A Polyhedral Approach for the Graph Equitable Coloring Problem*

Daniel Severin[1] Isabel Méndez Diaz[2] Graciela Nasini[1,3]

[1] Fac. de Ciencias Exactas, Ingeniería y Agrim., Universidad Nacional de Rosario, Pellegrini 250, 2000 Rosario, Argentina, {nasini, daniel}@fceia.unr.edu.ar
[2] Fac. de Ciencias Exactas y Naturales, Universidad de Buenos Aires, Pabellón I - Ciudad Universitaria, C1428EGA Capital Federal, Argentina, imendez@dc.uba.ar
[3] CONICET, Argentina

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Introduction

Standard coloring

Let $G = (V, E)$ be a simple graph. A k-coloring is a partition C_1, C_2, ..., C_k of V such that:

- $u, v \in C_j \implies uv \notin E \qquad \forall\, j = 1, \ldots, k$

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Equitable coloring

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Equitable chromatic number

$\chi_{eq}(G) = \min\{k : G \text{ admits a } k\text{-eqcol}\}$

- Find $\chi_{eq}(G)$ is **NP-Hard** [Furmanczyk H., Kubale M. (2005)].

Coloring vs. Equitable coloring

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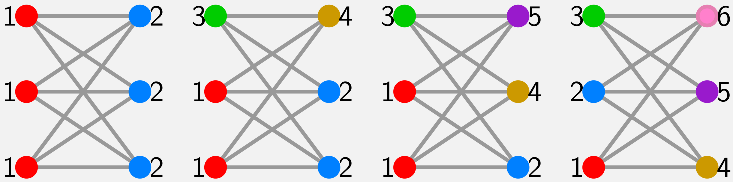

$2 - eqcol$ $4 - eqcol$ $5 - eqcol$ $6 - eqcol$

$$\mathscr{A}(K_{3,3}) = \{2, 4, 5, 6\}$$

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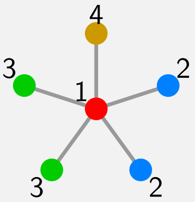
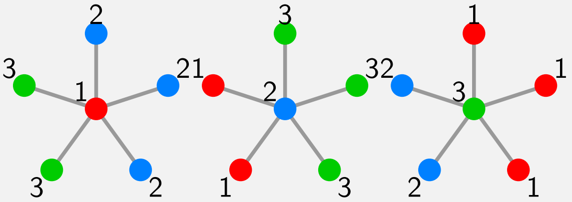


$\chi_{eq}(K_{1,5}) = 4$



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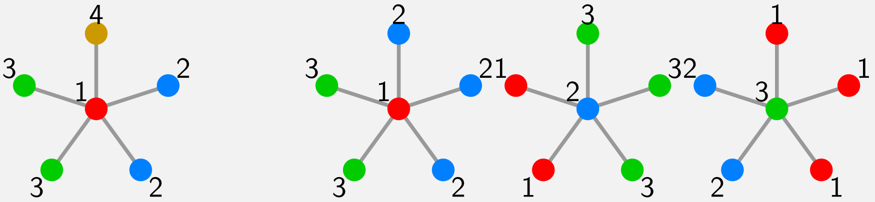

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- We cannot restrict ourselves to connected graphs as in the case of standard coloring.

Goals and Results

Goals:

- Formulate an IP model for the equitable coloring problem.
- Study its polyhedral structure.
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In this work:

- Standard coloring problem + *equity constraints*.
- First polyhedral results.
- Preliminary computational experiences.

IP Model for the standard coloring problem

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$$x_{vj} = \begin{cases} 1 & \text{color } j \text{ is assigned to } v, \\ 0 & \text{otherwise,} \end{cases} \qquad w_j = \begin{cases} 1 & x_{vj} = 1 \text{ for some } v, \\ 0 & \text{otherwise.} \end{cases}$$

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(x, w) *coloring* $\leftrightarrow$ (x, w) *binary and satisfies* :

$$\sum_{j=1}^{n} x_{vj} = 1, \qquad \forall\, v \in V \qquad (1)$$

$$x_{uj} + x_{vj} \leq w_j, \qquad \forall\, uv \in E,\ j = 1, \ldots, n \qquad (2)$$

$$w_{j+1} \leq w_j, \qquad \forall\, j = 1, \ldots, n-1 \qquad (3)$$

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- $\mathcal{CP} = conv \; \{ \; (x, w) : (x, w) \; coloring \; \}$
- $\chi(G) = min \; \{ \; w_1 + w_2 + \ldots + w_n \; : \; (x, w) \in \mathcal{CP} \; \}$

Equity constraints

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- Linearize "$w_i = w_j = 1 \implies -1 \leq |C_i| - |C_j| \leq 1$"

$$-1 - M(2 - w_i - w_j) \leq \sum_{v=1}^{n} x_{vi} - \sum_{v=1}^{n} x_{vj} \leq 1 + M(2 - w_i - w_j) \;\; \forall i, j \quad (1)$$

