized with some positive definite matrix P_0, corresponds to the *Recursive Least Squares* algorithm, where the constant λ is known as the 'forgetting factor'. Finally, the case

$$L_k = \frac{\mu P_{k-1}\phi_k}{\sigma^2 + \mu\phi_k^T P_{k-1}\phi_k} \tag{6.9}$$

where P_k satisfies the Riccati equation

$$P_k = P_{k-1} - \mu\frac{P_{k-1}\phi_k\phi_k^T P_{k-1}}{\sigma^2 + \mu\phi_k^T P_{k-1}\phi_k} + \mu\Sigma \tag{6.10}$$

with $\Sigma > 0$ and symmetric matrix, corresponds to the *Kalman Filter* algorithm. The KF gives an estimate of the parameter vector θ_k, when its time variation is modeled via a random walk as

$$\theta_{k+1} = \theta_k + \rho\, w_k \tag{6.11}$$

where w_k is a stationary zero mean vector white noise process with covariance matrix $\mathbf{E}\left\{w_k w_k^T\right\} = Q$. The estimate is optimal in a mean-square sense under Gaussian assumptions if $\mu = \rho, \sigma^2 = \sigma_\nu^2$ and $\Sigma = Q$. In the sequel it is assumed that $\{w_k\}$ is weakly uncorrelated with $\{u_k\}$ in the sense that $|\mathbf{E}\{u_k w_{k-\tau}\}| \to 0$ as $\tau \to \infty$.

The design variables of the three algorithms are the gain μ for the LMS algorithm, the forgetting factor λ for the RLS algorithm, and the matrices σ^2 and Σ, and the gain μ for the Kalman Filter.

A central question when using these algorithms is the accuracy of the estimate. The most common way to quantify this accuracy is to compute the so-called Mean Square Error (MSE) of the transfer function estimate

$$\text{Mean Square Error} \triangleq \mathbf{E}\left\{|G_k(q) - G(q,\widehat{\theta}_k)|^2\right\}, \tag{6.12}$$

as a measure of how well the estimate $G(q,\widehat{\theta}_k)$ approximates the 'true' transfer function $G_k(q)$. To do that, we need first to examine the accuracy of the parameter estimate $\widehat{\theta}_k$ itself [SK95, GS84]. This may be achieved by defining the parameter estimation error $\widetilde{\theta}_k$ as

$$\widetilde{\theta}_k \triangleq \theta_k - \widehat{\theta}_k. \tag{6.13}$$

where θ_k represents the true parameter vector that allows the model structure to exactly describe the underlying time varying dynamics as $G(q,\theta_k) = G_k(q)$. Of course this can only be done if the system belongs to the model set. In the general case, the model structure $G(q,\theta_k)$ will be too simple to exactly describe the true dynamics $G_k(q)$ for any value of θ_k, and so there will be no 'true' parameter θ_k. However, as pointed out in [GL89, LY85], we still can define θ_k as the 'best' approximation (for example in L_2 sense) of a given order, and use this value to calculate $\widetilde{\theta}_k$. In this case, the true transfer function $G_k(q)$ can be expressed as

$$G_k(q) = G(q,\theta_k) + \Delta_p(q)$$

where $\Delta_p(q)$ represents the undermodelling error. As the model order increases to infinity this error tends to zero ($\Delta_p(q) \to 0$ as $p \to \infty$). It is reasonable then (following [GL89, LY85]) to perform an analysis asymptotic in model order. Since this analysis will involve deriving expressions that are asymptotic in p, there is no point in continually include the term $\Delta_p(q)$ (that tends to zero as p goes to infinity) throughout the derivations, and so we will ignore this term from the beginning. The validity of this strategy is confirmed by simulation example in Section 6.7. For the case in which $G_k(q)$ is represented using the orthonormal bases with fixed poles of Section 3.5, the rate at which $\Delta_p(e^{\mathrm{j}\omega})$ tends to zero with increasing model order p is established via Theorem 4.3.1.

Substituting (6.13) into the general update equation (6.5), and considering that the time-variation of the parameters is given by the random walk (6.11), gives that the parameter error satisfies the following difference equation [GL95b]

$$\widetilde{\theta}_{k+1} = \left(I - L_k\phi_k^T\right)\widetilde{\theta}_k + \rho\,\omega_k - L_k\nu_k. \tag{6.14}$$

Equation (6.14) could now be used to calculate the covariance matrix of the parameter error $\mathbf{E}\left\{\widetilde{\theta}_k\widetilde{\theta}_k^T\right\}$ as a preliminary step for the computation of the MSE of the transfer function estimate, which ultimately is the magnitude we have chosen to measure the quality of the estimation. Unfortunately, as pointed out in [GL89, GL95b, Gun88, LG90], the exact expression for this covariance will be very complicated except in very special circumstances. To obtain simple expressions approximating this covariance matrix we resort to the main result of [GL95b] which establishes that, under certain assumptions on the regressors

$\{\phi_k\}$ (the so-called ϕ-mixing condition), and on the noise ν_k and parameter drift ω_k, the covariance $\mathbf{E}\left\{\widetilde{\theta}_k\widetilde{\theta}_k^T\right\}$ may be approximated by the matrix Π_k given by the linear deterministic difference equation

$$\Pi_{k+1} = (I - \mu S_k R)\Pi_k(I - \mu S_k R)^T + \mu^2\sigma_\nu^2 S_k R S_k + \rho^2 Q \tag{6.15}$$

where $R \triangleq \mathbf{E}\left\{\phi_k\phi_k^T\right\}$ and S_k is defined as

LMS:

$$S_k = I, \tag{6.16}$$

RLS:

$$S_k = (1+\mu)S_{k-1} - \mu S_{k-1}RS_{k-1}; \quad S_0 = P_0, \tag{6.17}$$

Kalman Filter:

$$S_k = S_{k-1} - \mu S_{k-1}RS_{k-1} + \frac{\mu}{\sigma^2}\Sigma; \quad S_0 = \frac{1}{\sigma^2}P_0, \tag{6.18}$$

The degree of the approximation is quantified in [GL95b] by a result of the form[1]

$$\left\|\mathbf{E}\left\{\widetilde{\theta}_k\widetilde{\theta}_k^T\right\} - \Pi_k\right\| \leq \kappa(\mu)$$

where $\kappa(\mu)$ is a bounded function that tends to zero as μ tends to zero, where μ is a measure of the step size of the algorithm.

However, as mentioned earlier in this section, our interest is not in the quantification of the accuracy in the parameter space, but in quantifying how close the estimated model $G(q,\widehat{\theta}_k)$ is to the true system $G_k(q)$ in terms of the error

$$\widetilde{G}_k(e^{\mathrm{j}\omega}) \triangleq G(e^{\mathrm{j}\omega},\widehat{\theta}_k) - G_k(e^{\mathrm{j}\omega})$$

in the estimated frequency response, and ultimately in terms of the MSE of the transfer function estimate (6.12).

In order to be able to compute this frequency response estimation error, we need to relate it with the error in the parameter space. Since we are assuming a linear regression model for the system, we are restricted to considering model structures $G(q,\theta_k)$ for which the estimated frequency response depends linearly on the estimated parameters, i.e. of the form

$$G(e^{\mathrm{j}\omega},\widehat{\theta}_k) = \Gamma_p^T(e^{\mathrm{j}\omega})\widehat{\theta}_k, \tag{6.19}$$

[1] Here $\|\cdot\|$ stands for the 2-norm of matrices or operator norm

where

$$\Gamma_p(q) \triangleq [\mathcal{B}_0(q), \mathcal{B}_1(q), \cdots, \mathcal{B}_{p-1}(q)]^T \tag{6.20}$$

is a vector of p rational-stable-causal transfer functions $\mathcal{B}_n(q)$. For example, $\mathcal{B}_n(q) = q^{-n}$ corresponds to an FIR model structure.

Using equations (6.15), (6.19) and (6.20), an approximate expression for the Mean Square Error (MSE) of the transfer function estimate can then be computed as

$$\mathbf{E}\left\{|\widetilde{G}_k(e^{\mathrm{j}\omega})|^2\right\} = \Gamma_p^\star(e^{\mathrm{j}\omega})\mathbf{E}\left\{\widetilde{\theta}_k\widetilde{\theta}_k^T\right\}\Gamma_p(e^{\mathrm{j}\omega}) \approx \Gamma_p^\star(e^{\mathrm{j}\omega})\Pi_k\Gamma_p(e^{\mathrm{j}\omega}) \tag{6.21}$$

This MSE of the transfer function estimate can then be used as a frequency domain quantification of the performance of the adaptive algorithms. Unfortunately again, this expression will in general be too complicated to give any practical insight. Following the ideas in [GL89, LY85, Gun88], we will derive simple approximations for (6.21) that are increasingly accurate for increasing model order p. Through simulation experiments we will show that the simplified expressions are good approximations even for relatively small model order.

To be able to proceed with this asymptotic analysis we need to be more explicit about the expression for $\Gamma_p(e^{\mathrm{j}\omega})$ in (6.21). That is, to be more specific on the exact formulation of the transfer functions $\{\mathcal{B}_n(q)\}$ determining the model structure $G_k(q,\theta)$.

6.4 Model Structures

In recent years, in an adaptive filtering context, Williamson and co-workers have proposed and studied, in a series of papers [Wil95, Wil93b, Wil93a, WZ96], a new class of Infinite Impulse Response (IIR) adaptive filters that have been termed 'Fixed Pole Adaptive Filters' (FPAF), or more generically 'Vector Space Adaptive Filters' (VSAF). For the case of real poles, these adaptive filters can be formulated as

$$G(q,\theta_k') = \sum_{n=0}^{p-1}\theta_k'(n)\prod_{i=0}^{n}\frac{1}{(q-\xi_i)} = \sum_{n=0}^{p-1}\theta_k'(n)\mathcal{A}_n(q), \tag{6.22}$$

where the 'fixed poles' $\{\xi_n\}_{n=0}^{p-1}$ are chosen to reflect the available prior information about the likely pole positions of the true time varying system $G_k(q)$, and where $\theta_k'(n)$ are adjustable filter parameters. In passing to the last equality we have defined

$$\mathcal{A}_n(q) \triangleq \prod_{i=0}^{n}\frac{1}{(q-\xi_i)}.$$

To contemplate the case of complex conjugate poles, the formulation is modified to

$$G(q,\theta_k') = \sum_{n=0}^{p-1} (\theta_k'(n)q + \theta_k''(n)) \prod_{i=0}^{n} \frac{1}{(q-\xi_i)(q-\overline{\xi_i})}. \tag{6.23}$$

A special case of the structure (6.22) arises when all the poles $\{\xi_n\}$ are chosen at the origin in which case (6.22) is an FIR model structure.

Although IIR filters have several advantages over FIR filters when used to model very long impulse responses, FIR models have been preferred almost exclusively for the adaptive applications [WS85], mainly due to the fact that the global convergence of the adaptive algorithms can be ensured, in contrast to IIR filters for which theoretical guarantee of global convergence is difficult to provide due to the possibility of existence of multimodal error surfaces[2] [WS85, Reg95].

Williamson and co-workers [WZ96] show that the FPAF structure (6.22)-(6.23), though being an IIR model structure, preserves the global convergence characteristics of FIR filters. They also provide some simulation results showing that significant improvements in the estimation accuracy can be achieved if the prior information about the dominant modes of the system is used to chose the fixed poles in the model structure close to these dominant poles. Later in this chapter, we will give some theoretical support to this observations by resorting to the results on the undermodelling error for the OBFP we presented in Theorem 4.3.1.

