# A POLYHEDRAL APPROACH FOR THE GRAPH EQUITABLE COLORING PROBLEM

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**Keywords:** equitable graph coloring, facet-defining inequalities, cutting plane algorithm.

**Abstract.** An equitable coloring is a way of coloring the vertices of a graph such that a pair of adjacent vertices do not share the same color and any pair of color classes differ in size by at most one. Given a graph $G$, the equitable coloring problem is to find the minimum number of colors needed so as to have an equitable coloring of $G$.

It is known that Branch & Cut algorithms based on the polyhedral study of linear integer programming (IP) models have proven to be an important tool to deal with traditional coloring problems.

The goal of this work is to give an IP formulation for the equitable coloring problem, studying its polyhedral structure, and develop a cutting plane algorithm. These are the first steps to make a further Branch & Cut algorithm.

## 1 Introduction

Many applications require to split a set of conflicting elements into *balanced* and no conflictive classes (Pemmaraju, 2001; Tucker, 1973). These kind of applications are usually modeled as a graph coloring problem with additional restrictions on the color class sizes. In particular, in the *equitable coloring problem* it is required that the difference between the sizes of any pair of color classes is at most one. The equitable coloring problem was first studied by Meyer (1973). Like many graph optimization problems, the equitable coloring problem belongs to the class of NP-Hard problems (Furmańczyk and Kubale, 2005).

From now on, we assume that $G = (V, E)$ is a simple graph where $V = N = \{1, \dots, n\}$ and $E$ is the set of edges of $G$. Given a $k$-coloring of $G$, we denote by $C_j$ the $j$th color class, for each $j \in N$. A $k$-coloring is an *equitable $k$-coloring* (or just $k$-eqcol) if and only if $||C_i| - |C_j|| \leq 1$, for $i, j = 1, \dots, k$.

The *equitable chromatic number* of $G$, $\chi_{eq}(G)$, is the minimun $k$ for which there exists a $k$-eqcol in $G$. Whenever it is clear from the context, we write $\chi_{eq}$ rather than $\chi_{eq}(G)$.

The equitable coloring problem presents some additional pitfalls with respect to the coloring problem. First, a graph admiting a $k$-eqcol may not admit a $(k + 1)$-eqcol. For example, $\chi_{eq}(K_{3,3}) = 2$ but there is not a 3-eqcol in $K_{3,3}$. Then, we denote by $\mathscr{A}(G)$ the *set of admissible equitable colorings* of $G$, i.e

$$\mathscr{A}(G) = \{k \in N : G \text{ admits a } k\text{-eqcol}\}.$$

In this way, $\mathscr{A}(K_{3,3}) = \{2, 4, 5, 6\}$.

Also, the equitable chromatic number of a graph can be smaller than the equitable chromatic number of one of its subgraphs. For instance, considering the disconnected graph $G = K_{1,5} \cup$

$K_{1,5} \cup K_{1,5}$, $\chi_{eq}(G) = 3$ but $\chi_{eq}(K_{1,5}) = 4$. So, we can not restrict ourselves to connected graphs as in the case of traditional coloring problems.

Let $\Delta$ be the maximum degree of vertices in $G$. It is known that $\Delta + 1, \ldots, n \in \mathscr{A}$ (Hajnal and Szemerédi, 1970). Moreover, Kierstead and Kostochka (2008) presents a polynomial algorithm for finding a $(\Delta+1)$-eqcol.

Greedy heuristics for the problem can be found in (Furmańczyk and Kubale, 2005). However, as far as we know, there are no previous works on IP based exact algorithms as in the case of coloring problem (Méndez-Díaz and Zabala, 2006; Figueiredo et al., 2002; Campêlo et al., 2004; Mehrotra and Trick, 1996).

Clearly, IP models for the coloring problem can be adapted for the equitable coloring problem by addition of the *equity constraints*. In particular, we consider the model for the coloring problem presented by Méndez-Díaz and Zabala (2006), which has shown a good performance in the context of a Branch & Cut algorithm based on its polyhedral study.
In that work, the authors use binary variables $X \in \{0,1\}^{n^2}$ and $W \in \{0,1\}^n$ to represent colorings as follows: for each vertex $v \in V$ and color $j \in N$,

$$x_{vj} = \begin{cases} 1 & \text{if color } j \text{ is assigned to vertex } v, \\ 0 & \text{otherwise,} \end{cases} \qquad w_j = \begin{cases} 1 & \text{if } x_{vj} = 1 \text{ for some vertex } v, \\ 0 & \text{otherwise.} \end{cases}$$

