Polyhedral results for the Equitable Coloring Problem

Isabel Méndez-Díaz[2] Graciela Nasini[1,3] Daniel Severin[1,3]

[1] FCEIA, UNR, {nasini, daniel}@fceia.unr.edu.ar

[2] FCEyN, UBA, imendez@dc.uba.ar

[3] CONICET, Argentina

LAGOS, Bariloche, 2011

Definitions

Classic coloring

Let $G = (V, E)$. A k-coloring is a partition of V in C_1, C_2, ..., C_k such that:

- $uv \in E, \ u \in C_j \implies v \notin C_j$

Equitable coloring

A k-eqcol is a k-coloring such that:

- $||C_i| - |C_j|| \leq 1 \qquad \forall\, i, j = 1, \ldots, k$

Equitable chromatic number

$\chi_{eq}(G) = \min\{k : G \text{ admits a } k\text{-eqcol}\}$

- ECP consists in finding $\chi_{eq}(G)$.

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A bit of history about ECP...

- Definition and first results [Meyer (1973)]
- Applications:
  - Municipal garbage collection service [Tucker 1973]
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- ECP is NP-Hard [Kubale, Furmanczyk (2005)]
- IP model and C&B [Méndez-Díaz, Nasini, S.- (Alio/Euro 2008)]
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$$x_{vj} = \begin{cases} 1 & \text{color } j \text{ is assigned to } v, \\ 0 & \text{otherwise,} \end{cases} \qquad w_j = \begin{cases} 1 & x_{vj} = 1 \text{ for some } v, \\ 0 & \text{otherwise.} \end{cases}$$

$$\min \sum_{j=1}^{n} w_j$$

$$s.t. \sum_{j=1}^{n} x_{vj} = 1, \qquad \forall\, v \in V$$

$$x_{uj} + x_{vj} \leq w_j, \qquad \forall\, uv \in E,\ j = 1, \ldots, n$$

$$x_{vj} \leq w_j, \qquad \forall\, v \text{ isolated},\ j = 1, \ldots, n$$

$$w_{j+1} \leq w_j, \qquad \forall\, j = 1, \ldots, n-1$$

Coloring polytope

$\mathcal{CP} =$ convex hull of colorings of G

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Dimension

Equitable coloring polytope

$\mathcal{ECP} =$ convex hull of equitable colorings of G

$\mathscr{A}(G) = \{k \ : \ G \text{ does not admit any } k\text{-eqcol}\}$

- Example: $K_{3,3}$ only admits 2,4,5 and 6-eqcols $\longrightarrow \quad \mathscr{A}(K_{3,3}) = \{1,3\}$

Dimension of $\mathcal{ECP}$

$dim(\mathcal{ECP}) = n^2 - (|\mathscr{A}(G)| + 2)$

- Example: $dim(K_{3,3}) = 32$

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Known valid inequalities for $\mathcal{CP}$

Clique inequalities

Let $j \leq n-1$ and Q be maximal clique of G such that $|Q| \geq 2$. Then, the clique inequality

$$\sum_{v \in Q} x_{vj} \leq w_j,$$

defines a **facet** of $\mathcal{ECP}$.

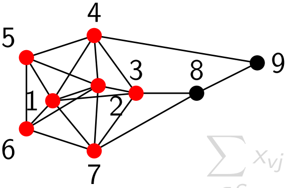
Block inequalities

Let $v \in V$ and $j \leq n-2$. Then, the block inequality

$$\sum_{k=j}^{n} x_{vj} \leq w_j,$$

is **valid** for $\mathcal{ECP}$. If $j-1 \notin \mathscr{A}(G)$, it defines a **facet** of $\mathcal{ECP}$.

Known valid inequalities for $\mathcal{CP}$



$j \leq n - 1$

$S \subset V$ with $\alpha(S) = 2$

$$\sum_{v \in S} x_{vj} \leq 2w_j \quad \text{valid for } \mathcal{CP}$$

(S, Q)-2-rank inequalities

Let $Q = \{q : q \in S,\ S \subset N[q]\}$. The *$(S, Q)$-2-rank inequality* defined as

$$\sum_{v \in S \setminus Q} x_{vj} + 2 \sum_{v \in Q} x_{vj} \leq 2w_j,$$

is **valid** for $\mathcal{ECP}$.