In spite of the above mentioned advantages of the model structure (6.22), its generality (as compared to an FIR structure) makes the frequency domain analysis of the adaptive algorithms much more difficult. To overcome this difficulty and make the analysis more tractable, we replace the model structure (6.22) with an orthonormal re-parameterization of the form

$$G(q,\theta_k) = \sum_{n=0}^{p-1} \theta_k(n)\mathcal{B}_n(q) \tag{6.24}$$

where $\{\mathcal{B}_n(q)\}$ are the Orthonormal Bases with Fixed Poles introduced in Section 3.5, whose formulation we rewrite here for convenience,

$$\mathcal{B}_n(q) = \left(\frac{\sqrt{1-|\xi_n|^2}}{q-\xi_n}\right) \prod_{k=0}^{n-1} \left(\frac{1-\overline{\xi_k}q}{q-\xi_k}\right). \tag{6.25}$$

As shown for the time-invariant case in Chapter 4, the model structure

[2]This basically means that the surface describing the mean square tracking error in the (adaptive) parameter space may have multiple local minima.

(6.24) can be written in the linear regression form (6.4) by defining

$$\phi_k \triangleq \Gamma_p^T(q)u_k, \tag{6.26}$$

$$\Gamma_p(q) \triangleq [\mathcal{B}_0(q), \mathcal{B}_1(q), \cdots, \mathcal{B}_{p-1}(q)]^T, \tag{6.27}$$

$$\theta_k \triangleq [\theta_k(0), \theta_k(1), \cdots, \theta_k(p-1)]^T. \tag{6.28}$$

Before going further with the analysis, a natural question we should answer is the following: How do the estimates with the orthonormal re-parameterization (6.24) relate to the estimates with the original model structure (6.22), when using the different adaptive algorithms?

To answer this question, let us notice first that since the poles of the model structures (6.24) and (6.22) are identical, we can always find a nonsingular (constant) matrix $J \in \mathbb{R}^{p\times p}$ such that the matrices $\Gamma_p(q)$ and $\Gamma'_p(q)$ associated with each parameterization are related according to

$$\Gamma_p(q) = J^T\Gamma'_p(q), \tag{6.29}$$

with $\Gamma_p(q)$ defined in (6.27), and where $\Gamma'_p(q)$ is defined analogously as

$$\Gamma'_p(q) \triangleq [\mathcal{A}_0(q), \mathcal{A}_1(q), \cdots, \mathcal{A}_{p-1}(q)]^T$$

It is then clear that the regressor vectors ϕ_k and ϕ'_k associated respectively to the model structures (6.24) and (6.22), are related according to

$$\phi_k = J^T\phi'_k \tag{6.30}$$

Based on this observations, we are now able to prove the following result showing that the estimations using RLS and KF algorithms are invariant under the linear re-parameterization (6.29), while the estimation using LMS algorithm is not.

Lemma 6.4.1. *The RLS and KF algorithms are invariant under the linear re-paramaterization (6.29), which means that the estimates corresponding to the model structures (6.24) and (6.22) are related according to*

$$\widehat{\theta}'_k = J\widehat{\theta}_k \tag{6.31}$$

provided that the initialization is consistent with the linear re-parameterization, i.e.

$$P'_0 = JP_0J^T$$

for the RLS algorithm, and

$$\Sigma' = J\Sigma J^T$$

for the KF. On the other hand, the LMS algorithm is not invariant under linear re-parameterization and consequently equation (6.31) doesn't hold for this case.

Proof: See Appendix 6.A. ■

It is clear now that, since the RLS and KF algorithms are frequency domain invariant to linear model re–parameterization, the analysis using the orthonormal structure will provide results that are valid for any fixed denominator model structure such as (6.22). Unfortunately, the LMS algorithm is not invariant under linear re–parameterisations [EJLW92, Wil95], and so any subsequent results for this algorithm will only pertain to the orthonormal model structure (6.24). This is not considered a significant limitation, since for the LMS algorithm the orthonormal structure (6.24) is attractive from the viewpoint of numerical robustness and enhanced convergence rate under white noise excitation.

6.5 Transient analysis

In this section, we study the transient behavior of the frequency response estimation error for the Least Mean Squares algorithm.

The dynamic behaviour of the frequency response estimation error can be studied by using the approximation (6.21) substituted into (6.15) to obtain

$$\begin{aligned}\mathbf{E}\left\{|\widetilde{G}_{k+1}(e^{\mathrm{j}\omega})|^2\right\} \;\approx\; & \Gamma_p^\star(e^{\mathrm{j}\omega})(I-\mu S_k R)\Pi_k(I-\mu S_k R)^T\Gamma_p(e^{\mathrm{j}\omega}) \\ & + \mu^2\sigma_\nu^2\Gamma_p^\star(e^{\mathrm{j}\omega})S_k R S_k\Gamma_p(e^{\mathrm{j}\omega}) \\ & + \rho^2\Gamma_p^\star(e^{\mathrm{j}\omega})\,Q\,\Gamma_p(e^{\mathrm{j}\omega}). \end{aligned} \tag{6.32}$$

As the reader can see, this is a very complicated expression from which it is difficult to extract useful design insights. However, if the model order p is assumed to be large, then equation (6.32) can be simplified considerably.

Here again, an important rôle in the analysis will be played by the reproducing kernel $\gamma_p(\omega) \triangleq K_p(\omega,\omega)$ associated to the orthonormal bases with fixed poles $\{\mathcal{B}_n(q)\}$ in (6.25). We recall that $\gamma_p(\omega)$ can be computed as

$$\gamma_p(\omega) = \sum_{n=0}^{p-1} |\mathcal{B}_n(e^{\mathrm{j}\omega})|^2. \tag{6.33}$$

For the case of FIR model structures, it is well known that the error in the estimated models induced by the measurement noise is inversely proportional to the model order p. This has been rigorously proved by Ljung and Yuan in [LY85] for off-line identification in the time-invariant case, and by Gunnarsson and Ljung in [GL89] for the case of recursive identification. The frequency dependent quantity $\gamma_p(\omega)$ will serve to capture how this phenomenon generalizes to the fixed–denominator model structures (6.22),(6.24). As pointed out in Chapter 4, for the case $\xi_n = 0$ that corresponds to (6.24) being an FIR model structure, the factor $\gamma_p(\omega)$ equals p.

It is also necessary to define a quantity which reflects the time varying nature of the system $G_k(q)$ in the frequency domain, and is also commensurate with the parameter space model for this time variation (6.11). Following [GL89], this may be achieved by using the linear relationship (6.19) in combination with (6.11) to obtain the model for the time variation as

$$G_{k+1}(e^{\mathrm{j}\omega}) = G_k(e^{\mathrm{j}\omega}) + \rho\,\Gamma_p^T(e^{\mathrm{j}\omega})w_k$$

so that

$$\mathbf{E}\left\{|G_{k+1}(e^{\mathrm{j}\omega}) - G_k(e^{\mathrm{j}\omega})|^2\right\} = \rho^2\Gamma_p^\star(e^{\mathrm{j}\omega})Q\Gamma_p(e^{\mathrm{j}\omega}) = \rho^2\delta_p(\omega). \tag{6.34}$$

where we have defined

$$\delta_p(\omega) \triangleq \Gamma_p^\star(e^{\mathrm{j}\omega})Q\Gamma_p(e^{\mathrm{j}\omega}). \tag{6.35}$$

In the sequel, simplifications for equation (6.32) will be obtained by considering increasing model order p. To facilitate this analysis it is necessary (as in [GL89]) to assume that as p grows, the covariance matrix Q is extended accordingly in such a way that it is always positive definite and of bounded norm. In this case, the limit

$$\delta(\omega) \triangleq \lim_{p\to\infty}\frac{\delta_p(\omega)}{\gamma_p(\omega)} \tag{6.36}$$

exists and is non-zero. Given these definitions, the following theorem provides a simple frequency domain characterisation of the tracking characteristics of the LMS adaptation scheme when using the general fixed denominator model structure (6.24).

Theorem 6.5.1. *For the LMS algorithm and the model structure (6.24), then using the approximation (6.15) for the covariance matrix of the parameter estimation error, the following limit result can be established*

$$\lim_{p\to\infty}\left|\frac{1}{\gamma_p(\omega)}\mathbf{E}\left\{|\widetilde{G}_{k+1}(e^{\mathrm{j}\omega})|^2\right\} - \left(\frac{[1-\mu\Phi_u(\omega)]^2}{\gamma_p(\omega)}\mathbf{E}\left\{|\widetilde{G}_k(e^{\mathrm{j}\omega})|^2\right\} + \mu^2\sigma_\nu^2\Phi_u(\omega) + \rho^2\delta(\omega)\right)\right| = 0$$

Proof: See Appendix 6.A. ■

The interpretation of this result is that for large model order p, the equation governing the behaviour of the MSE of the tranfer function estimate for the

LMS algorithm and fixed denominator model structures can be approximated by the following first order difference equation

$$\begin{aligned}\mathbf{E}\left\{|\widetilde{G}_{k+1}(e^{\mathrm{j}\omega})|^2\right\} &\approx \\ &\approx \quad [1-\mu\Phi_u(\omega)]^2\mathbf{E}\left\{|\widetilde{G}_k(e^{\mathrm{j}\omega})|^2\right\} + \mu^2\sigma_\nu^2\Phi_u(\omega)\gamma_p(\omega) + \rho^2\delta(\omega)\gamma_p(\omega).\end{aligned} \tag{6.37}$$

The transient response is then governed by the homogeneous part of this difference equation. That is

$$\mathbf{E}\left\{|\widetilde{G}_{k+1}(e^{\mathrm{j}\omega})|^2\right\} \approx [1-\mu\Phi_u(\omega)]^2\mathbf{E}\left\{|\widetilde{G}_k(e^{\mathrm{j}\omega})|^2\right\}, \tag{6.38}$$

which for small μ can be approximated by

$$\mathbf{E}\left\{|\widetilde{G}_{k+1}(e^{\mathrm{j}\omega})|^2\right\} \approx [1-2\mu\Phi_u(\omega)]\mathbf{E}\left\{|\widetilde{G}_k(e^{\mathrm{j}\omega})|^2\right\}, \tag{6.39}$$

and it is then characterized by the (frequency dependent) time constant for the decay of the MSE

$$\tau(\omega) \triangleq [1-2\mu\Phi_u(\omega)].$$

From equation (6.39) it is clear that the condition on the stepsize μ for stability of the algorithm for all ω is

$$\mu < \frac{1}{\sup_\omega \Phi_u(\omega)}. \tag{6.40}$$

These results are similar to the ones presented in [GL89, Gun88] for the FIR case. A substantial difference between the MSE expression (6.37) and those in [GL89, Gun88, EJLW92], is the presence of the factor $\gamma_p(\omega)$ in the driving terms, making explicit how the tracking error is affected by the choice of poles in the model structure (6.24).

Similarly to the results in [GL89, Gun88], equation (6.37) shows the design trade-off (for the LMS algorithm) between the tracking ability and the noise rejection properties of the algorithm. To be more specific, from (6.37) it can be seen that the tracking error decays like $[1-\mu\Phi_u(\omega)]^{2k} \approx [1-2\mu\Phi_u(\omega)]^k$, so that tracking would be better at frequencies where $\mu\Phi_u(\omega)$ is large, but at these frequencies the 'noise driving' term $\sigma_\nu^2\mu^2\Phi_u(\omega)\gamma_p(\omega)$ would also be large. However, unlike the expressions in [GL89, Gun88], the presence of the modulation factor $\gamma_p(\omega)$ in the driving terms in (6.37) suggests that the tracking performance could be improved, but not necessarily at the expense of a deterioration in the noise rejection properties of the algorithm, by an appropriate choice of the poles in the model structure.