The restrictions in this model are:

- *assignment constraints*, saying that each vertex has to be painted by an unique color, i.e.

$$\sum_{j=1}^{n} x_{vj} = 1, \qquad \forall\, v \in V, \tag{1}$$

- *edge constraints*, saying that two adjacent vertices can not share the same color, i.e.

$$x_{uj} + x_{vj} \leq w_j, \qquad \forall\, uv \in E,\ j \in N, \tag{2}$$

- *color order constraints*, avoiding to consider solutions corresponding to some classes of symmetric $k$-colorings, i.e.

$$w_{j+1} \leq w_j, \qquad \forall\, j \in N - \{n\}. \tag{3}$$

The *coloring polytope*, $\mathcal{CP}(G)$, is the convex hull of $(X, W) \in \{0,1\}^{n^2+n}$ satisfying the constraints (1), (2) and (3). The chromatic number can be computed by minimizing $\sum_{j=1}^{n} w_j$ on $\mathcal{CP}$.

In order to obtain an IP model for the equitable coloring problem, we need to add *equity constraints*, i.e. we have to linearize the non linear constraints $|\sum_{v=1}^{n} x_{vi} - \sum_{v=1}^{n} x_{vj}| \leq 1$, for $i, j$ such that $w_i = w_j = 1$.
Firstly, let us observe that the natural constraint "if color $j$ is not used in the coloring, no vertex can be painted by $j$" is implied by (2) for non isolated vertices. However, since in the equitable coloring problem the graph could be disconnected, we are forced to add these constraints for isolated vertices. Let $I \subseteq V$ be the set of isolated vertices. Then, we have to add the constraints

$$x_{ij} \leq w_j, \qquad \forall\, i \in I,\ j \in N. \tag{4}$$

In the next section we analyze several ways for modelling equity constraints.

## 2 Modelling equity constraints

We first analyze the formulation proposed by Bahiense et al. (2007), where the equity constraints are modeled by using a *big M* constant: for all $i, j \in N$ such that $i < j$,

$$-1 - M(2 - w_i - w_j) \leq \sum_{v=1}^{n} x_{vi} - \sum_{v=1}^{n} x_{vj} \leq 1 + M(2 - w_i - w_j).$$

It is known that this kind of constraints usually give bad linear relaxations. In order to reduce the use of the *big M* constant, we propose to introduce a new free variable $y$ and to model the equity constraint as follows:

$$y + M(1 - w_j) \leq \sum_{v=1}^{n} x_{vj} \leq y + 1, \qquad \forall\, j \in N.$$

Finally, we propose a third formulation based in the following results. Let us observe that, in a $k$-eqcol, the color class sizes have only two possible values. Moreover,

**Lemma 1.** *Given $k \in N$, let $y = \lfloor n/k \rfloor$ and $p = n \mod k$. Then, every $k$-eqcol has $p$ color classes of size $y + 1$ and $k - p$ color classes of size $y$.*

Let us note that, if we restrict ourselves to $k$-eqcols such that $|C_i| \geq |C_j|$ when $i \leq j$, $t_j^k = |C_j|$ is well defined and

$$t_k^j = \begin{cases} \lfloor n/k \rfloor + 1 & \text{if } j \leq p, \\ \lfloor n/k \rfloor & \text{if } p < j \leq k, \\ 0 & \text{if } k < j. \end{cases}$$

From now on, a $k$-eqcol is a $k$-eqcol satisfying $|C_i| \geq |C_j|$ when $i \leq j$. Thus, we have

**Proposition 2.** *Let $(X, W)$ be an integer point in $\mathcal{CP}$ satisfying (4) and*

$$\sum_{v=1}^{n} x_{vj} = \sum_{k=2}^{n} (t_k^j - t_{k-1}^j) w_k, \qquad \forall\, j \in N - \{1\}. \tag{5}$$

*Then, the $(\sum_{j=1}^{n} w_j)$-coloring associated with $(X, W)$ is equitable.*

Let us observe that in (5) we do not need to consider color 1 because, for this case, the constraint can be obtained as a linear combination of constraints (1) and (5). Furthermore, constrains (2) and (4) with $j > \lfloor n/2 \rfloor$ are dominated by (5) and the nonnegativity of $x_{vj}$.

We tested the previous three formulations by solving small instances using a pure Branch & Bound method and we concluded that the third one outperforms the others. In this way, we decided to work with the formulation where equity contraints are modeled by (5). Then, our model for the equitable coloring problem uses $n^2 + n$ variables and $\big(|E| + |I|\big)\lfloor n/2 \rfloor + 3n - 2$ constraints.

