Polyhedral results for the Equitable Coloring Problem

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Abstract

In this work we study the polytope associated with a 0/1 integer programming formulation for the Equitable Coloring Problem. We find several families of valid inequalities and derive sufficient conditions in order to be facet-defining inequalities. We also present computational evidence of the effectiveness of including these inequalities as cuts in a Branch & Cut algorithm.

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1 Introduction and preliminary results

The *Equitable Coloring Problem* (ECP), originally presented in [2], is a variation of the widely studied *Graph Coloring Problem* (GCP) with additional constraints imposing that any pair of color classes has to differ in size by at most one. Further references and applications can be seen in [1].

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A *k-coloring* of a graph $G = (V, E)$ is a partition of V in k stable sets, C_j, with $1 \leq j \leq k$. The stable set C_j is the *class of color j*. An *equitable k-coloring* (or just k-eqcol) of G is a k-coloring satisfying the *equity constraints*, i.e. $\lfloor n/k \rfloor \leq |C_j| \leq \lceil n/k \rceil$ for each $1 \leq j \leq k$, where $n = |V|$.

Unlike GCP, a graph admiting a k-eqcol may not admit a $(k+1)$-eqcol. This leads us to define $\mathscr{A}(G)$ as the set of $k \leq n$ such that G does not admit any k-eqcol. For instance, $\mathscr{A}(K_{3,3}) = \{1, 3\}$.

The *equitable chromatic number* of G, $\chi_{eq}(G)$, is the minimum k for which G has a k-eqcol. Computing $\chi_{eq}(G)$ for arbitrary graphs is an NP-hard problem [1].

Although many integer programming formulations are known for GCP, as far as we know, just two of these models were adapted for ECP. One case is the model in [3], adapted in [5]. Preliminary results concerning a Branch & Cut algorithm based on one of the models in [4] were presented in [6]. The algorithm turns out to be competitive compared to the one presented in [5]. This encouraged us to delve into a polyhedral study with the aim of finding strong inequalities that allow us to improve the performance of our algorithm.

2 The polytope $\mathcal{ECP}$

From now on, we assume that G is a graph with n vertices such that $n \geq 5$ and $2 \leq \chi_{eq}(G) \leq n - 2$. Other cases are trivial.

In [4], colorings of G are identified with binary vectors $(x, w) \in \{0,1\}^{n^2+n}$ where $x \in \{0,1\}^{n^2}$ and $w \in \{0,1\}^n$, satisfying the following constraints:

$$
\begin{array}{lll}
\displaystyle\sum_{j=1}^{n} x_{vj} = 1 & \forall\, v \in V & \text{(assign a unique color to each vertex)} \\
x_{uj} + x_{vj} \leq w_j & \forall\, uv \in E,\ j = 1, \ldots, n & \text{(adjacent vertices do not share the same color)} \\
w_{j+1} \leq w_j & \forall\, j = 1, \ldots, n-1. & \text{(eliminate some symmetric colorings)}
\end{array}
$$

where $x_{vj} = 1$ if color j is assigned to vertex v and $w_j = 1$ if color j is used, i.e. $C_j \neq \varnothing$. The *coloring polytope* $\mathcal{CP}$ is defined as the convex hull of colorings of G. In this work, equitable colorings are identified with binary vectors defining colorings which also satisfy

$$
x_{vj} \leq w_j, \qquad \forall\, v \text{ isolated},\ j = 1, \ldots, n, \quad (1)
$$

$$
\sum_{k=j}^{n} \left\lfloor \frac{n}{k} \right\rfloor (w_k - w_{k+1}) \leq \sum_{v \in V} x_{vj} \leq \sum_{k=j}^{n} \left\lceil \frac{n}{k} \right\rceil (w_k - w_{k+1}), \qquad \forall\, j = 1, \ldots, n-1, \quad (2)
$$

where w_{n+1} is a dummy variable set to 0, constraints (1) ensure that isolated vertices use enabled colors and (2) are the equity constraints. The *Equitable Coloring Polytope* $\mathcal{ECP}$ is the convex hull of the equitable colorings of G.