In order to be able to apply these results –which are asymptotic in model order– in practical problems we need to evaluate the accuracy of the approximation (6.37) for the low model orders likely to be used in these cases. This is examined in Section 6.7 where for small enough step size μ, (6.37) is shown to faithfully predict the frequency domain convergence properties of the LMS algorithm – see figure 6.7.

6.6 Steady State Analysis

In this section we are interested in the quantification of the steady state behavior of the frequency response estimation error. That is, we are interested in the error $\mathbf{E}\left\{|\widetilde{G}_k(e^{\mathrm{j}\omega})|^2\right\}$ for large values of k. In order to compute this steady-state error we need first to analyze the limiting behavior of the solution Π_k of (6.15) for large k; that is

$$\lim_{k\to\infty} \Pi_k \triangleq \Pi.$$

Of course, for this limit to exist (and indeed for the approximation (6.32) to hold) it is necessary that the adaptive algorithm (6.5) be stable. As mentioned in the introduction to this chapter, sufficient conditions for the exponential stability of the general adaptation algorithm (6.5), for any of the choices (6.6),(6.7)-(6.8), or (6.9)-(6.10) have been established by Guo and Ljung in [GL95a]. These conditions are very technical but can be summarized as follows:

- Persistency of excitation: the sum of the covariance matrices of the regressors $\mathbf{E}\left\{\phi_k\phi_h^T\right\}$ over a finite time span of arbitrary length has full rank. In general this condition is satisfied if the input spectral density $\Phi_u(\omega)$ is positive definite for almost all frequencies ω. That is, if the input sequence is persistently exciting of any order.

- Weak dependance of the regressors: the dependance between the regressors ϕ_k and $\{\phi_i, \nu_{i-1}, w_i\}$ decays to zero as the time distance $(k-i)$ tends to infinity. This is the so-called 'ϕ-mixing condition' [GL95a, GL95b].

- The measurement noise $\{\nu_k\}$ and the parameter drift $\{w_k\}$ are zero mean white noise sequences.

Once the stability of the algorithm has been ensured, the steady-state parameter error Π may be evaluated by determining the steady-state solutions

$$S \triangleq \lim_{k\to\infty} S_k$$

of (6.16)–(6.18) and then substituting them into (6.32) before examining its own steady state solution with terms of order $\mu^2\Pi$ discarded [GL95b]. The results of this strategy for the three adaptive algorithms are as follows.

LMS: Here $S = I$ so that Π is the solution of the Lyapunov equation

$$\Pi R + R\Pi = \mu\sigma_\nu^2 R + \frac{\rho^2}{\mu}Q. \tag{6.41}$$

RLS: Here $S = R^{-1}$ so that Π is given by

$$\Pi = \frac{\mu\sigma_\nu^2}{2}R^{-1} + \frac{\rho^2}{2\mu}Q. \tag{6.42}$$

Kalman Filter: This case is more difficult. S is the solution of

$$\sigma^2 SRS = \Sigma,$$

which may be expressed as

$$S = \frac{1}{\sqrt{\sigma^2}}R^{-1/2}\left(R^{1/2}\Sigma R^{1/2}\right)^{1/2}R^{-1/2}, \tag{6.43}$$

so that Π is given by the solution of

$$SR\Pi + \Pi RS = \frac{\mu\sigma_\nu^2}{\sigma^2}\Sigma + \frac{\rho^2}{\mu}Q. \tag{6.44}$$

For the special case of $\Sigma = Q$ this system has solution

$$\Pi = \frac{\sigma^2}{2}\left(\mu\frac{\sigma_\nu^2}{\sigma^2} + \frac{\rho^2}{\mu}\right)S. \tag{6.45}$$

with S given in (6.43).

Based on these expressions for the steady-state parameter error Π together with equation (6.21) it is now possible to quantify the steady state estimation error in the frequency domain as

$$\mathbf{E}\left\{|\widetilde{G}(e^{\mathrm{j}\omega})|^2\right\} \triangleq \lim_{k\to\infty}\mathbf{E}\left\{|\widetilde{G}_k(e^{\mathrm{j}\omega})|^2\right\} \approx \Gamma_p^\star(e^{\mathrm{j}\omega})\Pi\Gamma_p(e^{\mathrm{j}\omega}). \tag{6.46}$$

However, the resulting expression is so complicated that it is difficult to extract useful design insight from it. As in the previous section, we will perform an asymptotic analysis with increasing model order p in order to simplify (6.46). The resulting expressions will be more tractable, and will provide some insight on how the steady-state estimation error is affected by factors such as the step size, the measurement noise energy, the input spectral density, and perhaps most interestingly, the choice of fixed pole position. The effect of this choice of poles will be quantified by the term $\gamma_p(\omega)$.

The simplified error equations for the LMS, RLS and KF algorithms are presented in Theorems 6.6.1, 6.6.2 and 6.6.3 respectively.

- **LMS Algorithm:** In this case, the simplified error quantification is as follows.

  Theorem 6.6.1. *For the LMS algorithm and the model structure (6.24), the following limit result holds,*

$$\lim_{p\to\infty} \frac{1}{\gamma_p(\omega)} \mathbf{E}\left\{|\widetilde{G}(e^{\mathrm{j}\omega})|^2\right\} = \frac{1}{2}\left[\mu\sigma_\nu^2 + \frac{\rho^2\delta(\omega)}{\mu\Phi_u(\omega)}\right]$$

  Proof: See Appendix 6.A. ■

  The interpretation of this theorem is that for large model order p, and after the algorithm has converged (large k), the steady-state MSE can be approximated by

$$\mathbf{E}\left\{|\widetilde{G}_k(e^{\mathrm{j}\omega})|^2\right\} \approx \frac{\gamma_p(\omega)}{2}\left[\mu\sigma_\nu^2 + \frac{\rho^2\delta(\omega)}{\mu\Phi_u(\omega)}\right]. \tag{6.47}$$

  For the case of all the poles $\{\xi_n\}$ in the model structure (6.24) chosen at the origin, then $\gamma_p(\omega) = p$ and the above expression specializes to that derived in [GL89, Gun88] (although in [GL89], only the case $\rho = 1$ is considered). However, an important difference with the results for the FIR case in [GL89, Gun88], is the presence of the frequency dependent factor $\gamma_p(\omega)$ in the expression of the MSE (6.47) showing how pole choices other than FIR influence the frequency domain estimation error. As well, (6.47) illustrates that for time-invariant systems (corresponding to $\rho = 0$), then the error is proportional to the step size μ and the measurement noise variance σ_ν^2, while for time varying systems ($\rho \neq 0$), another error component arises due to the parameter drift, which is inversely proportional to step size, and is also inversely proportional to input spectral density $\Phi_u(\omega)$. In this latter case, we have a compromise in the choice of the step size μ, since a small value of μ will reduce the contribution of the measurement noise to the MSE, but simultaneously will increase the contribution of the parameter drift to that error.

  A fundamental question that now arises is one concerning the reliability of using the approximation (6.47) –that has been derived considering large model order– for the relatively small model orders used in practical applications. The most suitable way to deal with this issue would be to quantify the convergence rate in Theorem 6.6.1. This appears to be extremely difficult. Instead, the approach used in [GL89] is taken wherein the validity of (6.47) for finite p is examined via a simulation study. This is done in Section 6.7 , where it is shown (see Figures 6.3 and 6.5) that for a tenth order model, (6.47) is quite an accurate approximation; in fact, as shown in Figure 6.4, this holds even for as low as a fourth order model.

- **RLS Algorithm:** The same sort of analysis can also be performed in this case, with the results as follows.

  Theorem 6.6.2. *For the RLS algorithm and the model structure (6.22) or (6.24), the following limit result holds*

$$\lim_{p\to\infty} \frac{1}{\gamma_p(\omega)} \mathbf{E}\left\{ |\widetilde{G}(e^{j\omega})|^2 \right\} = \frac{1}{2}\left[\frac{\mu\sigma_\nu^2}{\Phi_u(\omega)} + \frac{\rho^2}{\mu}\delta(\omega) \right] \quad ; \mu = 1-\lambda.$$

  Proof: See Appendix 6.A. ■

  A similar interpretation to the previous theorem can be given also in this case. Namely, this theorem means that for large model order p, and after the algorithm has converged (large k), the steady state MSE can be approximated by

$$\mathbf{E}\left\{ |\widetilde{G}_k(e^{j\omega})|^2 \right\} \approx \frac{\gamma_p(\omega)}{2}\left[\frac{\mu\sigma_\nu^2}{\Phi_u(\omega)} + \frac{\rho^2}{\mu}\delta(\omega) \right]. \tag{6.48}$$

  Comparing this approximation to (6.47) illustrates a fundamental difference between the steady state behavior of LMS and RLS in terms of how the input spectral density affects the noise and tracking performance. Specifically, (6.48) shows that for stationary systems ($\rho = 0$), the RLS estimation error is inversely proportional to the input spectral density $\Phi_u(\omega)$, while (6.47) shows that the LMS estimation error is invariant to the size of this spectral density; see Figure 6.5 for a simulation study validation of this phenomenon. Conversely when $\rho \neq 0$, (6.47) shows that for LMS the tracking ability increases with increasing input spectral density, while for RLS the tracking ability is invariant to this factor, and only depends on step size.

  These observations, in the context of FIR model structures have already been made in [GL89, Gun88], and as per the LMS case, when all the poles are chosen at the origin and hence $\gamma_p(\omega) = p$, then (6.48) is identical to the expressions for RLS steady state error presented in [GL89, Gun88]. However, again as per the LMS case, the inclusion of the frequency dependent factor $\gamma_p(\omega)$ in (6.48) shows how the choice of fixed denominator pole position in the model structure $G(q,\theta)$ affects the estimation error in the frequency domain (Figure 4.2).

  The question of the validity of using an asymptotic result as a finite data and model order approximation in (6.48) again arises, and again this is dealt with in Section 6.7 via a simulation study. For example in Figure 6.4, (6.48) is shown to be quite accurate even for only a fourth order model and 300 data points.

- **Kalman Filter Algorithm:** The same strategy of considering large model order can be used in this case. However, as already mentioned, there are particular difficulties in solving for the steady state parameter covariance Π, and this leads to the treatment of only a specialised case in which $\Sigma = Q$.

  Theorem 6.6.3. *For the Kalman Filter algorithm, the model structure (6.22) or (6.24) and under the assumption that $\Sigma = Q$, then*

$$\lim_{p\to\infty} \frac{1}{\gamma_p(\omega)} \mathsf{E}\left\{|\widetilde{G}(e^{\mathrm{j}\omega})|^2\right\} = \frac{1}{2}\left(\mu\frac{\sigma_\nu^2}{\sigma^2} + \frac{\rho^2}{\mu}\right)\sqrt{\frac{\sigma^2\delta(\omega)}{\Phi_u(\omega)}}$$

  Proof: See Appendix 6.A. ■

  Similarly to the previous results, the interpretation of this theorem is that for large model order p and after the algorithm has converged (large k) then the MSE can be approximated by

$$\mathsf{E}\left\{|\widetilde{G}_k(e^{\mathrm{j}\omega})|^2\right\} \approx \frac{\gamma_p(\omega)}{2}\left(\mu\frac{\sigma_\nu^2}{\sigma^2} + \frac{\rho^2}{\mu}\right)\sqrt{\frac{\sigma^2\delta(\omega)}{\Phi_u(\omega)}}. \tag{6.49}$$

  In terms of how input spectral density affects noise sensitivity and tracking ability, (6.49) shows that Kalman Filter based algorithms sit between the LMS and RLS algorithms in that instead of being affected separately, both noise and tracking performance are affected equally (but to a lesser extent due to the $\sqrt{\cdot}$ operation–see Figure 6.5) by the size of the input spectral density $\Phi_u(\omega)$. Again, the accuracy of the approximation (6.49) is validated experimentally in Section 6.7 to show that in fact it is meaningful for low model orders; see Figure 6.4.