## 3 The equitable coloring polytope

We define the *equitable coloring polytope* of $G$, $\mathcal{ECP}(G)$, as the convex hull of integer points of $\mathcal{CP}$ satisfying (4) and (5).

Let us observe that colorings of $K_n$ are equitable colorings and correspond to the set of assignments between colors in $N$ and vertices in $V$. Hence, the linear relaxation of $\mathcal{ECP}$ is an integer polytope. So, in the sequel we will consider $G \neq K_n$ (or equivalently, $\chi_{eq}(G) < n$).

We first study the dimension of $\mathcal{ECP}$. We prove that

**Proposition 3.** *The dimension of $\mathcal{ECP}$ is $n^2 - 2n + |\mathscr{A}|$ and a minimal equation system is defined by $w_{\chi_{eq}} = 1$, $w_k = w_{k+1}, \forall\, k \notin \mathscr{A}$ and the equalities in the formulation, i.e. (1), (5).*

In order to find facet-defining inequalities, we start analyzing valid inequalities in the formulation. We have that

**Proposition 4.** *The inequality $w_j \geq w_{j+1}$ defines a facet of $\mathcal{ECP}$ iff $j \in \mathscr{A} - \{n\}$. The inequality $w_j \leq 1$ defines a facet of $\mathcal{ECP}$ iff $j = \chi_{eq} + 1$. For all $j \in N$, the inequality $w_j \geq 0$ does not define a facet of $\mathcal{ECP}$.*

In the case of the nonnegativity constraints of $x_{vj}$, we have the following

**Proposition 5.** *Let $v \in V$ and $j \in N$. If, for every $k \in \mathscr{A} - \{n\}$, there exists a $k$-eqcol in which $v \notin C_j$, then the inequality $x_{vj} \geq 0$ defines a facet of $\mathcal{ECP}$.*

It is not hard to see that, if $j$ does not divide $n+1$, the sufficient condition in the previous proposition holds and we obtain

**Corollary 6.** *If $j$ does not divide $n+1$, then $x_{vj} \geq 0$ defines a facet of $\mathcal{ECP}$, for all $v \in V$.*

If $j \neq 1$ and $j$ divides $n+1$, the sufficient condition in Proposition 5 can be reduced in the following way

**Corollary 7.** *Let $j \neq 1$ such that $j$ divides $n+1$. If $j \notin \mathscr{A}$, $x_{vj} \geq 0$ defines a facet of $\mathcal{ECP}$ for all $v \in V$. If $j \in \mathscr{A}$, $x_{vj} \geq 0$ defines a facet of $\mathcal{ECP}$ for all $v \in V$ such that there exists a $j$-eqcol in which $v \notin C_j$.*

Finally, for the first color we have

**Corollary 8.** *$x_{v1} \geq 0$ defines a facet of $\mathcal{ECP}$ for all $v \in V$ such that, if $k \in \mathscr{A} - \{n\}$ and $k$ divides $n-1$, there exists a $k$-eqcol in which $v \notin C_1$.*

Next, we show some cases where the nonnegativity constraints are not facet-defining inequalities. If $G = K_{3,2}$ and $u$ is one of the two vertices with degree 3, $x_{u2} \geq 0$ does not define a facet because it is dominated by the valid inequality $x_{u2} + w_3 \geq 1$. Another example is when $V - \{v\}$ is a clique. In this case, the nonnegativity constraint associated with $x_{v1}$ is dominated by the valid inequality $x_{v1} + w_n \geq 1$.

Next step in the polyhedral study of $\mathcal{ECP}$ is to find valid inequalities not included in the formulation. It is natural to start with those facet-defining inequalities for $\mathcal{CP}$.
Given a maximal clique $Q$ of $G$ and $j \in N$, only one vertex of $Q$ can be painted with color $j$. This condition is imposed by the *clique constraint*

$$\sum_{v \in Q} x_{vj} \leq w_j. \tag{6}$$

We denote by $\mathcal{F}_{Q,j}$ the face defined by (6).