Next we state the main results related to the polyhedral structure of $\mathcal{ECP}$.

Proposition 2.1 *The dimension of $\mathcal{ECP}$ is $n^2 - (|\mathscr{A}(G)| + 2)$.*

In [4], *clique inequalities* and *block inequalities* are proven to be facet-defining inequalities of $\mathcal{CP}$. In our case, we have:

Proposition 2.2 *(i) Let $j \leq n-1$ and Q be maximal clique of G such that $|Q| \geq 2$. Then, the clique inequality $\sum_{v \in Q} x_{vj} \leq w_j$ defines a facet of $\mathcal{ECP}$. (ii) Let $v \in V$ and $j \leq n-2$. Then, the block inequality $\sum_{k=j}^{n} x_{vj} \leq w_j$ is valid for $\mathcal{ECP}$ and defines a facet of $\mathcal{ECP}$ if $j-1 \notin \mathscr{A}(G)$.*

By lifting *rank inequalities* and *neighborhood inequalities*, also studied in [4], we obtain new families of valid inequalities which often define facets.

Proposition 2.3 *Let $j \leq n-1$, $S \subset V$ with $\alpha(S) = 2$ and $Q = \{q : q \in S,\ S \subset N[q]\}$. Then, the (S,Q)-*2-rank inequality* defined as*

$$\sum_{v \in S \backslash Q} x_{vj} + 2 \sum_{v \in Q} x_{vj} \leq 2w_j.$$

is valid for $\mathcal{ECP}$. Let us assume that $|Q| \geq 2$ and no connected component of the complement graph of $G[S \backslash Q]$ is bipartite. The inequality defines a facet of $\mathcal{ECP}$ if one of the following conditions holds:

- *for all $v \in V \backslash S$, $Q \cup \{v\}$ is not a clique,*
- *n is odd, $j \leq \lceil n/2 \rceil - 1$ and for all $v \in V \backslash S$ such that $Q \subset N(v)$, there exists a stable set H of size 3 such that $v \in H$ and $|H \cap S| = 2$, and the complement of $G - H$ has a perfect matching,*
- *n is even, $j \leq \lceil n/2 \rceil - 1$ and for all $v \in V \backslash S$ such that $Q \subset N(v)$, there exist two disjoint stable sets of size 3, H and H', such that $v \in H$ and $|H \cap S| = 2$, and the complement of $G - (H \cup H')$ has a perfect matching.*

If $Q = \varnothing$ or $Q = \{q\}$, the (S,Q)-2-rank inequality is respectively dominated by the inequalities

$$\sum_{v \in S} x_{vj} + \sum_{v \in V} x_{vn-1} \leq 2w_j + w_{n-1} - w_n, \quad \text{or}$$

$$\sum_{v \in S \backslash \{q\}} x_{vj} + 2x_{qj} + x_{qn} \leq 2w_j$$

which also usually define facets of $\mathcal{ECP}$.

Proposition 2.4 *Given $j \leq n-1$, $u \in V$ and $S \subset N(u)$ with $\alpha(S) \geq 2$, the (u,j,S)-*subneighborhood inequality* defined as*

$$\gamma_{jS} x_{uj} + \sum_{v \in S} x_{vj} + \sum_{k=j+1}^{n} (\gamma_{jS} - \gamma_{kS}) x_{uk} \leq \gamma_{jS} w_j,$$

where $\gamma_{kS} = \min\{\lceil n/k \rceil, \alpha(S)\}$, *is a valid inequality for* $\mathcal{ECP}$. *If* $S = N(u)$ *or* $\alpha(S) \leq \lceil n/j \rceil - 1$, *the inequality defines a facet of* $\mathcal{ECP}$ *when the following conditions hold:*

- *for all* $k \in \{\lceil \frac{n}{i} \rceil - 1 : 2 \leq i \leq \gamma_{jS} - 1\}$, *there exists a* k*-eqcol such that* $|C_j \cap S| = \gamma_{kS}$,
- *for all* $v \in N(u) \backslash S$, *there exists an equitable coloring such that* $|C_j \cap S| = \alpha(S)$ *and* $(C_j \cap N(u)) \backslash S = \{v\}$.