Note that the approximations (6.47), (6.48) and (6.49) can also be used to calculate the optimal step size μ^{opt} which will minimise the MSE at a particular frequency, with the results being

$$\mu^{\mathsf{opt}} = \frac{\rho}{\sigma_\nu}\sqrt{\frac{\delta(\omega)}{\Phi_u(\omega)}}$$

for the LMS and RLS algorithms, and

$$\mu^{\mathsf{opt}} = \frac{\rho\sigma}{\sigma_\nu}$$

for the Kalman Filter algorithm. It is clear from these expressions that for the LMS, and RLS algorithms a value μ^{opt} that minimizes the MSE for all

frequencies can not be obtained. On the other hand, for the Kalman Filter case the value of μ^{opt} is independent of frequency and so it will minimize the MSE at all frequencies. For all of these cases, the minimum MSE is

$$\mathsf{E}\left\{|\widetilde{G}_k(e^{\mathrm{j}\omega})|^2\right\}_{\min} = \rho\,\sigma_\nu\gamma_p(\omega)\sqrt{\frac{\delta(\omega)}{\Phi_u(\omega)}}.$$

The presence of the term $\gamma_p(\omega)$ in all these error quantifications shows that the orthonormal parameterisation (6.24), (6.25) is more than just an essential tool for the analysis of general fixed denominator model structures. Instead, the orthonormal 'basis functions' $\{\mathcal{B}_n(q)\}$ appear as an intrinsic part of adaptive estimation with any fixed denominator model structure $G(q,\theta)$.

For example, for RLS and Kalman Filtering schemes, then in steady–state whether or not the fixed denominator model structure is ab–initio cast in the orthonormal form (6.24), the complete contribution of the fixed pole choice to the frequency domain error properties is captured by the term $\gamma_p(\omega)$ which via (6.33) is itself completely described by the orthonormal basis $\{\mathcal{B}_n(q)\}$.

In other words, for any model structure with fixed denominator such as (6.22) and (6.24), the RLS and Kalman Filter frequency domain error quantification depends only on the location of the poles $\{\xi_n\}$ and is quantified via the factor $\gamma_p(\omega)$ associated with the orthonormal basis functions $\{\mathcal{B}_n(q)\}$.

6.7 Simulation Examples

In this section, the utility of the previous theoretical analysis will be demonstrated via several simulation studies. In all cases, it is assumed that there is an underlying continuous time system with transfer function

$$G(s) = \frac{1}{(s+1)(10s+1)}$$

from which input–output data is collected by sampling every one second. We begin the study by considering the case of stationary systems, but later on, time variations away from $G(s)$ will also be analyzed. It is assumed that the input $\{u_k\}$ is stationary and Gaussian with spectral density

$$\Phi_u(\omega) = \frac{10}{1.25 - \cos\omega},$$

and that the observed output $\{y_k\}$ is corrupted by a white Gaussian noise sequence $\{\nu_k\}$ of variance $\sigma_\nu^2 = 0.01$. Based on this observed data, the identification objective is to estimate the zero order hold equivalent [ÅW84] discrete time system

$$G(q) = \text{ZOH}\left\{\frac{1}{(s+1)(10s+1)}\right\} = \frac{0.0355q + 0.0247}{(q-0.9048)(q-0.3679)} \tag{6.50}$$

via the model structure (6.24) with poles $\{\xi_n\}$ chosen to correspond to continuous time guesses of 0.2 and 0.25 radians per second. Note that these poles, being far from either of the true poles at 0.1 and 1 rad/s, are particularly bad guesses. They have been chosen to dispel any suspicion in the sequel that the high accuracy of the approximations (6.47), (6.48) and (6.49) illustrated in Figures 6.3 to 6.7 derives from unreasonable prior knowledge or idealized conditions.

All three algorithms, the LMS with $\mu = 0.001$, RLS with $\lambda = 0.999$ and $P_0 = I$, and the Kalman Filter with $\mu = 0.001, \Sigma = 0.1, P_0 = I$ and $\sigma^2 = 0.01$ were employed with a tenth order model structure ($p = 10$). The parameter space convergence results are shown in the plots of Figure 6.2, the fast convergence illustrating that these examples do not represent a case of unreasonably slow adaptation. Again, this choice is made to illustrate the robustness of the theoretical analysis to the violation of certain assumptions (small μ) that it is performed under. These estimation experiments were performed five hundred

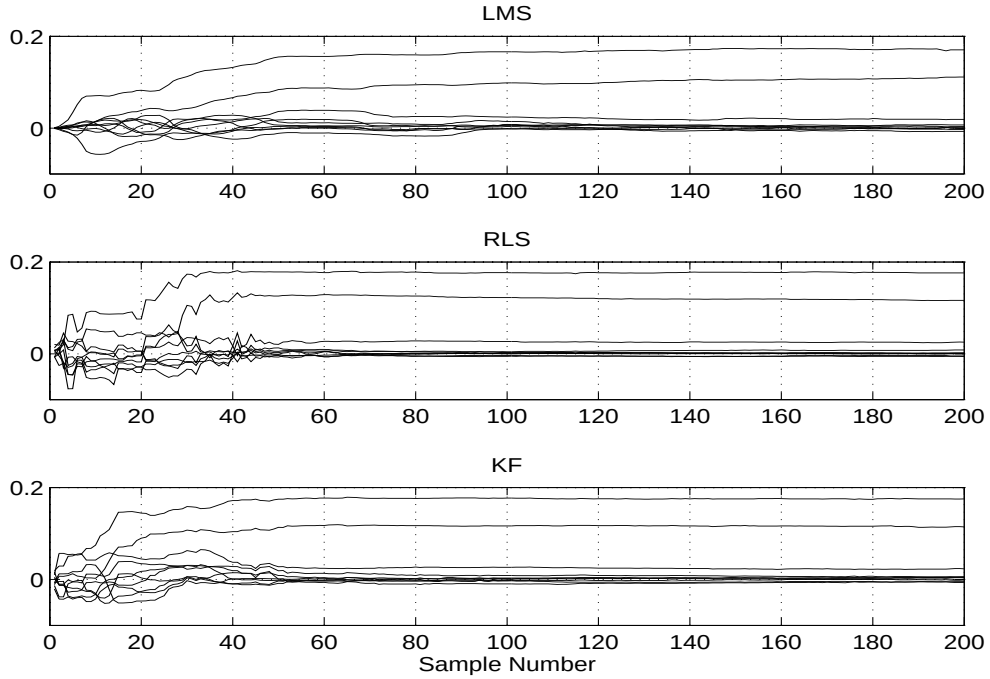

Figure 6.2: *Parameter space convergence for LMS, RLS and Kalman Filter.*

times with different realizations for the input and measurement noise. This allowed the true frequency domain estimation error $\mathbf{E}\left\{|\widetilde{G}_k(e^{j\omega})|^2\right\}$ to be estimated by calculating its sample value as an average over the 500 realizations. This is plotted as the solid line in Figures 6.3 to 6.5. The dash–dotted lines in these figures are the approximations (6.47), (6.48) and (6.49) derived from Theorems 6.6.1, 6.6.2 and 6.6.3, respectively.

To be more specific, in the left hand diagram of Figure 6.3, the LMS approxi-

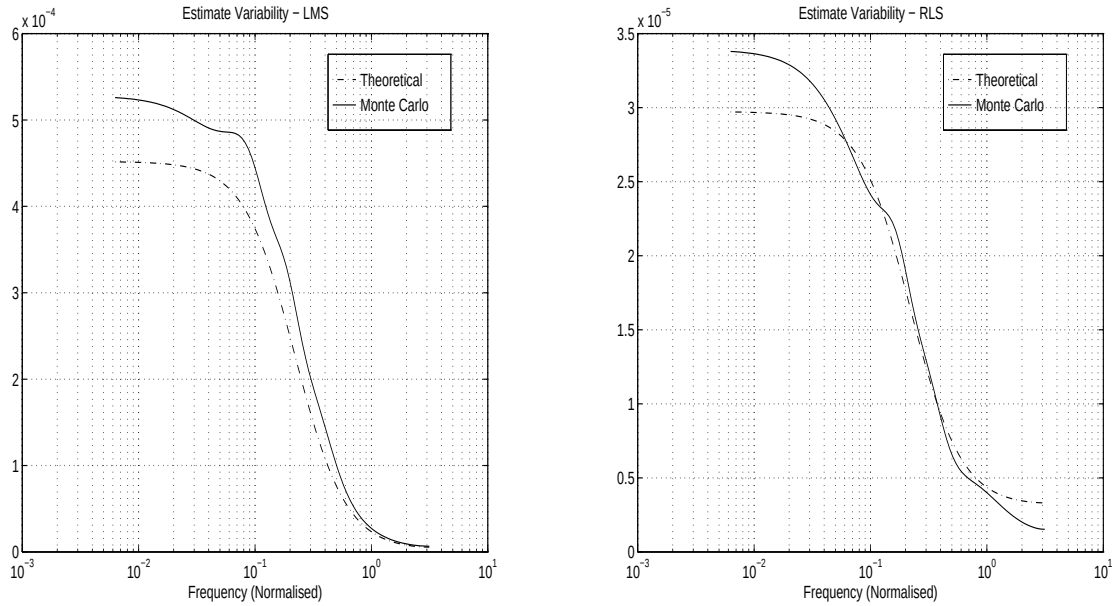

Figure 6.3: *Comparison of Sample Variability (over 500 experiments) vs theoretically derived approximations. On the left is the LMS algorithm vs the approximation (6.47). On the right is the RLS algorithm vs the approximation (6.48). In all cases a 10th order model and 800 data samples were used.*

mation (6.47) is profiled against the true average error distribution with respect to frequency, and appears to be highly accurate. This is in spite of the approximation (6.47) being derived from the asymptotic in p result in Theorem 6.6.1, but being applied in this simulation to only a $p = 10$-th order model.

Similarly, in the right hand diagram of Figure 6.3, the RLS approximation (6.48) is profiled against the true error distribution, and is again quite accurate. Finally, in the left hand diagram of Figure 6.4, the same validity of (6.49) as an approximant for the error distribution of the Kalman Filter algorithm is demonstrated.

To illustrate just how robust the approximations can be to the use of a low model order, the case for the RLS algorithm and a model order of only $p = 4$ and a data length of only 300 samples is shown in the right hand diagram of Figure 6.4. The approximation (dash–dotted) line still appears to be a highly informative indication of the true variability (solid line).