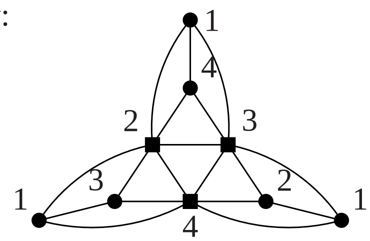
Although clique inequalities always define facets of $\mathcal{CP}$, they can define facets of $\mathcal{ECP}$ only if $j \leq \lfloor n/2 \rfloor$. In fact, when $j > \lfloor n/2 \rfloor$, (6) is dominated by (5) and the nonnegativity constraints. So, we have

**Proposition 9.** *Let $Q$ be a maximal clique of $G$ of size at least 2, and $j \leq \lfloor n/2 \rfloor$. Let $K = \{k \in \mathscr{A} : |Q| \leq p_k < j \leq k \vee j \leq p_k < k - |Q|\}$, where $p_k = n \mod k$, for each $k$. The clique inequality (6) is facet-defining of $\mathcal{ECP}$ if the following conditions hold:*

1. *For all $k \in K$, there exists a $k$-eqcol such that $|C_j \cap Q| = 1$.*

2. *$\{\lceil n/2 \rceil, \ldots, n-j\} \cap \mathscr{A} \neq \varnothing$.*

Condition 2 of Proposition 9 does not always hold. Let $G$ be $K_{3,3}$ and suppose two adjacent vertices $u, v$. Thus, $Q = \{u, v\}$ is a maximal clique but $x_{u3} + x_{v3} \leq w_3$ does not define a facet in $\mathcal{ECP}$. An example where condition 1 of Proposition 9 becomes false is the clique $Q$ denoted by squares in the graph $G$ given below:




Now, $G$ admits a 4-eqcol but no vertex of $Q$ can use color 1. In this case, $\mathcal{F}_{Q,1}$ does not define a facet of $\mathcal{ECP}$.

Even when a clique inequality does not define a facet, it defines a face of high dimension, as we state in the following

**Corollary 10.** $dim(\mathcal{F}_{Q,j}) \geq dim(\mathcal{ECP}) - \big(n + |\mathscr{A}| - |Q| - 1\big)$.

## 4 Computational experiments and conclusions

Our study in the previous section suggests that a cutting plane algorithm based on clique inequalities may be an effective way of strengthening the linear relaxation of $\mathcal{ECP}$. In order to know whether the clique cuts make an improvement or not, we compare the behavior of a pure Branch & Bound algorithm (B & B) with a Cut & Branch algorithm (C & B) that separates clique inequalities at the root node. We show the performance of both algorithms on 234 randomly generated graphs. The tests were performed on a Sun UltraSparc workstation, using CPLEX 10.1 as the optimizer.

Table 1 summarizes the results. The first column is the number of vertices. The second column is the graph density: *low* means 0-33%, *medium* means 33-66% and *high* means 66-100%, where the percentage is calculated with $100|E|/(n(n-1)/2)$. Columns 3 and 4 are the percentage of succesfully solved instances (s.s.i.) for each algorithm. An instance is not solved whether the optimizer exceeds the time limit (1 hour). Columns 5-8 are the averages of the evaluated nodes/elapsed time for each algorithm over instances solved by both algorithms, except the averages marked with (*) where they are calculated over s.s.i. of C & B.

In almost every instance, the addition of clique cuts have been improved the performance of the optimization. Furthermore, every solved instance by B & B has also been solved by C & B, but not conversely.

| n | Dens. | % of s.s.i. | | Evaluated nodes | | Time in sec. | |
|---|---|---|---|---|---|---|---|
| | | B & B | C & B | B & B | C & B | B & B | C & B |
| 15 | low | 100 | 100 | 6 | 3 | 0 | 0 |
| | med. | 100 | 100 | 81 | 15 | 0 | 0 |
| | high | 100 | 100 | 226 | 55 | 1 | 0 |
| 20 | low | 100 | 100 | 22 | 5 | 0 | 0 |
| | med. | 100 | 100 | 957 | 70 | 7 | 3 |
| | high | 100 | 100 | 515 | 16 | 5 | 2 |
| 25 | low | 100 | 100 | 119 | 34 | 3 | 2 |
| | med. | 87 | 100 | 10082 | 520 | 369 | 58 |
| | high | 50 | 90 | 25471 | 41 | 591 | 20 |
| 30 | low | 100 | 100 | 91 | 105 | 6 | 6 |
| | med. | 22 | 78 | 499 | 39 | 43 | 28 |
| | high | 0 | 33 | – | 553* | – | 424* |
| 35 | low | 100 | 100 | 397 | 140 | 51 | 26 |
| | med. | 22 | 33 | 10724 | 2254 | 2520 | 611 |
| | high | 0 | 13 | – | 52* | – | 181* |

Table 1: Benchmarks for graphs from 15 to 35 vertices.

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