[Bahiense et al. (2007)]

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- Our first proposal: "$w_j = 1 \implies y \leq |C_j| \leq y + 1$"

$$y + M(1 - w_j) \leq \sum_{v=1}^{n} x_{vj} \leq y + 1 \;\; \forall j \qquad (2)$$

Equity constraints

Lemma

If $|C_j| \geq |C_{j+1}|$ in a k-eqcol, then $|C_j|$ is uniquely determined by n and k.

$$|C_j| = \begin{cases} \lfloor n/k \rfloor + 1, & j = 1, \ldots, (n \mod k) \\ \lfloor n/k \rfloor, & j = (n \mod k) + 1, \ldots, k \\ 0 & j = k+1, \ldots, n. \end{cases}$$

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Proposition

(x, w) eqcol (such that $|C_j| \geq |C_{j+1}|$) $\Leftrightarrow$ (x, w) coloring such that:

$$\sum_{v \in V} x_{vj} = \sum_{k=2}^{n} (t_k^j - t_{k-1}^j) w_k \qquad (3)$$

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$$x_{ij} \leq w_j, \qquad \forall\ i\ isolated,\ j = 1, \ldots, \lfloor n/2 \rfloor$$

$$\mathcal{ECP} = conv\ \{\ (x,w) : (x,w)\ eqcol\ \}$$

$$\chi_{eq}(G) = min\ \{\ w_1 + w_2 + \ldots + w_n\ :\ (x,w) \in \mathcal{ECP}\ \}$$

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Dimension of the equitable coloring polytope

Proposition

$$dim(\mathcal{ECP}) = n^2 - 2n + |\mathscr{A}(G)|$$

System of equalities:

$$\sum_{j=1}^{n} x_{vj} = 1, \qquad \forall\ v \in V$$

$$\sum_{v \in V} x_{vj} = \sum_{k=2}^{n} (t_k^j - t_{k-1}^j) w_k, \qquad \forall\ j = 2, \dots, n$$

$$w_{\chi_{eq}(G)} = 1,$$

$$w_k = w_{k+1}, \qquad \forall\ k \notin \mathscr{A}(G)$$

Some facets of $\mathcal{CP}$ and $\mathcal{ECP}$

- $\mathcal{ECP} \subset \mathcal{CP}$.
- (π, π_0) facet-defining in $\mathcal{CP} \Rightarrow (\pi, \pi_0)$ valid inequality of $\mathcal{ECP}$.

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	$x_{vj} \geq 0$	$x_{vj} \geq \sum_{k \in \mathscr{K}_1} (w_k - w_{k+1})$
(clique)	$\sum_{v \in Q} x_{vj} \leq w_j$	$\sum_{v \in Q} x_{vj} \leq w_j - \sum_{k \in \mathscr{K}_2} (w_k - w_{k+1})$
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- Frequently, $\mathscr{K}_1 = \mathscr{K}_2 = \mathscr{K}_3 = \varnothing$.

Example: Clique inequalities

EqClique inequality

Q maximal clique, $\quad |Q| \geq 2, \quad j \leq \lfloor n/2 \rfloor$

$$\sum_{v \in Q} x_{vj} \leq w_j - \sum_{k \in \mathscr{K}_2} (w_k - w_{k+1})$$

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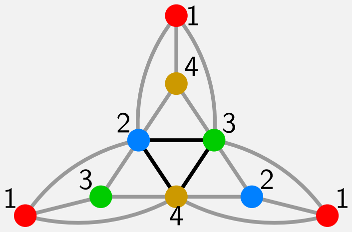

- No vertex of Q can use color 1 in a 4-eqcol $\implies$ $\mathscr{K}_2 = \{4\}$.

Testing clique cuts

- C & B: Generates clique cuts at the root node.
- Testbed: Over 230 random graphs (15-30 vertices)
  - ⋆ low density = 0-33% of edges
  - ⋆ medium density = 33-66% of edges
  - ⋆ high density = 66-100% of edges
- Compare C & B with pure B & B:
  - ⋆ evaluated nodes
  - ⋆ elapsed time
- Optimizer: CPLEX 10.1
- An instance is solved $\Leftrightarrow$ $time < 2hs$.

Testing clique cuts

n	Dens.	% of solv. inst.		Evaluated nodes		Time in sec.	
		B & B	C & B	B & B	C & B	B & B	C & B
20	low	100	100	22	5	0	0
	med.	100	100	957	70	7	3
	high	100	100	515	16	5	2
25	low	100	100	119	34	3	2
	med.	87	100	10082	520	369	58
	high	50	90	25471	41	591	20
30	low	100	100	91	105	6	6
	med.	22	78	499	39	43	28
	high	0	33	–	553*	–	424*
35	low	100	100	397	140	51	26
	med.	22	33	10724	2254	2520	611
	high	0	13	–	52*	–	181*

Further works

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- Identify new families of facets in $\mathcal{ECP}$.
- Study the behaviour of facets in $\mathcal{ECP}$ as cuts.

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