In Figure 6.5, all these tenth order Monte–Carlo simulation results are compared to their theoretical approximants (6.47), (6.48) and (6.49) in one diagram. The top plots are for the LMS algorithm, the middle ones are for the Kalman filter and the bottom ones are for RLS. The fact that the RLS variability is approximately an order of magnitude lower than that for LMS is supported theoretically by examining the approximants (6.47) and (6.48). Specifically,

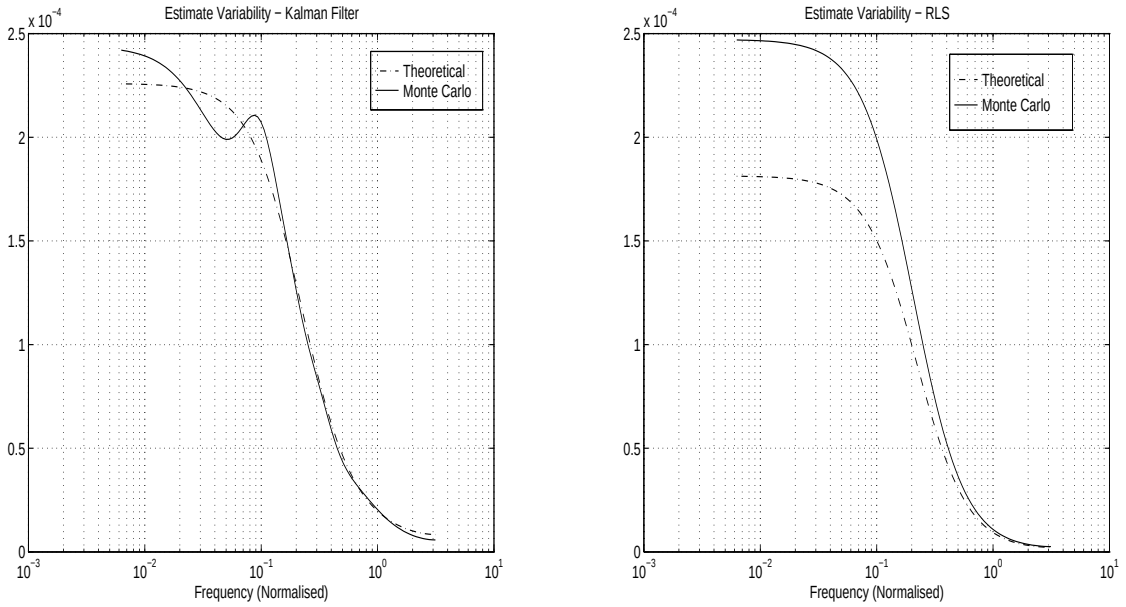

Figure 6.4: *Comparison of Sample Variability (over 500 experiments) vs theoretically derived approximations. On the left is the Kalman Filter algorithm vs the approximation (6.49). On the right is the RLS algorithm vs the approximation (6.48). For the Kalman Filter on the left, 10th order model and 800 data samples were used. For RLS on the right, only 300 data samples and only a 4th order model were used.*

(6.47) indicates that for $\rho = 0$, the LMS steady state error is unaffected by input spectral density $\Phi_u(\omega)$, while (6.48) illustrates that for RLS, this same steady state error is inversely proportional to $\Phi_u(\omega)$. Therefore, for stationary systems, the ratio of LMS error to RLS error should be equal to $\Phi_u(\omega)$, which is observed in Figure 6.5.

As well, a comparison of the theoretical expressions (6.49) and (6.48) indicates that the Kalman Filter variability should be larger than the RLS variability by a factor $\sqrt{\Phi_u(\omega)\delta(\omega)}/\sigma$, and this is also supported by the larger observed variability for the Kalman filter in Figure 6.5. Notice also the slower roll–off at high frequencies of the Kalman filter error as compared to the RLS or LMS error. This is due to the presence of the $\sqrt{\cdot}$ term in the Kalman filter error expression (6.49) which is not present in the LMS or RLS expressions (6.47) or (6.48).

Turning away from steady–state (large k) performance, the LMS (with μ decreased to $\mu = 0.0003$ and $\Phi_u(\omega) = (1.57 - 1.75\cos\omega)^{-1}$) transient analysis approximation in equation (6.37) was also tested via simulation with the results shown in Figure 6.7. In this figure, the variance $\mathbf{E}\left\{|\widetilde{G}_k(e^{j\omega})|^2\right\}$ was estimated as the sample variance calculated over 500 simulations with different input and

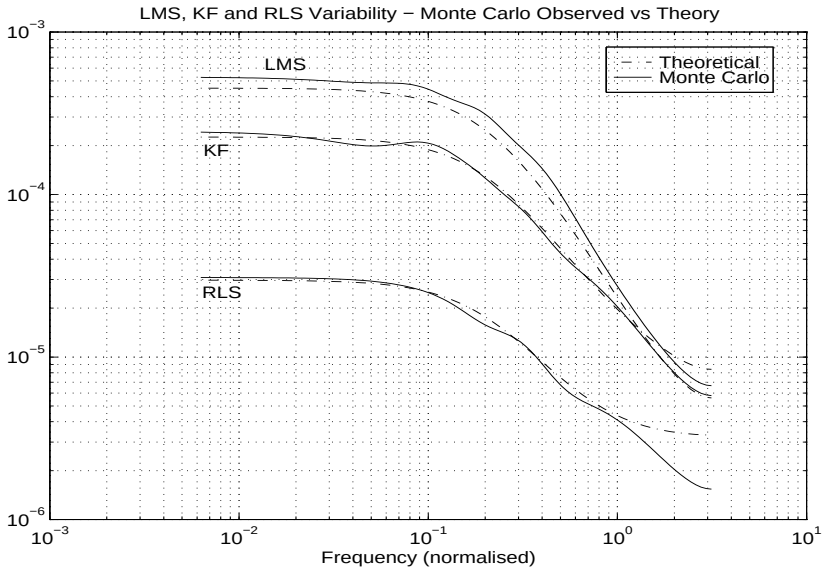

Figure 6.5: *Comparison of observed variability (solid line) vs theory (dash–dotted line) for LMS (top), Kalman Filter (middle) and RLS (bottom).*

measurement noise realisations. This estimate is shown as the solid lines. The top plot with the faster decay rate is the error at $\omega = 0.1$ rad/s, and the bottom plot with the slower decay rate is the error at $\omega = 0.4$ rad/s. The dash–dotted lines in these two diagrams are the estimates for this transient response derived from (6.37). Again, the agreement between the observed error and the theoretical prediction is quite close in spite of the fact that the approximation (6.37) is derived from an asymptotic in p result and then applied to a small ($p = 10$) model order. In particular, note that (6.37) is able to clearly explain and predict the slower convergence at the higher frequency where the input spectral density is smaller.

Finally, the validity of the approximations (6.47), (6.48) and (6.49) for the case of non–stationary plants ($\rho \neq 0$) was tested by starting with the system (6.50) and then perturbing it according to the random walk model (6.34) with Q chosen so as to imply a $\rho^2 \delta_p(\omega)$ shown in the right hand diagram of Figure 6.6. In the left hand diagram of that figure is shown (solid line) the sample mean square variability of the RLS estimate in steady state ($k = 800$) versus that predicted (dash–dotted line) via the theoretically derived approximation (6.48). As in previous simulations, a tenth order model was used, and also as in previous simulations the agreement between sample observation and theoretical approximation is good.

If fact, counter–intuitively the agreement between observation and theory appears better in the time varying plant case than in the previous time invariant

case. This can be explained by noting that in the time invariant case, the poles $\{\xi_k\}$ in the model were deliberately chosen to be far from the true plant poles in order to test the robustness of the approximations (6.47), (6.48) and (6.49). In the time varying case under the model (6.50) this is not possible, so that there is no under–modeling component in the results shown in Figure 6.6.

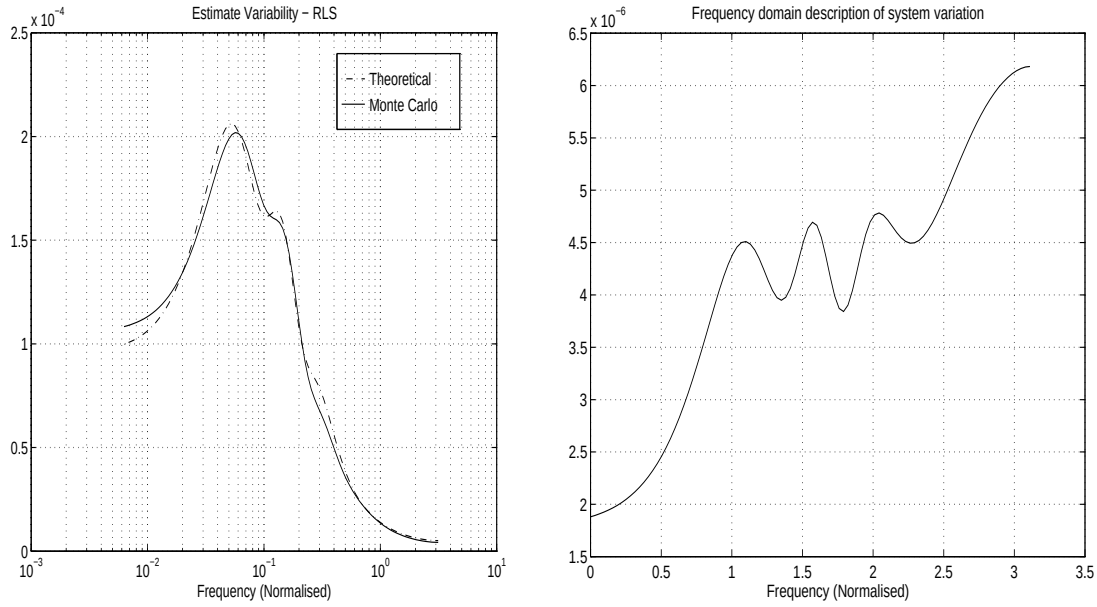

Figure 6.6: *Comparison of Sample Variability (over 500 experiments) vs theoretically derived approximation for RLS algorithm and non–stationary plant. On the left is the sample estimation error after $k = 800$ iterations (solid line) compared to the theoretical approximation (6.48) as a dash–dotted line. On the right is the non-stationarity modeled in the frequency domain by $\rho^2\delta_p(\omega)/\mu$.*

6.8 Conclusions

In this chapter, a frequency domain analysis of the tracking performance of several adaptive estimation schemes has been carried out for the case in which the system is represented by fixed denominator model structures. The transient, as well as the steady-state behaviour of the frequency response tracking error was analyzed. The main contribution of the chapter was to extend known results for FIR model structures, where the poles are fixed at the origin, to more general model structures where the poles may be placed arbitrarily. The key tool in this analysis was to re-parameterize the system using orthonormal bases with fixed poles that generalize the classical FIR bases. The analysis