Finally, we obtain three new families of valid inequalities for $\mathcal{ECP}$, which were not derived from any of the valid inequalities given in [4].

Proposition 2.5 *Let* $S \subset \{1, \dots, n\}$. *The* S-color inequality *defined as*

$$\sum_{j \in S} \sum_{v \in V} x_{vj} \leq \sum_{k=1}^{n} b_{Sk} (w_k - w_{k+1}),$$

where $d_{Sk} = |S \cap \{1, \dots, k\}|$ *and* $b_{Sk} = d_{Sk} \lfloor \frac{n}{k} \rfloor + \min\{d_{Sk}, n - k \lfloor \frac{n}{k} \rfloor\}$, *is a valid inequality for* $\mathcal{ECP}$. *In addition, if* $3 \leq |S| \leq n - 2$, S *contains all the colors greater than* $n - \lceil \frac{|S|+1}{2} \rceil$ *and the complement of* G *has a matching of size* $\lceil \frac{|S|+1}{2} \rceil$, *then the* S*-color inequality defines a facet of* $\mathcal{ECP}$.

Proposition 2.6 *Given* u *a non universal vertex of* G *and* $j \leq \lfloor n/2 \rfloor$ *such that* $\alpha(N(u)) \geq \lfloor n/j \rfloor$, *the* (u, j)-outside-neighborhood inequality *defined as*

$$(\lfloor n/j \rfloor - 1) x_{uj} - \sum_{v \in V \backslash N[u]} x_{vj} + \sum_{k=j+1}^{n} b_{jk} x_{uk} \leq \sum_{k=j+1}^{n} b_{jk} (w_k - w_{k+1}),$$

where $b_{jk} = \lfloor n/j \rfloor - \lfloor n/k \rfloor$, *is valid for* $\mathcal{ECP}$ *and defines a facet of* $\mathcal{ECP}$ *if the following conditions hold:*

- *there exists* $v \in V \backslash N[u]$ *such that* $N(u) \backslash N(v) \neq \varnothing$,
- *if* n *is odd, the complement of* $G - u$ *has a perfect matching,*
- *for all* $v \in V \backslash N[u]$, *there exists a* $\lfloor n/2 \rfloor$*-eqcol such that* $C_j = \{u, v\}$,
- *for all* k *such that* $j \leq k \leq \lfloor n/2 \rfloor$ *and* $\lfloor \frac{n}{k} \rfloor > \lfloor \frac{n}{k+1} \rfloor$, *there exists a* k*-eqcol such that* $|C_j \cap N(u)| = \lceil n/k \rceil$, *and a* k*-eqcol such that* $u \in C_j$ *and* $|C_j \backslash N[u]| = \lfloor n/k \rfloor - 1$,
- *for all* $k \in \{j, \dots, n-3\} \backslash \mathscr{A}(G)$, *there exists a* k*-eqcol lying on the face defined by the inequality.*

Proposition 2.7 *Given $u \in V$, Q be a clique of G such that $Q \cap N[u] = \varnothing$ and j, k such that $j \leq k \leq n-2$ and $\alpha(N(u)) \geq \lceil n/k \rceil - 1$. The (u, j, k, Q)-*clique-neighborhood inequality *defined as*

$$\big(\lceil n/k \rceil - 1\big) x_{uj} + \sum_{v \in N(u) \cup Q} x_{vj} + \sum_{l=k+1}^{n} \big(\lceil n/k \rceil - \lceil n/l \rceil\big) x_{ul} + \sum_{v \in V} x_{vn-1} + \sum_{v \in V \setminus \{u\}} x_{vn}$$
$$\leq \sum_{l=j}^{k-1} b_{ul}(w_l - w_{l+1}) + \sum_{l=k}^{n-2} \lceil n/k \rceil (w_l - w_{l+1}) + \sum_{l=n-1}^{n} (\lceil n/k \rceil + 1)(w_l - w_{l+1}),$$

where $b_{ul} = \min\{\lceil n/l \rceil, \alpha(N(u)) + 1\}$, is a valid inequality for $\mathcal{ECP}$. If there exists $v \in Q$ such that $N(u) \backslash N(v) \neq \varnothing$, the inequality defines a facet of $\mathcal{ECP}$ when the following conditions hold:

- *for all $l \in \{j, \ldots, n-3\} \backslash \mathscr{A}(G)$, there exists an l-eqcol lying on the face defined by the inequality,*
- *for all $v \in V \backslash (N[u] \cup Q)$, there exist two k-eqcols lying on the face defined by the inequality, with $v \in C_j$ in the first one and where the second one is obtained from the first by only changing the color of v, i.e. $v \notin C_j$,*
- *for all $1 \leq i \leq \lceil n/k \rceil - 1$, if $l = \max\{\lceil \frac{n}{i} \rceil - 1, n-2\}$, there exist two l-eqcols lying on the face defined by the inequality such that $u \in C_j$ in one of them and $u \in C_l$ in the other.*

Although the sufficient conditions in the previous results are strong, we find several cases where they hold. Moreover, even when the inequalities do not define facets, the dimension of the faces defined by them is quite high. For example, if $k \leq \lceil n/2 \rceil - 1$, it can be proved that the dimension of the face defined by the (u, j, k, Q)-clique-neighborhood inequality is at least $dim(\mathcal{ECP}) - \big(3n - |\mathscr{A}(G)| - \lfloor n/2 \rfloor - |N(u)| - |Q| - 5\big)$.

3 Computational performance of valid inequalities

In this section, we report on the computational performance of the families of valid inequalities studied in the previous section, embedded as cuts in a B&C algorithm for solving ECP.

In order to strengthen the formulation and avoid considering classes of symmetric colorings, constraints $x_{vj} = 0$, $\forall\ 1 \leq v < j \leq n$ are considered within the initial relaxation, and $x_{vj} \leq \sum_{u=j-1}^{v-1} x_{uj-1}$, $\forall\ 2 \leq j \leq v \leq n$ are handled as cuts during the optimization.

The cutting process consists in looking for violated clique and (S, Q)-2-rank inequalities with a greedy algorithm. During the separation of clique inequalities, it attempts to find violated (u, j, k, Q)-clique-neighborhood inequalities by scanning vertices u not adjacent to a given clique Q. Whenever not enough cuts were generated, it tries to add block, $(u, j, N(u))$-subneighborhood and (u, j)-outside-neighborhood inequalities, handled by enumeration, and S-color inequalities with a greedy algorithm. Separation routines for clique and block inequalities are exposed in [4]. The B&C algorithm also includes an initial heuristic, a primal heuristic and a custom branching rule.

Experiments were carried out over random instances of 70 vertices with different density percentages and 2 hours time limit. We compare our B&C algorithm with (BC^+) and without (BC) our new inequalities against the general purpose IP-solver CPLEX 12.1 and results reported in [5].

%	% solved inst.				Nodes (average)				Time in sec. (average)			
dens.	BC^+	BC	CPX	[5]	BC^+	BC	CPX	[5]	BC^+	BC	CPX	[5]
10	100	100	100	100	3.4	4	13.3	57	0.3	0.3	4	109
30	90	90	0	0	2135	3949	–	–	276	224	–	–
50	70	70	0	0	7932	21595	–	–	1354	2145	–	–
70	80	80	10	100	525	2970	214	678	128	446	4380	273
90	100	100	100	100	5.1	14.5	30	9.4	2.6	2.8	29	11

As one may appreciate from the table, the addition of our cutting planes has shown to be particularly useful in substantially decreasing the number of Branch-and-Bound nodes and the CPU time was significantly reduced on medium and high density instances.

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