Example: $Q = \{1, 2\}$

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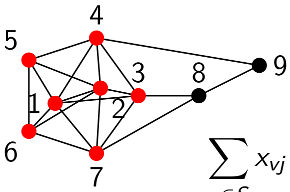

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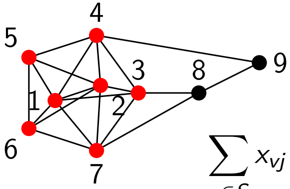

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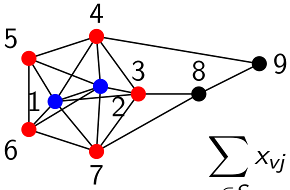

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(S, Q)-2-rank inequalities

The (S, Q)-2-rank inequality defines a **facet** of $\mathcal{ECP}$ if:

- $|Q| \geq 2$
- no connected component of $\overline{G[S \backslash Q]}$ is bipartite
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- $j \leq \lceil n/2 \rceil - 1$
- $\forall\ v \in V \backslash S$ such that $Q \cup \{v\}$ **is** a clique, $\exists\ H$ such that:
  - H stable set with $|H| = 3$
  - $v \in H$
  - $|H \cap S| = 2$
  - n odd $\longrightarrow$ $\overline{G - H}$ has a perfect matching
  - n even $\longrightarrow$ $\exists$ another stable set H' such that:
    - $|H'| = 3$
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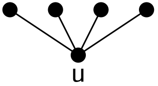

$j \leq n-1$
$u \in V$
$S \subset N(u)$ not a clique

$$\alpha(S)x_{uj} + \sum_{v \in S} x_{vj} \leq \alpha(S)w_j \quad \text{valid for } \mathcal{CP}$$

- in $\mathcal{ECP} \longrightarrow |C_j| \leq \lceil n/j \rceil$
- $\gamma_{kS} = \min\{\lceil n/k \rceil, \alpha(S)\}$

Subneighborhood inequalities

The (u, j, S)-*subneighborhood inequality* defined as

$$\gamma_{jS}x_{uj} + \sum_{v \in S} x_{vj} + \sum_{k=j+1}^{n} (\gamma_{jS} - \gamma_{kS})x_{uk} \leq \gamma_{jS}w_j,$$

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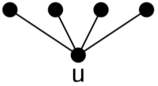

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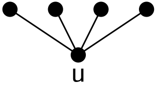

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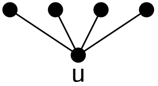

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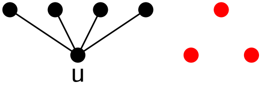
New valid inequalities for $\mathcal{ECP}$



$j \leq \lfloor n/2 \rfloor$
$u \in V$ not universal in G
$N(u)$ not a clique
we focus on $V \setminus N[u]$

- $\forall$ j-eqcol $\longrightarrow$ $|C_j| \geq \lfloor n/j \rfloor$
- $x_{uj} = 1 \longrightarrow \lfloor n/j \rfloor - 1$ vertices of $V \setminus N[u]$ uses color j

$$(\lfloor n/j \rfloor - 1)x_{uj} - \sum_{v \in V \setminus N[u]} x_{vj} \leq 0 \quad \text{valid for } j\text{-eqcols}$$

Outside-neighborhood inequalities

The (u, j)-*outside-neighborhood inequality* defined as

$$(\lfloor n/j \rfloor - 1)x_{uj} - \sum_{v \in V \setminus N[u]} x_{vj} + \sum_{k=j+1}^{n} b_{jk} x_{uk} \leq \sum_{k=j+1}^{n} b_{jk}(w_k - w_{k+1}),$$

where $b_{jk} = \lfloor n/j \rfloor - \lfloor n/k \rfloor$, is **valid** for $\mathcal{ECP}$.

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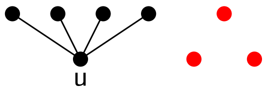

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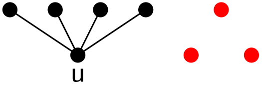

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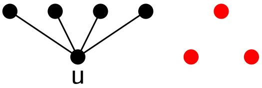

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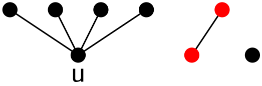

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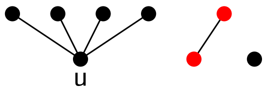

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More results

- For every family of valid inequalities presented previously:
  - Sufficient conditions in order to be facet-defining inequalities.
  - If $\mathscr{F}$ = face defined by the inequality, $\dim(\mathscr{F}) = o(n^2) = o(\dim(\mathcal{ECP}))$
- Separation routines:
  - Heuristics:
    - Clique ineq.
    - (S, Q)-2-rank ineq.
    - S-color ineq.
  - Enumeration:
    - Block ineq.
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- Comparison between:
  - BC^+ = B&C with all inequalities
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- 50 instances of 70 vertices
- 2 hours time limit

% dens.	% solved inst.				Nodes (average)				Time sec. (average)			
	BC^+	BC	CPX	LF_2	BC^+	BC	CPX	LF_2	BC^+	BC	CPX	LF_2
10	100	100	100	100	3.4	4	13.3	57	0.3	0.3	4	109
30	90	90	0	0	2135	3949	–	–	276	**224**	–	–
50	70	70	0	0	7932	21595	–	–	**1354**	2145	–	–
70	80	80	10	100	525	2970	214	678	**128**	446	4380	273
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