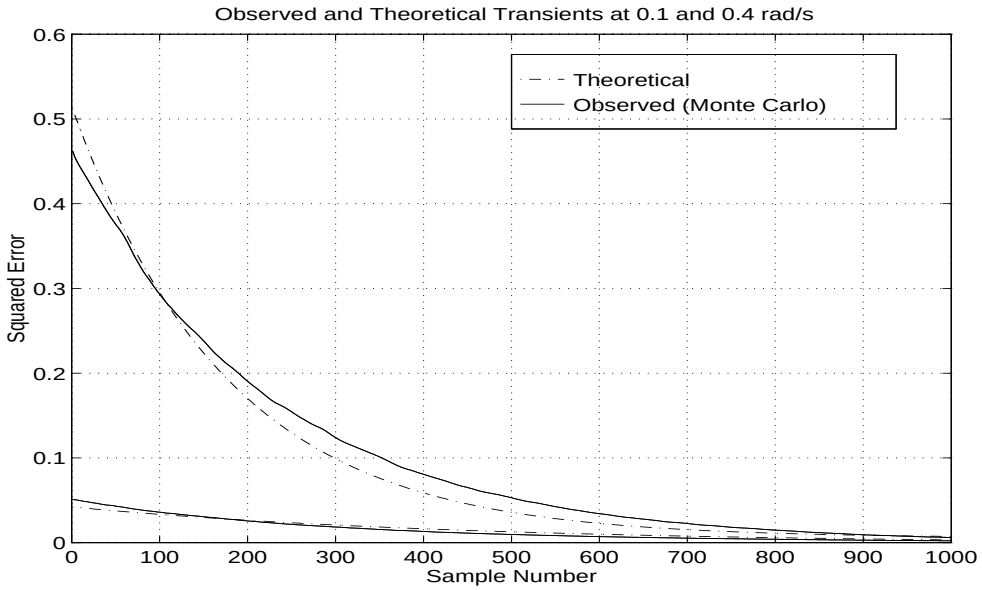

Figure 6.7: *Transient behavior of LMS at two different frequencies - observed (Monte Carlo Average) vs Theoretical prediction via approximation (6.37). Top plot is error* $\mathsf{E}\left\{|\widetilde{G}_k(e^{\mathrm{j}\omega})|^2\right\}$ *at* $\omega = 0.1$ *rad/s. Bottom plot is error* $\mathsf{E}\left\{|\widetilde{G}_k(e^{\mathrm{j}\omega})|^2\right\}$ *at* $\omega = 0.4$ *rad/s.*

showed how the choice of the poles in the orthonormal structure affects the tracking and noise rejection abilities of the algorithms. A fundamental rôle in this frequency domain error characterization was shown to be played by the reproducing kernel $\gamma_p(\omega)$ associated with the orthonormal bases. The validity of using the results derived for infinite model order in a finite model setting was examined via simulation. The simulation experiments showed that the results, which are exact for infinite model order were shown to provide good approximations even for relatively small model orders.

Appendices

Throughout the Appendices in this chapter we will use the Toeplitz-like matrix form $M_p(f)$ introduced in Chapter 4, as defined in equation (4.A.4).

6.A Proof for Chapter 6

Proof of Lemma 6.4.1

- **RLS algorithm:** The parameter update law is given by

$$\widehat{\theta}_{k+1} = \widehat{\theta}_k + P_k\phi_k(y_k - \phi_k^T\widehat{\theta}_k), \tag{6.A.1}$$

$$P_k = \frac{1}{\lambda}\left\{P_{k-1} - \frac{P_{k-1}\phi_k\phi_k^T P_{k-1}}{\lambda + \phi_k^T P_{k-1}\phi_k}\right\}; \quad \lambda \in (0,1). \tag{6.A.2}$$

Considering that $\phi_k = J^T\phi_k'$, equations (6.A.1) and (6.A.2) can be written as

$$\widehat{\theta}_{k+1} = \widehat{\theta}_k + P_k J^T\phi_k'(y_k - \phi_k'^T J\widehat{\theta}_k), \tag{6.A.3}$$

$$P_k = \frac{1}{\lambda}\left\{P_{k-1} - \frac{P_{k-1}J^T\phi_k'\phi_k'^T JP_{k-1}}{\lambda + \phi_k'^T JP_{k-1}J^T\phi_k'}\right\}. \tag{6.A.4}$$

Multiplying equation (6.A.3) on the left by J, and equation (6.A.4) on the left by J and on the right by J^T, we obtain

$$J\widehat{\theta}_{k+1} = J\widehat{\theta}_k + JP_kJ^T\phi_k'(y_k - \phi_k'^T J\widehat{\theta}_k), \tag{6.A.5}$$

$$JP_kJ^T = \frac{1}{\lambda}\left\{JP_{k-1}J^T - \frac{JP_{k-1}J^T\phi_k'\phi_k'^T JP_{k-1}J^T}{\lambda + \phi_k'^T JP_{k-1}J^T\phi_k'}\right\}. \tag{6.A.6}$$

Now, the result for the RLS algorithm follows by defining $P_k' \triangleq JP_kJ^T$.

- **KF algorithm:** The parameter update law in this case is given by

$$\widehat{\theta}_{k+1} = \widehat{\theta}_k + \frac{\mu P_{k-1}\phi_k}{\sigma^2 + \mu\phi_k^T P_{k-1}\phi_k}(y_k - \phi_k^T\widehat{\theta}_k),$$

$$P_k = P_{k-1} - \mu\frac{P_{k-1}\phi_k\phi_k^T P_{k-1}}{\sigma^2 + \mu\phi_k^T P_{k-1}\phi_k} + \mu\Sigma.$$

Considering that $\phi_k = J^T\phi_k'$, these equations can be written as

$$\widehat{\theta}_{k+1} = \widehat{\theta}_k + \frac{\mu P_{k-1}J^T\phi_k'}{\sigma^2 + \mu\phi_k'^T JP_{k-1}J^T\phi_k'}(y_k - \phi_k'^T J\widehat{\theta}_k), \tag{6.A.7}$$

$$P_k = P_{k-1} - \mu\frac{P_{k-1}J^T\phi_k'\phi_k'^T JP_{k-1}}{\sigma^2 + \mu\phi_k'^T JP_{k-1}J^T\phi_k'} + \mu\Sigma. \tag{6.A.8}$$

Multiplying equation (6.A.7) on the left by J, and equation (6.A.8) on the left by J and on the right by J^T, we obtain

$$J\widehat{\theta}_{k+1} = J\widehat{\theta}_k + \frac{\mu JP_{k-1}J^T\phi_k'}{\sigma^2 + \mu\phi_k'^T JP_{k-1}J^T\phi_k'}(y_k - \phi_k'^T J\widehat{\theta}_k),$$

$$JP_kJ^T = JP_{k-1}J^T - \mu\frac{JP_{k-1}J^T\phi_k'\phi_k'^T JP_{k-1}J^T}{\sigma^2 + \mu\phi_k'^T JP_{k-1}J^T\phi_k'} + \mu J\Sigma J^T.$$

Now, the result for the KF algorithm follows by defining $P_k' \triangleq JP_kJ^T$ and $\Sigma' \triangleq J\Sigma J^T$.

- **LMS algorithm:** In this case the parameter update law is given by

$$\widehat{\theta}_{k+1} = \widehat{\theta}_k + \mu\phi_k(y_k - \phi_k^T\widehat{\theta}_k).$$

Considering that $\phi_k = J^T\phi'_k$, we can write

$$\widehat{\theta}_{k+1} = \widehat{\theta}_k + \mu J^T\phi'_k(y_k - \phi_k'^T J\widehat{\theta}_k).$$

From this equation we can easily conclude that the LMS estimate is not invariant under the linear re-parameterization $\phi_k = J^T\phi'_k$.

■

Proof of Theorem 6.5.1 Considering equation (6.32) with $S_k = I$, and the definition (6.35), we can write

$$\begin{aligned}\frac{1}{\gamma_p(\omega)}\mathbf{E}\left\{|\widetilde{G}_{k+1}(e^{\mathrm{j}\omega})|^2\right\} = \\ = \quad \frac{1}{\gamma_p(\omega)}\Gamma_p^\star(e^{\mathrm{j}\omega})(I-\mu R)\Pi_k(I-\mu R)^T\Gamma_p(e^{\mathrm{j}\omega}) \\ + \quad \mu^2\sigma_\nu^2\frac{1}{\gamma_p(\omega)}\Gamma_p^\star(e^{\mathrm{j}\omega})R\Gamma_p(e^{\mathrm{j}\omega}) + \rho^2\frac{\delta_p(\omega)}{\gamma_p(\omega)}. \qquad (6.A.9)\end{aligned}$$

By Parseval's Theorem matrix R can be given the following spectral representation

$$R \triangleq \mathbf{E}\left\{\phi_k\phi_k^T\right\} = \frac{1}{2\pi}\int_{-\pi}^{\pi}\Gamma_p(e^{\mathrm{j}\omega})\Phi_u(\omega)\Gamma_p^\star(e^{\mathrm{j}\omega})\mathrm{d}\omega \qquad (6.A.10)$$

Now, using the notation (4.A.4), equation (6.A.9) can be written as

$$\begin{aligned}\frac{1}{\gamma_p(\omega)}\mathbf{E}\left\{|\widetilde{G}_{k+1}(e^{\mathrm{j}\omega})|^2\right\} = \\ = \quad \frac{1}{\gamma_p(\omega)}\Gamma_p^\star(e^{\mathrm{j}\omega})M_p(1-\mu\Phi_u)\Pi_k M_p(1-\mu\Phi_u)\Gamma_p(e^{\mathrm{j}\omega}) \\ + \quad \mu^2\sigma_\nu^2\frac{1}{\gamma_p(\omega)}\Gamma_p^\star(e^{\mathrm{j}\omega})M_p(\Phi_u)\Gamma_p(e^{\mathrm{j}\omega}) + \rho^2\frac{\delta_p(\omega)}{\gamma_p(\omega)}. \qquad (6.A.11)\end{aligned}$$

Finally, taking the limit as $p \to \infty$, and applying Theorem 4.B.1, Lemma 6.B.1, and the definition of $\delta(\omega)$ in (6.36), then gives the result. ■

Proof of Theorem 6.6.1 Using the formulation (6.A.10) together with the notation (4.A.4) in (6.41) gives that in the limit as $k \to \infty$

$$\Pi M_p(\Phi_u) + M_p(\Phi_u)\Pi = \mu\sigma_\nu^2 M_p(\Phi_u) + \frac{\rho^2}{\mu}Q$$

so that

$$\frac{\Gamma_p^\star(e^{\mathrm{j}\omega})\Pi M_p(\Phi_u)\Gamma_p(e^{\mathrm{j}\omega})}{\gamma_p(\omega)} + \frac{\Gamma_p^\star(e^{\mathrm{j}\omega})M_p(\Phi_u)\Pi\Gamma_p(e^{\mathrm{j}\omega})}{\gamma_p(\omega)} =$$
$$= \frac{\mu\sigma_\nu^2\Gamma_p^\star(e^{\mathrm{j}\omega})M_p(\Phi_u)\Gamma_p(e^{\mathrm{j}\omega})}{\gamma_p(\omega)} + \frac{\rho^2\delta_p(\omega)}{\mu\gamma_p(\omega)}$$

Now the result follows by taking the limit of both sides as $p \to \infty$ while using Theorem 4.B.1, Lemma 6.B.2 and the definition of $\delta(\omega)$ in (6.36). ■

Proof of Theorem 6.6.2 Using the formulation (6.A.10) together with the notation (4.A.4) in (6.42) gives that in the limit as $k \to \infty$

$$\frac{1}{\gamma_p(\omega)}\mathbf{E}\left\{|\widetilde{G}(e^{\mathrm{j}\omega})|^2\right\} = \frac{\Gamma_p^\star(e^{\mathrm{j}\omega})\Pi\Gamma_p(e^{\mathrm{j}\omega})}{\gamma_p(\omega)} =$$
$$= \frac{\mu\sigma_\nu^2}{2\gamma_p(\omega)}\Gamma_p^\star(e^{\mathrm{j}\omega})M_p^{-1}(\Phi_u)\Gamma_p(e^{\mathrm{j}\omega}) + \frac{\rho^2\delta_p(\omega)}{2\mu\gamma_p(\omega)}$$

Taking the limit of both sides as $p \to \infty$ while using Theorem 4.B.2 and the definition of $\delta(\omega)$ in (6.36) then gives the result. ■

Proof of Theorem 6.6.3 Substituting the value of S from (6.45) into (6.44) with $\Sigma = Q$ gives that Π is the solution of

$$\Pi R \Pi = \frac{\sigma^2}{4}\left(\mu\frac{\sigma_\nu^2}{\sigma^2} + \frac{\rho^2}{\mu}\right)^2 Q. \tag{6.A.12}$$

Define two positive definite, $p \times p$ dimensional real matrices A_p and B_p to be asymptotically equivalent $A_p \sim B_p$ if they leave quadratic forms with $\Gamma_p(e^{\mathrm{j}\omega})$ invariant:

$$A_p \sim B_p \Leftrightarrow \lim_{p\to\infty}\frac{1}{\gamma_p(\omega)}\Gamma_p^\star(e^{\mathrm{j}\omega})(A_p - B_p)\Gamma_p(e^{\mathrm{j}\omega}) = 0.$$

Using the definition

$$\alpha^2 \triangleq \frac{\sigma^2}{4}\left(\frac{\mu\sigma_\nu^2}{\sigma^2} + \frac{\rho^2}{\mu}\right)^2$$

and the notation (4.A.4), the matrix

$$\alpha M_p\left(\sqrt{\frac{\delta(\omega)}{\Phi_u(\omega)}}\right)$$

is an asymptotically equivalent solution to Π given by (6.A.12) since using the representation (6.A.10)

$$\frac{1}{\gamma_p(\omega)}\Gamma_p^\star(e^{\mathrm{j}\omega})\left(\Pi R\Pi - \alpha^2 M_p(\sqrt{\delta/\Phi_u})M_p(\Phi_u)M_p(\sqrt{\delta/\Phi_u})\right)\Gamma_p(e^{\mathrm{j}\omega}) =$$

$$= \frac{\alpha^2}{\gamma_p(\omega)}\Gamma_p^\star(e^{\mathrm{j}\omega})\left(M_p(\delta) - M_p(\sqrt{\delta/\Phi_u})M_p(\Phi_u)M_p(\sqrt{\delta/\Phi_u})\right)\Gamma_p(e^{\mathrm{j}\omega}) + \tag{6.A.13}$$

$$+ \frac{\alpha^2}{\gamma_p(\omega)}\Gamma_p^\star(e^{\mathrm{j}\omega})\left(Q - M(\delta)\right)\Gamma_p(e^{\mathrm{j}\omega}). \tag{6.A.14}$$

Now, considering the term (6.A.13), by Theorem 4.B.1 and Lemma 6.B.3

$$\lim_{p\to\infty}\frac{1}{\gamma_p(\omega)}\Gamma_p^\star(e^{\mathrm{j}\omega})\left(M_p(\delta) - M_p(\sqrt{\delta/\Phi_u})M_p(\Phi_u)M_p(\sqrt{\delta/\Phi_u})\right)\Gamma_p(e^{\mathrm{j}\omega}) =$$

$$= \delta(\omega) - \sqrt{\frac{\delta(\omega)}{\Phi_u(\omega)}}\Phi_u(\omega)\sqrt{\frac{\delta(\omega)}{\Phi_u(\omega)}} = 0.$$

Considering the term (6.A.14), and using the definition of $\delta(\omega)$ and Theorem 4.B.1

$$\lim_{p\to\infty}\frac{1}{\gamma_p(\omega)}\Gamma_p^\star(e^{\mathrm{j}\omega})\left(Q - M(\delta)\right)\Gamma_p(e^{\mathrm{j}\omega}) = \delta(\omega) - \delta(\omega) = 0.$$

Therefore, since $\Pi \sim \alpha M_p(\sqrt{\delta(\omega)/\Phi_u(\omega)})$, then again by Theorem 4.B.1

$$\begin{aligned}\lim_{p\to\infty}\frac{1}{\gamma_p(\omega)}\mathsf{E}\left\{|\widetilde{G}(e^{\mathrm{j}\omega})|^2\right\} &= \lim_{p\to\infty}\frac{1}{\gamma_p(\omega)}\Gamma_p^\star(e^{\mathrm{j}\omega})\Pi\Gamma_p(e^{\mathrm{j}\omega}) \\ &= \lim_{p\to\infty}\frac{\alpha}{\gamma_p(\omega)}\Gamma_p^\star(e^{\mathrm{j}\omega})M_p(\sqrt{\delta/\Phi_u})\Gamma_p(e^{\mathrm{j}\omega}) \\ &= \alpha\sqrt{\frac{\delta(\omega)}{\Phi_u(\omega)}}.\end{aligned}$$

■

6.B Technical Results

Lemma 6.B.1. *Let $Q_p \in \mathbb{R}^{p\times p}$ be a symmetric, positive definite matrix with $\|Q_p\|_2 < \infty$ for all p, and let $f(\omega)$ be a real valued function, continuous on $[-\pi,\pi]$ and having a finite dimensional spectral factorization. Suppose that*

$$\sum_{k=0}^{\infty}(1 - |\xi_k|) = \infty.$$

Then

$$\lim_{p\to\infty}\frac{1}{\gamma_p(\omega)}\Gamma_p^\star(\omega)M_p(f)Q_pM_p(f)\Gamma_p(\omega)=f^2(\omega)\lim_{p\to\infty}\frac{1}{\gamma_p(\omega)}\Gamma_p^\star(\omega)Q_p\Gamma_p(\omega).$$

Proof: For simplicity of notation, define the function $g(\sigma)\triangleq f(\sigma)-f(\omega)$. Then

$$\begin{aligned}\frac{1}{\gamma_p(\omega)}\left|\Gamma_p^\star(\omega)M_p(f)Q_pM_p(f)\Gamma_p(\omega)-f^2(\omega)\Gamma_p^\star(\omega)Q_p\Gamma_p(\omega)\right|\leq\\ \leq\ \frac{1}{\gamma_p(\omega)}\left|\Gamma_p^\star(\omega)M_p(g)Q_pM_p(g)\Gamma_p(\omega)\right|+\\ +\ \frac{|f(\omega)|}{\gamma_p(\omega)}\left|\Gamma_p^\star(\omega)[Q_pM_p(g)+M_p(g)Q_p]\Gamma_p(\omega)\right|.\end{aligned}\tag{6.B.15}$$

Considering the first term in this upper bound, we can write

$$\begin{aligned}&\frac{1}{\gamma_p(\omega)}\left|\Gamma_p^\star(\omega)M_p(g)Q_pM_p(g)\Gamma_p(\omega)\right|\leq\\ &\leq\ \frac{\|Q_p\|_2}{\gamma_p(\omega)}\left|\Gamma_p^\star(\omega)M_p^2(g)\Gamma_p(\omega)\right|\\ &\leq\ \frac{\|Q_p\|_2}{\gamma_p(\omega)}\left|\Gamma_p^\star(\omega)M_p(g^2)\Gamma_p(\omega)\right|+ &\text{(6.B.16)}\\ &+\ \frac{\|Q_p\|_2}{\gamma_p(\omega)}\left|\Gamma_p^\star(\omega)[M_p^2(g)-M_p(g^2)]\Gamma_p(\omega)\right|. &\text{(6.B.17)}\end{aligned}$$

Now, by Theorem 4.B.1, the term (6.B.16) tends to zero as p tends to infinity, since

$$\lim_{p\to\infty}\frac{1}{\gamma_p(\omega)}\Gamma_p^\star(\omega)M_p(g^2)\Gamma_p(\omega)=g^2(\omega)=0.$$

For the term (6.B.17) we have that since by construction the elements of $\Gamma_p(\omega)$ are bounded in magnitude by some finite number K_1 (as defined in Lemma 5.E.2), use of Lemma 4.C.2 give us that for some $|\eta|<1$

$$\begin{aligned}&\left|\Gamma_p^\star(\omega)[M_p^2(g)-M_p(g^2)]\Gamma_p(\omega)\right|\leq\\ &\leq\ \sum_{m=0}^{p-1}\sum_{n=0}^{p-1}\left|[\Gamma_p(\omega)^\star]_m\right|\times\\ &\times\ \left|\left[M_p^2(g)\right]_{m,n}-[M_p(g^2)]_{m,n}\right|\left|[\Gamma_p(\omega)]_n\right|\\ &\leq\ K_1K^2\sum_{m=0}^{p-1}\sum_{n=0}^{p-1}(\eta^{p-m}+\eta^m)(\eta^{p-n}+\eta^n)\\ &=\ K_1K^2\left(\frac{1-\eta^p}{1-\eta}\right)^2(\eta^p+\eta)^2<\infty.\end{aligned}$$

Considering that

$$\frac{1}{2}\sum_{k=0}^{p-1}(1-|\xi_k|) \leq \gamma_p(\omega),$$

then, under the conditions of the theorem, also the term (6.B.17) tends to zero as p tends to infinity, i.e.

$$\lim_{p\to\infty} \frac{\|Q_p\|_2}{\gamma_p(\omega)} \left|\Gamma_p^\star(\omega)[M_p^2(g) - M_p(g^2)]\Gamma_p(\omega)\right| = 0.$$

Using Lemma 6.B.2 to deal with the remaining term in the upper bound (6.B.15) then completes the proof. ■

Lemma 6.B.2. *Let $Q_p \in \mathbb{R}^{p\times p}$ be a symmetric, positive definite matrix with $\|Q_p\|_2 < \infty$ for all p, and let $f(\omega)$ be a real valued function, continuous on $[-\pi,\pi]$. Suppose that*

$$\sum_{k=0}^{\infty}(1-|\xi_k|) = \infty.$$

Then

$$\begin{aligned}\lim_{p\to\infty} \frac{1}{\gamma_p(\omega)}\Gamma_p^\star(\omega)M_p(f)Q_p\Gamma_p(\omega) = \\ = \lim_{p\to\infty} \frac{1}{\gamma_p(\omega)}\Gamma_p^\star(\omega)Q_pM_p(f)\Gamma_p(\omega) = \\ = f(\omega)\lim_{p\to\infty} \frac{1}{\gamma_p(\omega)}\Gamma_p^\star(\omega)Q_p\Gamma_p(\omega).\end{aligned}$$

Proof: For simplicity of notation, define the function $g(\sigma) \triangleq f(\sigma)-f(\omega)$. Then

$$\begin{aligned}\frac{1}{\gamma_p(\omega)}\left|\Gamma_p^\star(\omega)M_p(f)Q_p\Gamma_p(\omega) - f(\omega)\Gamma_p^\star(\omega)Q_p\Gamma_p(\omega)\right| = \\ = \frac{1}{\gamma_p(\omega)}\left|\Gamma_p^\star(\omega)M_p(g)Q_p\Gamma_p(\omega)\right|.\end{aligned}$$

Now define the functions $g^+(\sigma) \triangleq \max[g(\sigma),0]$ and $g^-(\sigma) \triangleq \min[g(\sigma),0]$ so that $g(\sigma) = g^+(\sigma) + g^-(\sigma)$ and hence

$$\begin{aligned}\frac{1}{\gamma_p(\omega)}\left|\Gamma_p^\star(\omega)Q_pM_p(g)\Gamma_p(\omega)\right| \leq \\ \leq \frac{1}{\gamma_p(\omega)}\left|\Gamma_p^\star(\omega)Q_pM_p(g^+)\Gamma_p(\omega)\right| + \\ + \frac{1}{\gamma_p(\omega)}\left|\Gamma_p^\star(\omega)Q_pM_p(g^-)\Gamma_p(\omega)\right|. \qquad (6.B.18)\end{aligned}$$

Considering only the first term in this upper bound, note that provided $f(\omega)$ is not equal to a constant (in which case the Lemma is trivial since $M_p(1) = I$), then for $x \in \mathbb{R}^p$ arbitrary

$$x^T M_p(g^+)x = \frac{1}{2\pi}\int_{-\pi}^{\pi} |x^T\Gamma_p(\sigma)|^2 g^+(\sigma)\,\mathrm{d}\sigma > 0$$

so that $M_p(g^+)$ is positive definite and hence

$$\begin{aligned}
&\frac{1}{\gamma_p(\omega)}\left|\Gamma_p^\star(\omega)Q_pM_p(g^+)\Gamma_p(\omega)\right| = \\
&\quad = \frac{1}{\gamma_p(\omega)}\left|\Gamma_p^\star(\omega)M_p^{1/2}(g^+)[M_p^{-1/2}(g^+)Q_pM_p^{1/2}(g^+)]M_p^{1/2}(g^+)\Gamma_p(\omega)\right| \\
&\quad \le \frac{\|Q_p\|_2}{\gamma_p(\omega)}\left|\Gamma_p^\star(\omega)M_p(g^+)\Gamma_p(\omega)\right|.
\end{aligned}$$

Finally, using Theorem 4.B.1

$$\lim_{p\to\infty}\frac{1}{\gamma_p(\omega)}\Gamma_p^\star(\omega)M_p(g^+)\Gamma_p(\omega) = g^+(\omega) = 0.$$

Using the same argument for the remaining term in the upper bound (6.B.18) then completes the proof. ■

Lemma 6.B.3. *Let $f, g \in L_2([-\pi,\pi])$ have finite dimensional spectral factorizations. Then provided $\sum_{k=0}^{\infty}(1-|\xi_k|) = \infty$,*

$$M_p(f)M_p(g)M_p(f) \sim M_p(f^2g) \quad \textit{as } p\to\infty.$$

with the notation '$\sim$' as introduced in the proof of Theorem 6.6.3.

Proof: Using Lemmas 4.C.2 and 4.C.3

$$\begin{aligned}
&\Gamma_p^\star(\omega)\left[M_p(f)M_p(g)M_p(f) - M_p(f^2g)\right]\Gamma_p(\omega) = \\
&\quad = \Gamma_p^\star(\omega)M_p(f)\left[M_p(g)M_p(f) - M_p(fg)\right]\Gamma_p(\omega) + \\
&\qquad +\Gamma_p^\star(\omega)\left[M_p(f)M_p(fg) - M_p(f^2g)\right]\Gamma_p(\omega) \\
&\quad \le \sum_{m=0}^{p-1}\sum_{n=0}^{p-1}\left|[\Gamma_p(\omega)^\star M_p(f)]_m\right|\left|[M_p(g)M_p(f)]_{m,n} - [M_p(fg)]_{m,n}\right|\left|[\Gamma_p(\omega)]_n\right| + \\
&\qquad + \sum_{m=0}^{p-1}\sum_{n=0}^{p-1}\left|[\Gamma_p(\omega)^\star]_m\right|\left|[M_p(f)M_p(fg)]_{m,n} - [M_p(f^2g)]_{m,n}\right|\left|[\Gamma_p(\omega)]_n\right| \\
&\quad \le K^3\sum_{m=0}^{p-1}\sum_{n=0}^{p-1}(\eta^{p-m}+\eta^m)(\eta^{p-n}+\eta^n), \\
&\quad = K^3\left(\frac{1-\eta^p}{1-\eta}\right)^2(\eta^p+\eta)^2 < \infty.
\end{aligned}$$

Noting that $2\gamma_p(\omega) \ge \sum_{k=0}^{p-1}(1-|\xi|)$ then completes the proof. ■

7

Conclusions

In this thesis we have analyzed several aspects regarding the use of rational orthonormal bases in identification of discrete-time linear systems from input-output data in the time domain.

We concentrated on the use of orthonormal bases with fixed poles that generalize the more common FIR, Laguerre and Kautz bases, and that allows the incorporation in the identification process of prior information about dominant dynamics of the system by an appropriate choice of the poles in the orthonormal structure.

Apart from the well known properties of orthonormal bases leading to linear regressor forms and guaranteeing worst case numerical conditioning of the least squares estimation, an aspect we have emphasized in this thesis is the use of orthonormal bases with fixed poles as an analysis tool that facilitates the quantification of the estimation accuracy of any (multivariable) fixed denominator model structure. The numerical robustness of the least squares estimation using orthonormal structures was also analyzed and compared to the case of using an equivalent non-orthonormal structure.

One of the main contributions of this thesis has been the extension to the multivariable setting and to general fixed denominator model structures of existent single-input single-output FIR results quantifying the undermodelling error and the asymptotic distribution of the transfer matrix estimate (the variance error). The variance error result establishes that the variance of the transfer matrix estimate at a given frequency, and for large model order and data-length, can be approximated by the generalized noise to signal ratio $(\Phi_u^{-1}(\omega) \otimes \Phi_\nu(\omega))$ weighted with a frequency dependent factor $(\gamma_p(\omega)/N)$ that depends on the particular bases. These results explicitly show how the choice of the poles of the bases influences the bias and variance errors. In addition, they illustrate an until now unknown phenomenom of bias/variance trade-off with respect to the location of the poles of the bases.

Another contribution of this thesis, in the framework of recursive identification, is the performance analysis of general adaptive algorithms using orthonor-

mal bases with fixed poles. Here again, the analysis illustrates how several factors, such as step size of the algorithm, excitation properties of the input signal and measurement noise, and choice of the fixed poles in the model structure, influence the disturbance rejection properties and the tracking ability of the algorithms.

A

Kronecker Product and vec-Operator

In this Appendix, some basic properties of Kronecker products and the vec-operator are summarized. A more detailed treatment of these topics can be found in [Bre78].

- **Kronecker Product:** Let $A = (a_{ij})$ and $B = (b_{ij})$ be $m \times n$ and $\ell \times p$ sized matrices. Then the Kronecker product of A and B, denoted $A \otimes B$, is defined as the $m\ell \times np$ sized matrix

$$A \otimes B = \begin{pmatrix} a_{11}B & a_{12}B & \cdots & a_{1n}B \\ a_{21}B & a_{22}B & \cdots & a_{2n}B \\ \vdots & & & \vdots \\ a_{m1}B & a_{m2}B & \cdots & a_{mn}B \end{pmatrix}$$

- **vec-Operator:** is an operator which turns an $m \times n$ matrix A into a vector by stacking the columns of A on top of one another:

$$\mathrm{vec}\, A = \begin{pmatrix} A_1 \\ A_2 \\ \vdots \\ A_n \end{pmatrix} \quad (m \times 1) \text{ vector}$$

  where A_k is the k-th column of A.

- **Operations with Kronecker Products:** The following identities hold

  1. $(A \otimes B)(C \otimes D) = AC \otimes BD$
  2. $(A \otimes B)^T = A^T \otimes B^T$
  3. $(A \otimes B)^{-1} = A^{-1} \otimes B^{-1}$

  Finally a useful property of the vec-operator is that when A is $m \times n$ and B is $n \times \ell$ then

$$\mathrm{vec}\, AB = (I_\ell \otimes A)\, \mathrm{vec}\, B = (B^T \otimes I_m)\, \mathrm{vec}\, A.$$

Lemma A.0.4. *The eigenvalues of $A \otimes B$ are $\lambda_i \mu_j$ where λ_i are the eigenvalues of A and μ_j are the eigenvalues of B.*

Proof: See [Bel60], Theorem 3, page 235. ■

Lemma A.0.5. *Let σ_i^A and σ_j^B denote the singular values of the matrices A and B respectively. Then the matrix $A \otimes B$ has singular values $\sigma_i^A \sigma_j^B$.*

Proof: By definition, the singular values of $A \otimes B$ are the square root of the eigenvalues of the matrix $(A \otimes B)(A \otimes B)^\star$. But

$$(A \otimes B)(A \otimes B)^\star = AA^\star \otimes BB^\star$$

so that the result follows by applying Lemma A.0.4 to the RHS of the previous equation. ■

Lemma A.0.6. *The following identity holds*

$$\|A \otimes B\|_2 = \|A\|_2 \|B\|_2$$

Proof: The results follows directly by applying Lemma A.0.5 and considering that the 2-norm of a matrix equals its maximum singular value. ■

B

Notation

q	forward shift operator .
$\mathbb{D}$	the open unit disk in the complex plane $\{z \in \mathbb{C} : \|z\| < 1\}$.
$\mathbb{T}$	the unit circle $\{z : \|z\| = 1\}$.
$\mathbb{E}$	the open region outside the unit disk $\{z : \|z\| > 1\}$.
$\mathbb{C}$	complex numbers.
$\mathbb{R}$	real numbers.
$\mathbb{Z}$	set of integer numbers .
$\mathbb{N}_0$	set of non-negative integer numbers.
$\ell_2(\mathbb{Z})$	Hilbert space of (two-sided) square summable sequences with support in $\mathbb{Z}$.
$\ell_2(\mathbb{N}_0)$	Hilbert space of (one-sided) square summable sequences with support in $\mathbb{N}_0$.
$L_2(\mathbb{T})$	Hilbert space of Lebesgue square-integrable functions on the unit circle.
$H_2(\mathbb{T})$	Hardy space of Lebesgue square-integrable functions on the unit circle $\mathbb{T}$ that are analytic outside the unit circle (i.e., analytic on $\mathbb{E}$.
$H_2^{m\times n}(\mathbb{T})$	Hardy space of $(m \times n)$ transfer matrices whose entries are in $H_2(\mathbb{T})$.
$\{\mathcal{B}_k(z)\}_{k=0}^{p-1}$	set of orthonormal functions.
$K_p(z, \mu)$	Reproducing Kernel associated with an orthonormal set.

$\langle \cdot, \cdot \rangle$	inner product (in a Hilbert space).
$\langle \cdot, \cdot \rangle_H$	inner product in Hilbert space H.
$\| \cdot \|$	norm induced by the inner product.
$\| \cdot \|_2$	2-norm of matrices or spectral norm.
$\| \cdot \|_F$	Frobenius norm of matrices.
$A^T, A^\star$	Transpose and conjugate transpose of matrix A.

Acronyms

FIR	Finite Impulse Response (filter or model structure).
IIR	Infinite Impulse Response (filter or model structure).
SISO	Single-Input Single-Output (system).
MIMO	Multiple-Input Multiple-Output (system).
DT	Discrete Time (system).
LTI	Linear Time-Invariant (system).
OBFP	Orthonormal Bases with Fixed Poles.
OBGIF	Orthonormal Bases Generated fron Inner Functions.
PEM	Prediction Error Methods.
rkHs	reproducing kernel Hilbert space.
DFT	Discrete Fourier Transform.
BIBO	Bounded-Input Bounded-Output (stability).
LMS	Least Mean Squares.
RLS	Recursive Least Squares.
KF	Kalman Filter.
4SID	Subspace-based State-Space System IDentification methods.
N4SID	Numerical algorithms for Subspace State-Space System IDentification.
MOESP	Multivariable Output Error State sPace.
CVA	Canonical Variate Analysis.
SVD	Singular Value Decomposition.

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